ENGINEERING TRANSACTIONS • *Engng. Trans.* • **46**, 2, 205–215, 1998
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.652.1998

THE INFLUENCE OF GENERAL SURFACE LOADING ON PENETRATION OF A CIRCULAR PUNCH INTO AN ELASTIC STRATUM

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The first of the authors (B. R.) derived an expression for the pressure distribution inside a circular region with vanishing shear tractions and normal displacements, due to the application of the forces distributed along the circumference on an elastic transversely isotropic half-space. This result is used to derive the relationships between the forces, moments and indentations for a punch acting on an elastic half-space. The results are given in terms of elementary functions. The influence of an annular punch encircling a central punch is considered. The stress intensity factor of Mode I related to non-symmetric stress distribution in the vicinity of an external crack under general surface loading, symmetric with respect to the crack plane, is also presented in terms of elementary functions.

1. Introduction

The influence of a concentrated normal load applied to the surface of the half-space outside the circular punch on the displacement and rotation of the punch, was presented by GALIN [1]. References to the studies related to interaction between a system of circular punches, indenting an elastic half-space in frictionless contact problem, and elegant analytical formulation and approximate results obtained were presented by GLADWELL and FABRIKANT [2].

In this paper we show the influence of general surface loading (normal and tangential, symmetric and asymmetric) on displacement and rotation of a circular punch on an elastic transversely isotropic half-space. The relations are given in terms of closed-form expressions of elementary functions. The result is used to derive approximate relationships between forces and displacements, and moments and rotations for the central punch and annular punch encircling it. Also, we obtain the results for the stress intensity factor of Mode I concerning an external crack.

2. Theory

Consider a single rigid flat-ended punch of radius a in frictionless contact with a transversely isotropic elastic half-space $z \geq 0$ with planes of isotropy parallel to $z = 0$ (Fig. 1).

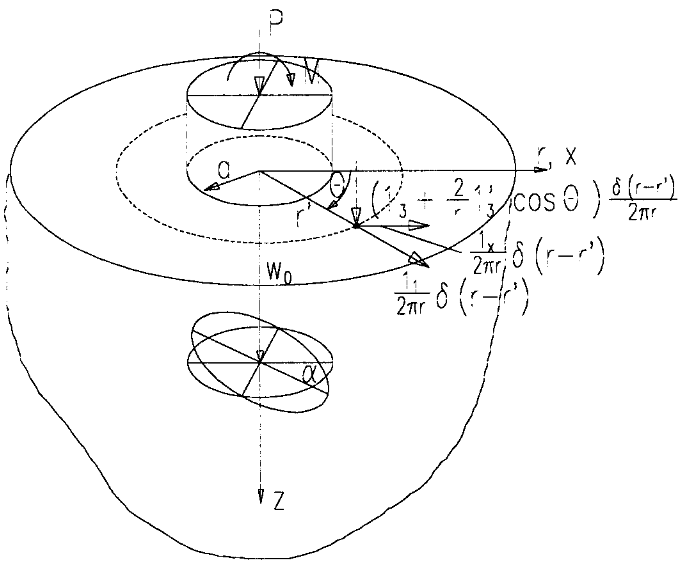

FIG. 1. Geometry, coordinate system and loading conditions.

If the normal displacement under the punch is

$$w(r,\theta) = w_0 + \alpha r \cos\theta, \tag{2.1}$$

then the normal pressure exerted by the punch is

$$p_0(r,\theta) = \frac{2}{\pi} G_z C \frac{w_0 + 2\alpha r\cos\theta}{\sqrt{a^2 - r^2}}, \tag{2.2}$$

where $G_z C = E_r/(1-\nu_{r\theta}^2)s_1 s_2(s_1+s_2)$ is an elastic constant which reduces to $E/2(1-\nu^2)$ when the half-space is an isotropic medium ($s_1 = s_2 = 1$) with Young's modulus E and Poisson's ratio ν [3]. The contact stiffness $G_z C$ of transversely isotropic material is defined by Young's modulus E_r and Poisson's ratio $\nu_{r\theta}$ in the isotropic plane and by two material parameters s_1, s_2, [3].

Now suppose that unit loads are applied to the surface of a half-space, and distributed over the circumference of a circle of radius r' ($r' > a$) (Fig. 1). This load will produce an additional contact pressure under the punch. This additional pressure, which is such that, together with concentrated load, it produces no additional normal displacement under the punch, was found by Rogowski; it is equal

$$p_e(r,\theta) = p_e(r) + q_e(r)\cos\theta, \tag{2.3}$$

where

$$\begin{aligned} p_e(r) &= -\frac{1}{\pi^2\sqrt{a^2-r^2}}\left(1_3\frac{\sqrt{r'^2-a^2}}{r'^2-r^2} - 1_1\frac{\vartheta_0}{r'}\right), \\ q_e(r) &= -2\frac{1'_3 r\sqrt{r'^2-a^2}}{\pi^2 r'^2(r'^2-r^2)\sqrt{a^2-r^2}}. \end{aligned} \tag{2.4}$$

The formulae for $p_e(r)$ and $q_e(r)$ are obtained from the results (3.14) and (3.15), (ROGOWSKI [3]), and (2.5), (3.10), (3.13), (ROGOWSKI [4]). We note that the pressure $p_e(r,\theta)$ associated with axial loadings $(1_3, 1'_3)$ is independent of the material constants, while the one associated with the radial ring load (1_1) depends on the material parameter

$$\vartheta_0 = \frac{G_zC}{\sqrt{c_{11}c_{33}} + c_{13}}. \tag{2.5}$$

Here c_{ij} are the elastic constants and G_zC is the contact stiffness of a transversely isotropic material. The shear loading in the x direction (1_x) does not produce normal surface tractions in the frictionless contact problem [3]. Equations (2.2), (2.3) and (2.4) show that if the displacement under the punch is given by Eq. (2.1), then the normal pressure under the punch is the following:

$$\begin{aligned} p(r,\theta) &= p_0(r,\theta) + p_e(r,\theta) \\ &= \frac{2}{\pi\sqrt{a^2-r^2}}\Bigg\{G_zCw_0 - \frac{1}{2\pi}\left(1_3\frac{\sqrt{r'^2-a^2}}{r'^2-r^2} - 1_1\frac{\vartheta_0}{r'}\right) \\ &\qquad + \left[2\alpha G_zCr - \frac{1'_3}{\pi}\frac{r\sqrt{r'^2-a^2}}{r'^2(r'^2-r^2)}\right]\cos\theta\Bigg\}. \end{aligned} \tag{2.6}$$

General equilibrium of the punch requires

$$\begin{aligned} P &= \int_0^{2\pi}\int_0^a p(r,\theta)r\,dr\,d\theta, \\ M &= \int_0^{2\pi}\int_0^a p(r,\theta)r^2\cos\theta\,dr\,d\theta. \end{aligned} \tag{2.7}$$

We now integrate this expression over the circle $r \le a$, $0 \le \theta \le 2\pi$ and use the integrals

$$
\begin{gathered}
\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi a,\\
\int_0^{2\pi}\int_0^a \frac{r}{(r'^2-r^2)\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi\frac{1}{\sqrt{r'^2-a^2}}\sin^{-1}\left(\frac{a}{r'}\right),\\
\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2-r^2}}\cos^2\theta\,dr\,d\theta = \frac{2}{3}\pi a^3,\\
\int_0^{2\pi}\int_0^a \frac{r}{(r'^2-r^2)\sqrt{a^2-r^2}}\cos^2\theta\,dr\,d\theta = \pi\left[\frac{r'^2}{\sqrt{r'^2-a^2}}\sin^{-1}\left(\frac{a}{r'}\right)-a\right],
\end{gathered}
\tag{2.8}
$$

to obtain

$$
\begin{aligned}
P &= 4G_zCw_0a - \frac{2}{\pi}\left[1_3\sin^{-1}\left(\frac{a}{r'}\right) - 1_1\vartheta_0\frac{a}{r'}\right],\\
M &= \frac{8}{3}G_zC\alpha a^3 - 2\frac{1_3'}{\pi}\left[\sin^{-1}\left(\frac{a}{r'}\right) - \frac{a}{r'}\sqrt{1-\frac{a^2}{r'^2}}\right].
\end{aligned}
\tag{2.9}
$$

Equations (2.9) are the main results of this paper. The condition of complete contact requires $p(r,\theta) \ge 0$ for all $r \le a$ and $0 \le \theta \le 2\pi$. This condition is satisfied if $p(a^-,\pi) \ge 0$, i.e.

$$
\begin{aligned}
G_zCw_0 - \frac{1}{2\pi}\left(\frac{1_3}{\sqrt{r'^2-a^2}} - 1_1\frac{\vartheta_0}{r'}\right)&\\
- 2G_zC\alpha a + \frac{1_3'}{\pi}\frac{a}{r'^2\sqrt{r'^2-a^2}} \ge 0,& \qquad r' > a.
\end{aligned}
\tag{2.10}
$$

3. Applications

3.1. Special cases of the loading conditions

When in an annular region $b \le r \le c$ $(b \ge a)$ axial forces $p_1(r')$ (symmetric part) and $p_2(r')$ (asymmetric part), and tangential radial forces $t_1(r')$ are applied, then from Eq. (2.9) we obtain

$$
\begin{aligned}
P &= 4G_z C w_0 a - 4\int_b^c r' p_1(r') \sin^{-1}\left(\frac{a}{r'}\right) dr' + 4a\vartheta_0 \int_b^c t_1(r')\, dr', \\
M &= \frac{8}{3} G_z C \alpha a^3 - 2\int_b^c r'^2 p_2(r') F\left(\frac{a}{r'}\right) dr',
\end{aligned}
\tag{3.1}
$$

where

$$
F(x) = \sin^{-1} x - x\sqrt{1-x^2}, \qquad x = \frac{a}{r'}. \tag{3.2}
$$

For example, if $p_1(r') = p_1$, $p_2(r') = p_2$ and $t_1(r') = t_1$ where p_1 and p_2 and t_1 are constants, integration yields

$$
\begin{aligned}
P &= 4G_z C w_0 a - 2p_1\left[c^2 F_1\left(\frac{a}{c}\right) - b^2 F_1\left(\frac{a}{b}\right)\right] + 4\vartheta_0 t_1 a(c-b), \\
M &= \frac{8}{3} G_z C \alpha a^3 - \frac{2}{3} p_2 \left[c^3 F\left(\frac{a}{c}\right) - b^3 F\left(\frac{a}{b}\right)\right. \\
&\qquad \left. + 2a^3\left(\operatorname{ch}^{-1}\left(\frac{c}{a}\right) - \operatorname{ch}^{-1}\left(\frac{b}{a}\right)\right)\right],
\end{aligned}
\tag{3.3}
$$

where

$$
F_1(x) = \sin^{-1} x + x\sqrt{1-x^2} \tag{3.4}
$$

and $F(x)$ is defined by Eq. (3.2).

In this case of loading the condition of complete contact requires

$$
Pa \geq 3M + 2(p_2 - p_1)\left[c^2(c-a) F\left(\frac{a}{c}\right) - b^2(b-a) F\left(\frac{a}{b}\right)\right]. \tag{3.5}
$$

Formulae (2.9) yield relations between forces, moments, displacements, and rotations for punch indentation, including normal and tangential loadings of a transversely isotropic half-space outside the punch. In the particular case of $\vartheta_0 = 0$, the radial shear loading (1_1) produces no additional pressure under the punch. For an isotropic material, the value $\vartheta_0 = 0$ corresponds to incompressible material ($\nu = 1/2$). The radial symmetric shear loadings do not influence the condition of complete contact for arbitrary function $t_1(r')$. If the inclination of the normal axisymmetric load to the radial direction is θ_0, we can find the results for this case by applying a normal ring load of magnitude $1_3 \sin\theta_0$, and a concentrated shear ring load of magnitude $1_3 \cos\theta_0$, and by superimposing the results. If the inclination of the normal asymmetric load to the x-axis is θ_1, we can find the results for this case applying a normal ring load of magnitude $1_3' \sin\theta_1$, since the shear ring load acting in the x-direction of magnitude $1_3' \cos\theta_0/r'$ produces no normally directed surface tractions under the punch in the frictionless contact problem. The general case of annular region of loading discussed above contains an interesting special case, namely $b = a$.

Consider two concentric rigid punches: circular punch of radius a and annular punch of radii b and c, encircling it. The axial force and bending moment applied to the annulus are denoted by P_1 and M_1. If $p_1(r')$ and $p_2(r')$ are the symmetric and asymmetric parts of the contact pressure under the annular punch, then equations (3.1) reduce to

$$\begin{aligned} P &= 4G_z C w_0 a - 4\int_b^c r' p_1(r') \sin^{-1}\left(\frac{a}{r'}\right) dr', \\ M &= \frac{8}{3} G_z C \alpha a^3 - 2\int_b^c r'^2 p_2(r') F\left(\frac{a}{r'}\right) dr'. \end{aligned} \tag{3.6}$$

If we now assume that a/R, $R = (b+c)/2$, is small, we may take

$$\begin{aligned} \sin^{-1}\left(\frac{a}{r'}\right) &\cong \sin^{-1}\left(\frac{a}{R}\right), \\ F\left(\frac{a}{r'}\right) &\cong F\left(\frac{a}{R}\right), \\ R &= \frac{b+c}{2} \end{aligned} \tag{3.7}$$

and obtain

$$\begin{aligned} P &= 4G_z C w_0 a - \frac{2}{\pi} P_1 \sin^{-1}\left(\frac{a}{R}\right), \\ M &= \frac{8}{3} G_z C \alpha a^3 - \frac{1}{\pi} M_1 F\left(\frac{a}{R}\right). \end{aligned} \tag{3.8}$$

A study of the errors made by using the approximations (3.7) shows that they are of small order and decrease with decreasing ratio of the thickness of annulus to its mean diameter $2(c-b)/(c+b)$. Formulae (3.8) yield the approximate relations between the axial forces and displacement, and between the bending moments and rotation of the cylindrical punch. If we expand the functions of r' in Eqs. (3.7) in the Taylor series in the neighborhood of the central point $r' = R$

of the annulus $b \le r' \le c$, then retaining two terms of this expansions we obtain

$$(3.9)\qquad \begin{aligned} \sin^{-1}\left(\frac{a}{r'}\right) &= \sin^{-1}\left(\frac{a}{R}\right) - \left(\frac{r'}{R} - 1\right)\frac{a}{R}\left(1 - \frac{a^2}{R^2}\right)^{-1/2}, \\ F\left(\frac{a}{r'}\right) &= F\left(\frac{a}{R}\right) - 2\left(\frac{r'}{R} - 1\right)\left(\frac{a}{R}\right)^3\left(1 - \frac{a^2}{R^2}\right)^{-1/2}. \end{aligned}$$

The second terms on the right-hand side of Eqs. (3.9) yield from Eqs. (3.6)

$$(3.10)\qquad \begin{aligned} \Delta P &= -\frac{a}{R}\left(1 - \frac{a^2}{R^2}\right)^{-1/2}\left[\frac{2}{\pi}P_1 - \frac{4}{\pi}\int\limits_b^c p_1(r')r'^2\,dr'\right], \\ \Delta M &= -\left(\frac{a}{R}^3\right)^3\left(1 - \frac{a^2}{R^2}\right)^{-1/2}\left[\frac{2}{\pi}M_1 - \frac{4}{\pi}\int\limits_b^c p_2(r')r'^3\,dr'\right]. \end{aligned}$$

In the cases in which $b \gg a$ and the ratio of annulus radii is near unity, ΔP and ΔM represent a small contribution to the total solution. To find ΔP and ΔM, we make use the solution for contact stresses under the annular punch, [5]. However, for those practical problems where a/R is small, the approximate relations (3.8) yield good results suitable for engineering applications.

3.2. Contact stresses related to the special cases of the loading functions

Equations (2.9) for the total applied forces are obtained directly, without previous determination of the stress distributions under the punch. We consider some special cases of the loading functions and determine the additional contact pressures $p_e(r)$ and $q_e(r)$. The cases in which the normal and tangential loadings are distributed uniformly over the circumference of a circle of radius r' $(r' > a)$ on the surface of a half-space with resultants 1_3, $1_3'$ and 1_1 – yield the additional pressures which are given by Eqs. (2.4).

Let us first consider the case in which the normal loadings outside the punch are uniform and equal p_1 – symmetric part, and p_2 – asymmetric part, and act over a circular ring of inner and outer radii b and c, respectively.

Equations (2.4) show that

$$(3.11)\qquad p_e(r) = -\frac{2p_1}{\pi}\left[\sqrt{\frac{c^2 - a^2}{a^2 - r^2}} - \sqrt{\frac{b^2 - a^2}{a^2 - r^2}} - \tan^{-1}\sqrt{\frac{c^2 - a^2}{a^2 - r^2}} + \tan^{-1}\sqrt{\frac{b^2 - a^2}{a^2 - r^2}}\right],$$

$$q_e(r) = -\frac{p_2}{\pi}\left[\frac{2r}{\sqrt{a^2-r^2}}\left(\mathrm{ch}^{-1}\left(\frac{c}{a}\right) - \mathrm{ch}^{-1}\left(\frac{b}{a}\right)\right) - \sin^{-1}\left(\frac{rc-a^2}{a(c-r)}\right)\right.$$
$$\left. - \sin^{-1}\left(\frac{rc+a^2}{a(c+r)}\right) + \sin^{-1}\left(\frac{rb-a^2}{a(b-r)}\right) + \sin^{-1}\left(\frac{rb+a^2}{a(b+r)}\right)\right]. \tag{3.11}$$

The stresses $p_e(r)$ and $q_e(r)$ increase with c for a fixed value of b. Therefore we cannot increase c indefinitely. This case contains an interesting special case, namely $b = a$. Let us consider the case in which the solid is loaded radially by the forces directed away from the origin, and the shearing action is variable over a circular ring of inner and outer radii b and c, respectively, as follows:

$$t_1(r') = t_1\left(\frac{a}{r'}\right)^2, \qquad b < r' < c, \tag{3.12}$$

where t_1 is constant.

Then

$$p_e(r) = \frac{2}{\pi}\frac{t_1\vartheta_0 a^2(c-b)}{bc\sqrt{a^2-r^2}}. \tag{3.13}$$

This problem contains three other problems as special cases, namely, (i) $b = a$ and c finite; (ii) $b > a$ and $c \to \infty$; (iii) $b = a$ and $c \to \infty$. Case (iii) is a case of radially decaying shear load on the boundary of a half-space. We can deduce the results for these three cases from Eq. (3.13) by setting $b = a$ and/or letting $c \to \infty$.

3.3. Application of normal and shear loadings on an external crack

We consider an external crack located outside of a circle of radius a in an infinite elastic body. The crack surfaces are subjected to the axisymmetric and asymmetric distributions (with respect to axis $\theta = \pi/2$) of normal and tangential tractions. These loadings are assumed to be symmetric with respect to the plane $z = 0$. Such a crack problem is a particular case of the more general case of the punch problem considered in the previous section and the formulae obtained there can give its immediate solution. In the external crack problem where both sides of the crack are loaded by arbitrary normal tractions acting in opposite directions and shear tractions acting in the same directions, due to the symmetry of the problem we obtain $w(r,\theta) = 0$ and $\sigma_{zr}(r,\theta) = 0$ for $r \le a$, $0 \le \theta \le 2\pi$. This implies that $w_0 = 0$ and $\alpha = 0$ in Eq. (2.1) and $p_0(r,\theta) = 0$ in Eq. (2.2). The tensile stresses in the region $r \le a$, $0 \le \theta \le 2\pi$ are $p_c(r) = -p_e(r)$ and $q_c(r) = -q_e(r)$, and

$$p_c(r,\theta) = p_c(r) + q_c(r)\cos\theta, \qquad r \le a, \qquad 0 \le \theta \le 2\pi, \tag{3.14}$$

where

$$
(3.15)\qquad
\begin{aligned}
p_c(r) &= \frac{2}{\pi\sqrt{a^2-r^2}}\left[\int_a^\infty \frac{r' p_1(r')\sqrt{r'^2-a^2}}{r'^2-r^2}dr' - \vartheta_0 \int_a^\infty t_1(r')\,dr'\right],\\
q_c(r) &= \frac{2r}{\pi\sqrt{a^2-r^2}}\int_a^\infty \frac{p_2(r')\sqrt{r'^2-a^2}}{r'^2-r^2}dr',
\end{aligned}
$$

and where $p_1(r')$, $p_2(r')$ and $t_1(r')$ denote the prescribed normal and tangential forces acting on the crack surfaces. Note that the shear loading in the x direction (1_x, Fig. 1) produces no stresses in the region $r \le a$, $0 \le \theta \le 2\pi$.

For $t_1(r') = 0$ the results (3.15) are in complete agreement with those obtained by LOWENGRUB and SNEDDON [6]. Defining the stress intensity factor of Mode I as follows:

$$
(3.16)\qquad K_I = \lim_{r\to a^-} \sqrt{2(a-r)}p_c(r,\theta),
$$

we obtain

$$
(3.17)\qquad K_I = \frac{2}{\pi\sqrt{a}}\left[\int_a^\infty \frac{r' p_1(r')}{\sqrt{r'^2-a^2}}dr' - \vartheta_0\int_a^\infty t_1(r')\,dr' + a\cos\theta\int_a^\infty \frac{p_2(r')}{\sqrt{r'^2-a^2}}dr'\right].
$$

Assuming the constant normal tractions and the shear radial traction in the form of Eq. (3.12) in the annular region $b \le r' \le c$, the stress intensity factor is obtained as follows:

$$
(3.18)\qquad K_I = \frac{2}{\pi\sqrt{a}}\Bigg[p_1\left(\sqrt{c^2-a^2}-\sqrt{b^2-a^2}\right) - \frac{t_1\vartheta_0 a^2(c-b)}{bc} + p_2 a\left(\mathrm{ch}^{-1}\left(\frac{c}{a}\right) - \mathrm{ch}^{-1}\left(\frac{b}{a}\right)\right)\Bigg].
$$

For $p_2 = 0$ this result agrees with the result obtained for an isotropic solid ($\vartheta_0 = (1-2\nu)/2(1-\nu)$) by PARIHAR and KRISHNA RAO ([7], Eqs. (45) and (70)). For real materials the quantity ϑ_0 is real and positive. For example, ϑ_0 takes the values: 0.1833; 0.2474; 0.4020 for cadmium, laminated composite consisting of alternating layers of two isotropic materials with $\mu/\overline{\mu} = 0.5$, $\overline{h}/h = 2$ and $\mu = 10^4$ MPa, and for E glass-epoxy composite, respectively. In the media having transversely isotropic material properties, the degenerate case for which $\vartheta_0 \to 0$ may occur, similar to the case of an incompressible isotropic material when $\nu = 1/2$ implies $\vartheta_0 = 0$. It is the case in which $1 - \nu_{r\theta} - 2\nu_{rz}\nu_{zr}$ tends to zero and, in consequence, c_{ij} increase to infinity (ν_{ij} – Poisson's ratios). In this case vanishes the following sum of the strains: $\varepsilon_{rr} + \varepsilon_{\theta\theta} + 2\nu_{rz}\varepsilon_{zz}$. If $\nu_{rz} = 1/2$ and $1 - \nu_{r\theta} - \nu_{rz} = 0$, it is an incompressible material. In this case ϑ_0 vanishes, as follows from Eq. (2.5). Note that constant normal tractions must be applied in a bounded annular region to obtain finite value of the stress intensity factor, while the radially decaying shear traction given by Eq. (3.12) may be applied in unbounded region ($c \to \infty$).

Assuming the radially decaying normal tractions in the form

$$p_1(r') = p_1\frac{a}{r'}, \qquad p_2(r') = p_2\frac{a}{r'}, \qquad a \le b < r' < c, \tag{3.19}$$

where p_1 and p_2 are constants, we obtain

$$K_I = \frac{2\sqrt{a}}{\pi}\left[p_1\left(\mathrm{ch}^{-1}\left(\frac{c}{a}\right) - \mathrm{ch}^{-1}\left(\frac{b}{a}\right)\right) + p_2\left(\sin^{-1}\left(\frac{a}{b}\right) - \sin^{-1}\left(\frac{a}{c}\right)\right)\cos\theta\right]. \tag{3.20}$$

In this case loading p_2 may be applied in an unbounded region, while p_1 in a bounded region. To obtain finite value of the stress intensity factor, the loadings applied in infinite region $r' \ge b$ ($b \ge a$) must be decaying as r'^{-2} for $p_1(r')$ and $t_1(r')$ and as r'^{-1} for $p_2(r')$. For the case of decaying tractions in the from of Eqs. (3.12) and (3.19) and for $b = a$, K_I takes the value

$$K_I = \frac{2\sqrt{a}}{\pi}\left[p_1\mathrm{ch}^{-1}\left(\frac{c}{a}\right) + \frac{\pi}{2}p_2\cos\theta - t_1\vartheta_0\right], \tag{3.21}$$

where c is bounded.

To obtain the crack opening displacement it is required that $K_I \ge 0$ for all of θ. This implies $K_I \ge 0$ for $\theta = \pi$. This condition and Eq. (3.21) yield

$$p_1\mathrm{ch}^{-1}\left(\frac{c}{a}\right) \ge \frac{\pi}{2}p_2 + t_1\vartheta_0. \tag{3.22}$$

The cases in which the loadings are distributed over the circumference of a circle of radius r' ($r' > a$) on both surfaces of the crack, with the total loads P ($= 1_3$), Q ($= 1_1$) and M ($= 1_3'$), are given by Eqs. (2.4), (3.14) and (3.16).

The stress intensity factors corresponding to these cases are given by

$$K_I = \frac{1}{\pi^2\sqrt{a}}\left[\frac{P}{\sqrt{r'^2 - a^2}} - \frac{Q\vartheta_0}{r'} + \frac{Ma}{r'^2\sqrt{r'^2 - a^2}}\cos\theta\right]. \tag{3.23}$$

For $M = 0$ and isotropic material this result agrees with the result obtained by Parihar and Krishna Rao ([7], Eqs. (60) and (79)). The formula (3.23) is valid if

$$P \ge \frac{Ma}{r'^2} + Q\vartheta_0\sqrt{1 - \frac{a^2}{r'^2}}. \tag{3.24}$$

In opposite case the stress intensity factor is negative, the crack may close and the boundary conditions are of unilateral nature. The general expressions related to the stress intensity factors in the crack problems have been given recently by the author in ROGOWSKI [3–4]. Of course, results of this subsection may be obtained from those fundamental solutions.

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Received October 13, 1997; new version August 24, 1998.