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Research Paper

Torsion of Functionally Graded Anisotropic Linearly Elastic Circular Cylinder

István ECSEDI, Attila BAKSA

University of Miskolc
Institute of Applied Mechanics
Hungary
e-mail: {istvan.ecsedi, attila.baksa}@uni-miskolc.hu

The object of this paper is Saint-Venant torsion of functionally graded anisotropic linearly elastic circular cylinder. The class of anisotropy considered has at least one plane of elastic symmetry normal to the axis of the circular cylinder. The elastic coefficients have radial dependence only. Here, we give the solution of Saint-Venant torsion problem for circular cylinder made of functionally graded anisotropic linearly elastic materials.

Key words: anisotropic; circular cylinder; elastic; functionally graded materials; Saint-Venant torsion.

1. Introduction

Saint-Venant torsion problem of cylindrical bars has been interest for many years and has been treated from numerous aspects. Recently there is a growing interest in the context of non-homogeneous and/or anisotropic bars such as Arghavan and Hematiyan [1], Batra [2], Horgan and Chan [4], Horgan [5], Lekhintskii [6, 7], Rand and Rovenski [9], Rooney and Ferrari [10]. In this paper the torsional deformation of functionally graded anisotropic linear elastic solid and hollow circular cylinders is studied. Functionally graded materials (FGMs) are microscopically inhomogeneous composite materials, in which the volume fraction of two or more materials is varied smoothly and continuously as a function of position along certain dimension(s) of the structure from one point to other [13]. This materials are mainly constructed to operate in high temperature enviroments [11]. The class of anisotropy considered has at least one plane of elastic symmetry normal to the axis of the circular cylinder. The elastic coefficients are smooth functions of the radial coordinate on the whole cross-section for the FGMs. The case of layered circular cross-section is also discussed when the elastic coefficients are piecewise smooth functions of the radial coordinate. This type of the cross-sections is called compound (composite) cross-section.

2. Formulation of Saint-Venant torsional problem

Let $B = A \times (0, L)$ be a right circular cylinder of length L. Let A_1 and A_2 be the bases and $A_3 = \partial A \times (0, L)$ the mantle of B. The cross-section A is given in the Cartesian coordinate frame $Oxyz$

$$A = \left\{ (x, y) \,|\, c_1^2 \leq x^2 + y^2 \leq c_2^2 \right\}. \tag{2.1}$$

The Cartesian coordinate frame $Oxyz$ is supposed to be chosen in a such way that the Oz axis is parallel to the generators of the cylindrical boundary surface segments $A_3^{'}$ and $A_3^{''}$ (Fig. 1). The plane Oxy contains the terminal cross-section A_1. The position of the end cross-section A_2 is given by $z = L$. A point P in $B = B \cup A_1 \cup A_2 \cup A_3^{'} \cup A_3^{''}$ is indicated by the vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = \mathbf{R} + z\mathbf{e}_z$, where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors of the coordinate system $Oxyz$ (Fig. 1). In the case of Saint-Venant torsion the displacement field of the twisted cylindrical bar has the form Lekhnitskii [6, 7]

$$\mathbf{u} = \vartheta z \mathbf{e}_z \times \mathbf{R} + \vartheta \omega(x, y) \mathbf{e}_z, \tag{2.2}$$

where ϑ is the rate of twist with respect to axial coordinate z and $\omega = \omega(x, y)$ is the torsion function furthermore the vectorial product of two vectors is denoted

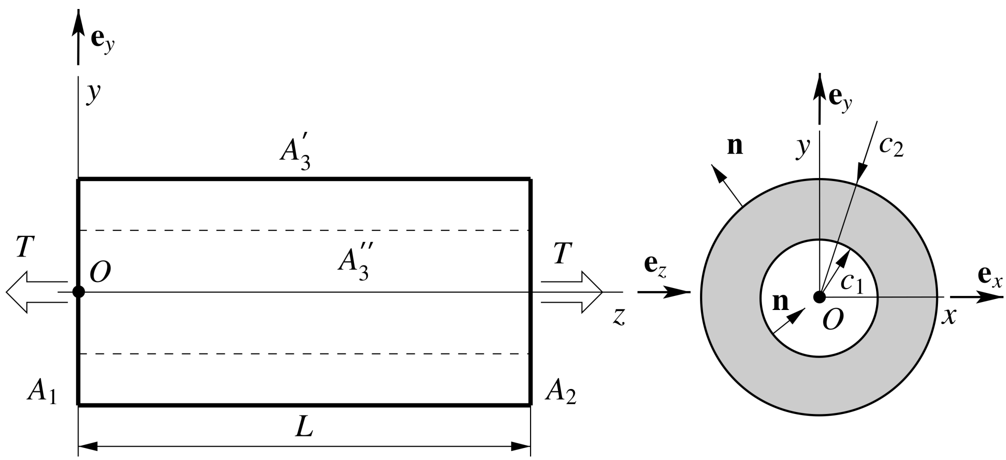

Fig. 1. Hollow circular cylinder.

by cross. For non-homogeneous, anisotropic, linearly elastic cylinder assuming that one plane of elastic symmetry normal to Oz axis we have [6, 7, 9]

$$(2.3)\quad \frac{\partial}{\partial x}\left[A_{55}\left(\frac{\partial\omega}{\partial x}-y\right)+A_{45}\left(\frac{\partial\omega}{\partial y}+x\right)\right] + \frac{\partial}{\partial y}\left[A_{45}\left(\frac{\partial\omega}{\partial x}-y\right)+A_{44}\left(\frac{\partial\omega}{\partial y}+x\right)\right]=0, \qquad (x,y)\in A,$$

$$(2.4)\quad \left[A_{55}\left(\frac{\partial\omega}{\partial x}-y\right)+A_{45}\left(\frac{\partial\omega}{\partial y}+x\right)\right]n_x + \left[A_{45}\left(\frac{\partial\omega}{\partial x}-y\right)+A_{44}\left(\frac{\partial\omega}{\partial y}+x\right)\right]n_y=0, \qquad (x,y)\in \partial A.$$

Here, $\partial A = \partial A_1 \cup \partial A_2$ is the boundary curve of A, the equation of ∂A_i $(i=1,2)$ is (Fig. 1)

$$(2.5)\qquad x^2+y^2=c_i^2 \qquad (i=1,2)$$

and n_x, n_y are the components of the unit normal vector $\mathbf{n}$ to the boundary curve ∂A_i $(i=1,2)$

$$(2.6)\qquad n_x=(-1)^i\frac{x}{c_i},\quad n_y=(-1)^i\frac{y}{c_i}, \qquad \text{on}\quad \partial A_i\ \ (i=1,2).$$

In our case the elastic coefficients (shear rigidities) A_{44}, $A_{45}=A_{54}$ and A_{55} depend only on the radial coordinate

$$(2.7)\qquad r=\sqrt{x^2+y^2}, \qquad c_1\le r\le c_2.$$

The dependence of elastic coefficients as a function of position is described by the next inhomogeneity function $f=f(r)$ such as

$$(2.8)\qquad A_{ij}=f(r)a_{ij} \qquad (i,j=4,5).$$

Here, we note $f=f(r)$ is unit free $f=f(r)>0$, $c_1\le r\le c_2$, and according to the positive definitness of strain energy density [6, 7] we have

$$(2.9)\qquad a_{44}>0 \quad \text{and} \quad a_{44}a_{55}-a_{45}^2>0.$$

It is obvious if inequalities (2.9) are satisfied then we have $a_{55} > 0$. The shearing stresses τ_{xz} and τ_{yz} can be obtained from the following equations [6, 7, 9]

$$\tau_{xz} = \vartheta \left[A_{55} \left(\frac{\partial \omega}{\partial x} - y \right) + A_{45} \left(\frac{\partial \omega}{\partial y} + x \right) \right], \tag{2.10}$$

$$\tau_{yz} = \vartheta \left[A_{45} \left(\frac{\partial \omega}{\partial x} - y \right) + A_{44} \left(\frac{\partial \omega}{\partial y} + x \right) \right], \tag{2.11}$$

and the next formula can be derived for the torque T

$$T = \int_A (x\tau_{yz} - y\tau_{xz})\,\mathrm{d}A = \vartheta S, \tag{2.12}$$

where S is the torsional rigidity of the considered non-homogeneous anisotropic cross-section according to Horgan and Chan [4], Horgan [5] and Lekhnitskii [6, 7]

$$S = \int_A \left\{ x \left[A_{45} \left(\frac{\partial \omega}{\partial x} - y \right) + A_{44} \left(\frac{\partial \omega}{\partial y} + x \right) \right] \right. \\ \left. - y \left[A_{55} \left(\frac{\partial \omega}{\partial x} - y \right) + A_{45} \left(\frac{\partial \omega}{\partial y} + x \right) \right] \right\} \mathrm{d}A. \tag{2.13}$$

Here, we note, assuming sufficient smoothness of inhomogeneity function $f = f(\sqrt{x^2 + y^2})$, standard results from the linear theory of second order partial differential equations show that the classical (strong) solutions to boundary value problem formulated by Eqs. (2.3) and (2.4) are unique to within a constant. Without loss of generality, we set this constant, which corresponds to a rigid translation along axis z, equal to zero.

3. Solution of the torsional problem

Theorem 1. The solution of the torsional boundary value problem formulated by Eqs. (2.3) and (2.4) under the condition (2.8) is

$$\omega(x, y) = \frac{1}{a_{44} + a_{55}} \left[(a_{55} - a_{44})xy + a_{45}(y^2 - x^2) \right] + C, \tag{3.1}$$

where C is an arbitrary constant.

Proof. By a direct substitution using next equations

$$\frac{\partial A_{ij}}{\partial x} = f'(r)\frac{x}{r}a_{ij}, \qquad \frac{\partial A_{ij}}{\partial y} = f'(r)\frac{y}{r}a_{ij}, \qquad f'(r) = \frac{\mathrm{d}f}{\mathrm{d}r} \tag{3.2}$$

we get, the function $\omega = \omega(x, y)$ given by Eq. (3.1) with arbitrary constant C satisfies Eqs. (2.3) and (2.4). Next we put for $C = 0$ according to the last sentence of previous section of this paper. The application of formulae (2.10) and (2.11) yields

$$\tau_{xz} = -2\vartheta \frac{a_{44}a_{55} - a_{45}^2}{a_{44} + a_{55}} f(r) y, \tag{3.3}$$

$$\tau_{yz} = 2\vartheta \frac{a_{44}a_{55} - a_{45}^2}{a_{44} + a_{55}} f(r) x. \tag{3.4}$$

From Eq. (2.13) we obtain for S

$$S = \frac{4(a_{44}a_{55} - a_{45}^2)\pi}{a_{44} + a_{55}} \int_{c_1}^{c_2} r^3 f(r)\,\mathrm{d}r. \tag{3.5}$$

For solid circular cross-section $c_1 = 0$. The expression of shear stresses in the polar coordinate system $Or\varphi z$ ($r = \sqrt{x^2 + y^2}$, $\varphi = \arctan(y/x)$) are as follows

$$\tau_{rz} = 0, \qquad \tau_{\varphi z} = T \frac{r f(r)}{2\pi \int_{c_1}^{c_2} \rho^3 f(\rho)\,\mathrm{d}\rho}. \tag{3.6}$$

The solution of Saint-Venant torsion problem presented by Eqs. (3.1), (3.6) has two important properties:

(a) The torsion function $\omega = \omega(x, y)$ does not depend on the inhomogeneity of the cross-section, it depends only on the anisotropy feature of the cross-section which is given by the stiffness coefficients a_{44}, a_{55}, $a_{45} = a_{54}$.

(b) For given torque T the stress field is independent of anisotropy, it depends only on the non-homogeneity of the considered circular cross-section. Formula (3.6) is the same as which was derived by Horgan and Chan [4] for isotropic non-homogeneous circular cross-section.

Here, we note, the expressions of the torsion function and of torsional rigidity for homogeneous and orthotropic solid circular cross-section was derived by Dubigeon [3] in cylindrical polar coordinate system. Dubigeon [3] obtained the next expression for the torsion function

$$\omega(r, \varphi) = \frac{A_{44} - A_{55}}{2(A_{44} + A_{55})} r^2 \sin 2\varphi. \tag{3.7}$$

The cited results of DUBIGEON [3] are obtained from Eqs. (3.1) and (3.5) of this paper by the next substitution

$$f(r)=1, \qquad a_{45}=0, \qquad c_1=0,$$
$$x=r\cos\varphi, \qquad y=r\sin\varphi. \tag{3.8}$$

The contour lines and the graph of the torsion function $\omega=\omega(r,\varphi)$ of solid circular cross-section for $a_{44}=1900$ MPa, $a_{55}=950$ MPa, $a_{45}=-730$ MPa and $c=c_2=10$ mm are shown in Fig. 2.

a) b)

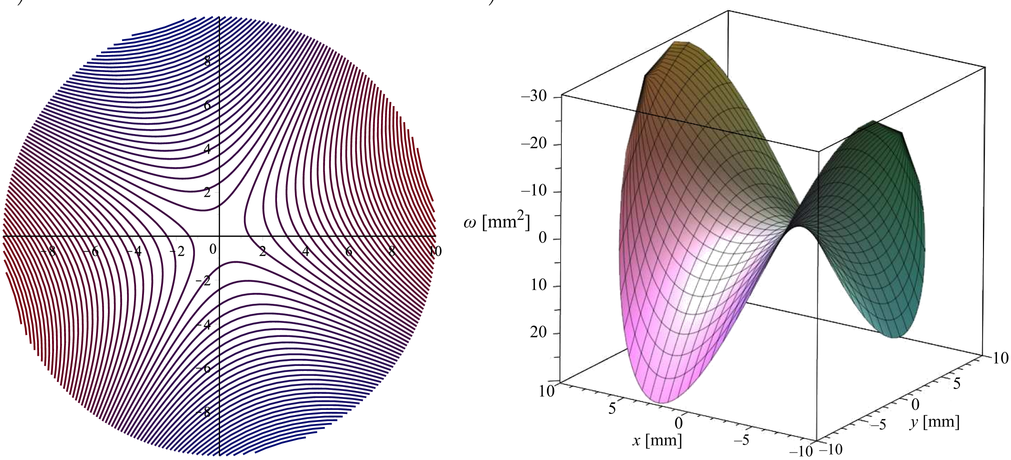

FIG. 2. Illustrations of the torsion function: a) contour lines, b) graph of the torsion function.

4. Prandtl's stress function

The determination of the Prandtl's stress function is based on the next equations

$$\tau_{rz} = \tau_{xz}\cos\varphi+\tau_{yz}\sin\varphi=0, \tag{4.1}$$

$$\tau_{\varphi z} = -\tau_{xz}\sin\varphi+\tau_{yz}\cos\varphi=2\vartheta\frac{a_{44}a_{55}-a_{45}^2}{a_{44}+a_{55}}rf(r) \tag{4.2}$$

according to Eqs. (3.3) and (3.4). The shearing stresses τ_{rz} and $\tau_{\varphi z}$ in terms of Prandtl stress function $U(r,\varphi)$ can be expressed as [6–9, 12]

$$\tau_{rz} = \frac{\vartheta}{r}\frac{\partial U}{\partial\varphi}, \tag{4.3}$$

$$\tau_{\varphi z} = -\vartheta\frac{\partial U}{\partial r}. \tag{4.4}$$

From Eqs. (4.1) and (4.3) it follows that the Prandtl stress function does not depend on the polar angle φ. The combination of Eq. (4.2) with Eq. (4.4) gives the next equation for the Prandtl's stress function

$$\frac{\mathrm{d}U}{\mathrm{d}r} = -2\frac{a_{44}a_{55} - a_{45}^2}{a_{44} + a_{55}} r f(r). \tag{4.5}$$

On the outer circular boundary curve the Prandtl's stress function vanishes [6, 7, 9] that is

$$U(c_2) = 0. \tag{4.6}$$

Integration of Eq. (4.5) under the condition (4.6) gives

$$U(r) = 2\frac{a_{44}a_{55} - a_{45}^2}{a_{44} + a_{55}} \int\limits_r^{c_2} \rho f(\rho)\,\mathrm{d}\rho. \tag{4.7}$$

For exponential FGM circular cylinder we have

$$f(r) = \exp(\alpha r). \tag{4.8}$$

In Eq. (4.8) α is a material constant. Substitution Eq. (4.8) into Eq. (4.7) leads to the expression of the Prandtl's stress function

$$U(r) = 2\frac{a_{44}a_{55} - a_{45}^2}{(a_{44} + a_{55})\alpha^2} \left[(1 - \alpha r)\exp(\alpha r) - (1 - \alpha c_2)\exp(\alpha c_2)\right]. \tag{4.9}$$

Knowing $U = U(r)$ the torsional rigidity of the exponential FGM anisotropic hollow circular cross-section can be obtained from the next formula [8, 9, 12]

$$S = 2\pi \int\limits_{c_1}^{c_2} rU(r)\,\mathrm{d}r + 2\pi U(c_1)c_1^2. \tag{4.10}$$

Application of formula (4.10) for the present case we obtain

$$S = 4\pi\frac{a_{44}a_{55} - a_{45}^2}{a_{44} + a_{55}}(H(c_1) - H(c_2)), \tag{4.11}$$

where

$$H(t) = \frac{6 - 6\alpha t + 3(\alpha t)^2 - (\alpha t)^3}{\alpha^4}. \tag{4.12}$$

It has to be noted, that the formula (4.11) can be obtained from Eq. (3.5). For homogeneous anisotropic cross-section $\alpha = 0$. From Eq. (4.9) and Eq. (4.12) for $\alpha \to 0$ the following results can be derived

$$U(r) = \frac{a_{44}a_{55} - a_{45}^2}{a_{44} + a_{55}}(c_2^2 - r^2), \tag{4.13}$$

$$S = \pi\frac{a_{44}a_{55} - a_{45}}{a_{44} + a_{55}}(c_2^4 - c_1^4). \tag{4.14}$$

For solid circular cross-section $c_1 = 0$, $c_2 = c = 10$ mm with the *'shear rigidities'* $a_{44} = 1900$ MPa, $a_{55} = 950$ MPa and $a_{45} = -730$ MPa for the three different values of α ($\alpha_1 = -0.1$ mm^{-1}, $\alpha_2 = 0$, $\alpha_3 = 0.1$ mm^{-1}) the graphs of the Prandtl's stress functions are shown in Fig. 3.

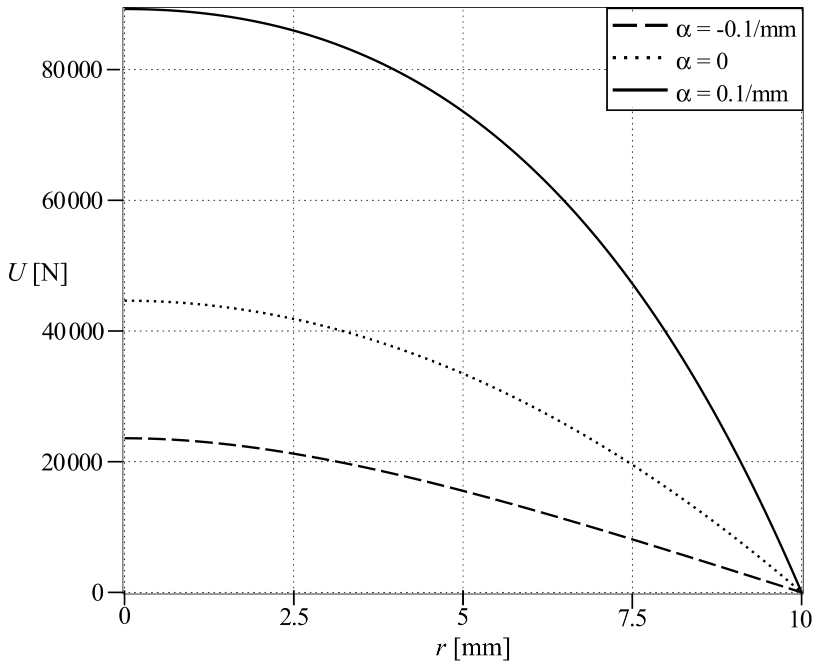

FIG. 3. The graphs of the Prandtl's stress functions.

The torsional rigidity S as a function of the material parameter α is illustrated in Fig. 4. Application of formulas (4.11) and (4.14) gives the following numerical results

$$S(\alpha_1) = 6\,390\,286.33 \text{ N} \cdot \text{mm}^2, \qquad S(\alpha_2) = 14\,022\,526.37 \text{ N} \cdot \text{mm}^2,$$

$$S(\alpha_3) = 31\,603\,209.2 \text{ N} \cdot \text{mm}^2.$$

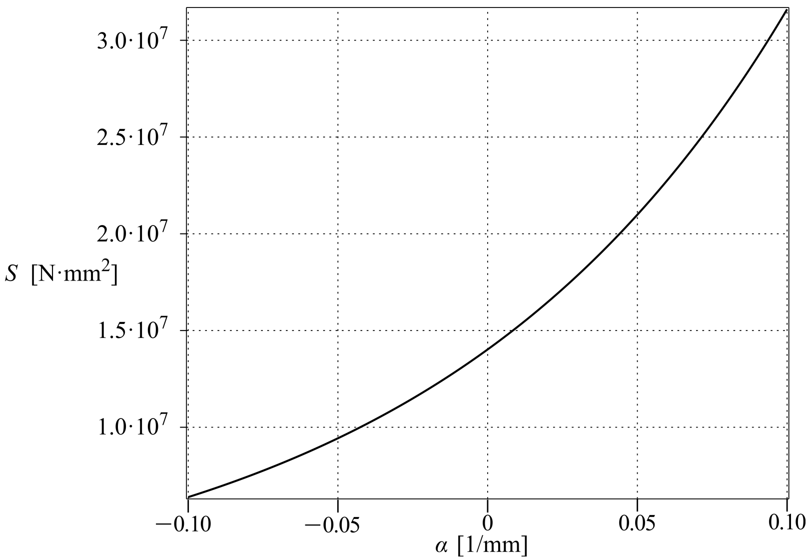

Fig. 4. Torsional rigidity as a function of α.

5. Layered non-homogeneous cross-section

Figure 5 shows a hollow circular cross-section which is layered in radial direction. In this case the inhomogeneity function is piecewise continuous on the cross-sectional domain and it is given by the next formula

$$f(r) = f_i(r) \qquad r_{i-1} \le r \le r_i \quad (i = 1 - n), \quad r_0 = c_1 \quad r_n = c_2, \tag{5.1}$$

where $f_i = f_i(r)$ is a positive value smooth function defined on the interval $r_{i-1} \le r \le r_i$ $(r = 1, ..., n)$.

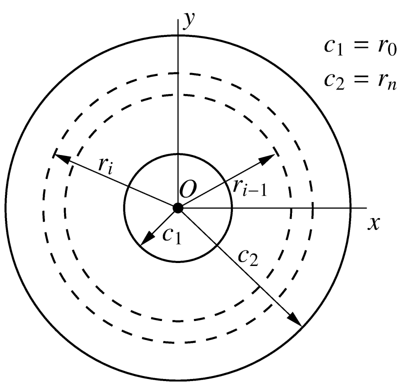

Fig. 5. Layered hollow circular cross-section.

It is evident, for layered non-homogeneous, anisotropic, hollow circular cross-section the torsional function is given by Eq. (3.1) and all formulae obtained before are valid and we have

$$\int_{c_1}^{c^2} \rho^3 f(\rho)\,\mathrm{d}\rho = \sum_{i=1}^{n} \int_{r_{i-1}}^{r_i} \rho^3 f_i(\rho)\,\mathrm{d}\rho. \tag{5.2}$$

6. Orthotropic circular cylinder

For orthotropic elastic material elasticity matrix $\mathbf{A}$ has the form the considered type of FGMs

$$\mathbf{A} = f(r)\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}. \tag{6.1}$$

In the general case we have three different shear moduli

$$A_{ii} = f(r)a_{ii} \quad (i = 4, 5, 6). \tag{6.2}$$

If the principal directions of orthotropy, which is perpendicular to the cross-section, is axis (x, y, z) then the corresponding torsion function and torsional rigidity are denoted by $(\omega_x, \omega_y, \omega_z)$ and (S_x, S_y, S_z) respectively. From Eqs. (3.1) and (3.2) we obtain

$$\omega_x = \omega_x(y, z) = \frac{a_{66} - a_{55}}{a_{55} + a_{66}} yz, \qquad S_x = \frac{4\pi F}{\frac{1}{a_{55}} + \frac{1}{a_{66}}}, \quad r = \sqrt{y^2 + z^2}, \tag{6.3}$$

$$\omega_y = \omega_y(x, z) = \frac{a_{44} - a_{66}}{a_{44} + a_{66}} xz, \qquad S_y = \frac{4\pi F}{\frac{1}{a_{44}} + \frac{1}{a_{66}}}, \quad r = \sqrt{x^2 + z^2}, \tag{6.4}$$

$$\omega_z = \omega_z(x, y) = \frac{a_{55} - a_{44}}{a_{44} + a_{55}} xy, \qquad S_z = \frac{4\pi F}{\frac{1}{a_{44}} + \frac{1}{a_{55}}}, \quad r = \sqrt{x^2 + y^2}, \tag{6.5}$$

where

$$F = \int_{c_1}^{c_2} \rho^3 f(\rho)\,\mathrm{d}\rho. \tag{6.6}$$

In all the three cases the shear stresses can be computed by application of Eq. (3.6), where $r = \overline{OP}$, O is the centre of the hollow circular cross-section and P is an arbitrary point of the considered cross-section. The shear stress vector is the perpendicular to the line segment $\overline{OP}$ and its magnitude τ can be obtained as

$$\tau = T\frac{rf(r)}{2\pi F}. \tag{6.7}$$

From the formulae (6.3)–(6.6) it follows that for homogeneous, solid circular cross-section ($f(r) = 1$) we have

$$a_{44} = \frac{2}{-s_x + s_y + s_z}, \qquad a_{55} = \frac{2}{s_x - s_y + s_z}, \qquad a_{66} = \frac{2}{s_x + s_y - s_z}. \tag{6.8}$$

Here

$$s_i = \frac{c^4\pi}{S_i} \qquad (i = x,\, y,\, z). \tag{6.9}$$

Equations (6.8), (6.9) show that the shear moduli of an orthotropic, homogeneous elastic material can be obtained from three torsional tests, assuming that the principal directions of the orthotropy are known [3].

7. Relationships between the torsional problems of solid circular and elliptical cross-sections

In this section, we consider homogeneous ($f(r) = 1$) orthotropic ($a_{45} = 0$) solid circular cross-section which is shown in Fig. 6a ($c_1 = 0$, $c_2 = c$). The torsional function and the torsional rigidity of this cross-section are

$$\omega(x, y) = \frac{a_{55} - a_{44}}{a_{44} + a_{44}}xy, \qquad S = \frac{a_{44}a_{55}}{a_{44} + a_{44}}c^4\pi. \tag{7.1}$$

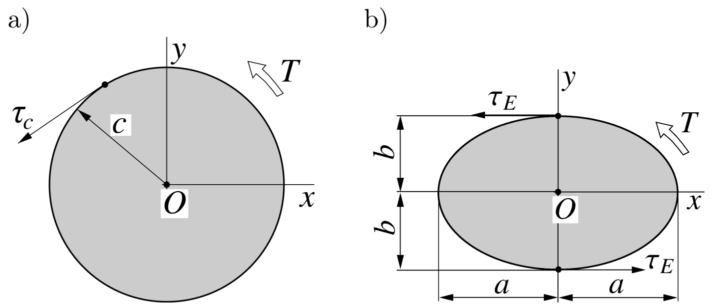

Fig. 6. Orthotropic circular (a) and isotropic elliptical (b) solid cross-sections.

We assume that $a_{44} \geq a_{55}$. We define a solid elliptical cross-section whose boundary contour is given by the next equation (Fig. 6b)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \qquad a^2 = ka_{44}, \qquad b^2 = ka_{55}. \tag{7.2}$$

In Eq. (7.2) k is an arbitrary positive constant with units $(\text{length})^4/(\text{force})$. The material of the solid elliptical cross-section is linearly elastic, homogeneous, isotropic with shear modulus G. By a simple computation and by the use of formulae of torsional function and torsional rigidity of a solid homogeneous, isotropic elliptical cross-section [8, 12] which are

$$\omega_E = \frac{b^2 - a^2}{a^2 + b^2}xy, \qquad S_E = G\frac{a^3b^3}{a^2 + b^2}\pi \tag{7.3}$$

we can prove the next theorem.

Theorem 2. If Eq. (7.2) and

$$k^2 = \frac{c^4}{G\sqrt{a_{44}a_{55}}} \tag{7.4}$$

meet then the torsion function and the torsional rigidity of orthotropic solid circular cross-section and the isotropic solid elliptical cross-section are the same, they are given by Eq. (7.1). For

$$G = \sqrt{a_{44}a_{55}} \tag{7.5}$$

we have

$$k = \frac{c^2}{\sqrt{a_{44}a_{55}}}, \qquad a^2 = c^2\sqrt{\frac{a_{44}}{a_{55}}}, \qquad b^2 = c^2\sqrt{\frac{a_{55}}{a_{44}}} \tag{7.6}$$

and in this case the cross-sectional area A_C of the orthotropic circular cross-section is the same as the cross-sectional area A_E of the isotropic elliptical cross-section, that is, we have

$$A_C = c^2\pi = A_E = ab\pi. \tag{7.7}$$

■

Denote the maximum shearing stresses τ_C and τ_E for orthotropic circular and isotropic elliptical cross-sections, respectively (Fig. 6). Assuming that Eqs. (7.2), (7.4) and (7.6) are satisfied then we have [8, 12]

$$\tau_E = \frac{2T}{ab^2\pi} = \tau_C\sqrt{\frac{a_{44}}{a_{55}}} \geq \tau_C = \frac{2T}{c^3\pi}. \tag{7.8}$$

8. Conclusion

The purpose of this paper is to investigate the effects of material inhomogeneity and anisotropy on the torsional response of a linearly elastic non-homogeneous anisotropic circular cylinder. The class of anisotropy considered has at least one plane of elastic symmetry normal to the axis of the circular cylinder. The elastic coefficients have radial dependence only. It is shown that Saint-Venant torsion problem analysed has two important properties:

- The torsional function does not depend on the cross-sectional inhomogeneity it depends only on the anisotropic features of the elastic circular cylinder.
- For given torque the stress field is independent of the material anisotropy, it depends only on the cross-sectional inhomogeneity.

A brief analysis deals with the torsional problems of layered anisotropic cross-sections. Some useful formulae for the torsional problem of orthotropic non-homogeneous circular cylinders are given. The connection between the torsional problems of solid orthotropic circular cross-section and of isotropic elliptical cross-section is also analysed. The presented exact analytical solutions can be used as benchmark solutions to verify the efficacy of the usual approximate methods such as finite element and boundary element methods.

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