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PERTURBED MOTIONS OF A ROTATING SYMMETRIC GYROSTAT

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The aim of the present paper is to provide analytical solutions for the perturbed problem of the behaviour of a symmetric gyrostat about a fixed point. This gyrostat is acted upon by a gyrostatic momentum $\ell_s(s = 1,2,3)$, variable restoring moment k and perturbing moments $M_i(i = 1,2,3)$. The moment k is introduced in view of the rotation of gyrostat under the action of electro-magnetic field. The solutions are achieved when the third component of the gyrostatic momentum is different from zero ($\ell_3 \neq 0$). The averaging method is applied to investigate the first order approximate solutions of resonant and non-resonant cases.

1. Introduction

The problem of rotatory motion of a gyrostat about a fixed point subject to external moments is one of the important problems dealt with in theoretical mechanics. This motion is governed by a system of six nonlinear differential equations [1]. In [2], the problem of earth's rotation is considered, a symmetrical gyrostat being used as a model. The first two components of the gyrostatic momentum are null, and the third component is chosen as a constant. In such a way the free polar motion has a period of 430 days (Chandler's period). This study has been extended and generalized in [3]. The Poincaré small parameter method is applied [4] to solve the problem of motion of a fast spinning rigid body about a fixed point. The problem of a perturbed rotational motion of a heavy solid with constant restoring moment close to regular precession was treated in [5]. The case when the restoring moment is dependent on the nutation angle θ was studied in [6, 7]. The case of perturbed motions of a heavy solid close to the Lagrangian case was considered in [8].

In this paper, the perturbed problem of rotary motion of a symmetric gyrostat about a fixed point with gyrostatic momentum ($\ell_3 \neq 0$) under the influence of the sum of constant and linear dissipative moments acting in the same direction of the principal axes of the gyrostat, and a restoring moment k which is the result of the electro-magnetic field H and the point charge e acting on the axis of symmetry Oz, has been considered. The equations of motion will be studied under certain boundary conditions of motion which means that the vector of the angular velocity of the gyrostat is very close to the symmetry axis. This velocity is very high and the values of the vectors of perturbed moments are less than or equal to the value of the restoring moment k. The moment k depends on the angular velocity vector and the Euler angles θ and ψ. These conditions give a small parameter ε which causes the perturbed motion. The averaging method [9, 10, 11] is applied to solve the system of equations of motion in the perturbed case. A theoretical description for this technique in both the resonant and non-resonant cases is given.

2. Formulation of the problem

Consider the rotational motion of a dynamically symmetrical gyrostat about a fixed point O, under the influence of a gyrostatic momentum $\ell_3 \neq 0$, a variable restoring moment k and perturbing moments $M_i (i = 1, 2, 3)$. Two systems of coordinates are considered at the fixed point O : a fixed one $OXYZ$ and another, rotating one $Oxyz$ which is fixed in the gyrostat and whose axes are directed along the principal axes of inertia of the gyrostat at O (see Fig. 1).

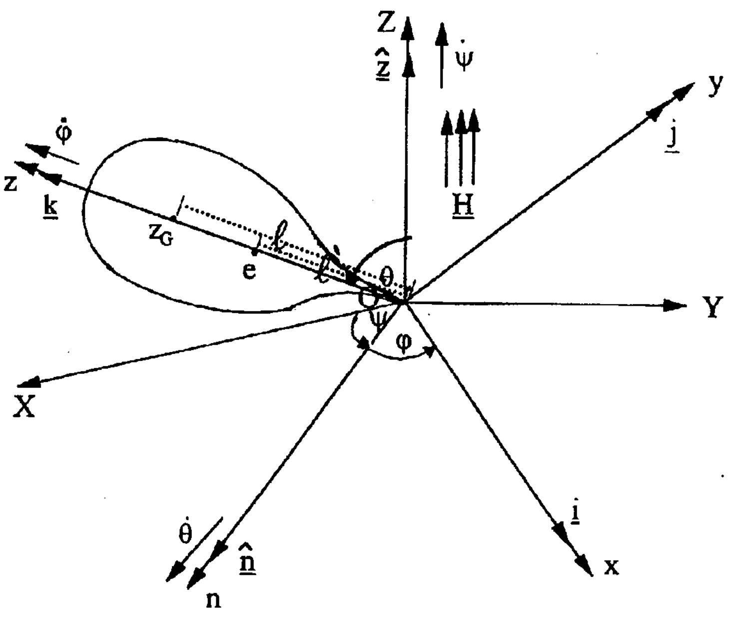

Fig. 1.

$$H_1 = H \sin\theta \sin\varphi i,\ H_2 = H \sin\theta \cos\varphi j,\ H_3 = H \cos\theta k.$$

Thus, supposing that

$$x_G = y_G = 0, \qquad z_G = \ell, \quad \ell_1 = \ell_2 = 0, \quad \ell_3 \neq 0, \quad A = B \neq C, \tag{2.1}$$

the equations of motion are

$$\begin{aligned} A\dot{p} + (C - A)qr + q\ell_3 &= k \sin\theta \cos\varphi + M_1, \\ A\dot{q} + (A - C)pr - p\ell_3 &= -k \sin\theta \sin\varphi + M_2, \\ C\dot{r} = M_3, \qquad M_i &= M_i(p, q, r, \psi, \theta, \varphi, t), \qquad (i = 1, 2, 3), \\ \dot{\theta} &= p \cos\varphi - q \sin\varphi, \\ \dot{\varphi} &= r - (p \sin\varphi + q \cos\varphi) \cot\theta, \\ \dot{\psi} &= (p \sin\varphi + q \cos\varphi) \operatorname{cosec}\theta, \end{aligned} \tag{2.2}$$

where (p, q, r) and $M_i, i = 1, 2, 3$ are the projections of the angular velocity of the body on the principal axes of inertia, and of the perturbing moment onto the principal axes of inertia of the gyrostat passing through O. Here A, B and C are the principal moments of inertia, x_G, y_G and z_G are the coordinates of the center of mass of the body, ℓ_1, ℓ_2, ℓ_3 are the components of the gyrostatic momentum, and θ, φ, ψ are the Eulerian angles such that ψ is the angle of precession, φ is the angle of self-rotations and θ is the angle of nutation.

Assume that the perturbing moments are 2π-periodic functions of Euler's angles, and that the gyrostat is acted upon by a variable restoring moment whose maximum value is equal to k such that

$$k = mg\ell, \tag{2.3}$$

where m is the mass of the gyrostat, g is the acceleration due to gravity, and ℓ is the distance from the fixed point O to the center of mass of the gyrostat.

For $M_i = 0$, $i = 1, 2, 3$, Eq. (2.2) corresponding to the equations of motion of a symmetric gyrostat whose two first components of the gyrostatic momentum are null [12], for $M_i = 0$, $(i = 1, 2, 3), \ell_3 = 0$, give the Lagrange-Poisson case [13], and for $\ell_3 = 0$, give the case of LESHCHENKO *et al.* [7].

Consider the following initial conditions:

$$p^2 + q^2 \ll r^2, \qquad r^2 \gg k, \qquad |M_i| \ll k, \qquad i = 1, 2, \qquad M_3 \approx k. \tag{2.4}$$

These conditions mean that the direction of the angular velocity of the gyrostat is close to the axis of the dynamic symmetry, the angular velocity is large so that the kinetic energy of the gyrostat is much greater than the potential energy resulting from the restoring moment, and two projections of the perturbing moment vector onto the principal axes of inertia of the gyrostat are small as compared with the restoring moment k while the third one is of the same order of magnitude as k. Inequalities (2.4) allow us to introduce the small parameter ε and to set

$$\begin{aligned} p &= \varepsilon P, \qquad q = \varepsilon Q, \qquad k = \varepsilon K, \\ M_i &= \varepsilon^2 M_i^*(P, Q, r, \psi, \theta, \varphi, t), \qquad i = 1, 2, \\ M_3 &= \varepsilon M_3^*(P, Q, r, \psi, \theta, \varphi, t). \end{aligned} \tag{2.5}$$

The new variables P and Q as well as the variables and constants r, ψ, θ, φ, $K, C, M_i^*, i = 1, 2, 3$, are assumed to be bounded quantities of order unity as ε tends to zero.

The aim of this work is to investigate the asymptotic behaviour of the solutions of system (2.2) for small ε, if conditions (2.4) and (2.5) are satisfied. This will be done by employing the averaging method which is extensively employed in problems of dynamics of gyrostats over the time interval of order ε^{-1}. This method was employed to investigate a variety of problems of dynamics, chiefly for gyrostats with dynamic symmetry. The perturbed motions close to Lagrange's motion are investigated in different works, such as e.g. [14].

The assumptions (2.4) and (2.5) adopted in this paper, enable us to obtain a relatively simple averaging scheme in the general case and to investigate thoroughly the following cases.

3. The case of variable restoring moment

The rotational motion of a symmetrical gyrostat about a fixed point in an electro-magnetic field of strength $\mathbf{H}$ ($\mathbf{H}$ is vertical) and a point charge e located on the axis of symmetry is considered when the restoring moment depends on the components of the angular velocity and further, on the angles θ and φ. Thus, this gyrostat rotates under the force of gravity, the gyrostatic momentum and the Lorentz force $e(\boldsymbol{\omega} \wedge \mathbf{H})$ where $\boldsymbol{\omega}$ is the angular velocity vector of that gyrostat. The resultant value of the restoring moment K, taking into account Eq. (2.3) and (2.4), can be written in the form:

$$K = mg\ell + eH\ell' \left[r - \frac{1}{2} r^{-1} (p \sin\varphi \sin\theta + q \cos\varphi \sin\theta + r\cos\theta)^2 \right], \tag{3.1}$$

where ℓ' is the distance of the position of the point charge e from a fixed point. Using (2.5) and (3.1) in (2.2) and cancelling ε on both sides of the first two equations, one gets:

$$
(3.2)\qquad
\begin{aligned}
&A\dot{P} + (C-A)Qr + Q\ell_3 = K\sin\theta\cos\varphi + \varepsilon M_1^*,\\
&A\dot{Q} + (A-C)Pr - P\ell_3 = -K\sin\theta\sin\varphi + \varepsilon M_2^*,\\
&C\dot{r} = \varepsilon M_3^*,\\
&\dot{\theta} = \varepsilon(P\cos\varphi - Q\sin\varphi),\\
&\dot{\varphi} = r - \varepsilon(P\sin\varphi + Q\cos\varphi)\cot\theta,\\
&\dot{\psi} = \varepsilon(P\sin\varphi + Q\cos\varphi)\operatorname{cosec}\theta.
\end{aligned}
$$

The last three equations of the system (3.2), with $\varepsilon = 0$, give the zero approximation, i.e.,

$$
(3.3)\qquad r = r_0, \qquad \psi = \psi_0, \qquad \theta = \theta_0, \qquad \varphi = r_0 t + \varphi_0.
$$

Here r_0, ψ_0, θ_0 and φ_0 are constants we obtain for initial values of the variables for $t = 0$. Substituting (3.3) into the first two equations of system (3.2) for $\varepsilon = 0$, we obtain

$$
(3.4)\qquad
\begin{aligned}
\ddot{P} + y_0^2 P &= z_0 k_0 \sin\theta_0 \sin(r_0 t + \varphi_0),\\
\ddot{Q} + y_0^2 Q &= z_0 k_0 \sin\theta_0 \cos(r_0 t + \varphi_0).
\end{aligned}
$$

Solving the above system (3.4) we arrive at

$$
(3.5)\qquad
\begin{aligned}
P &= a\cos\gamma_0 + b\sin\gamma_0 + E_0\sin\theta_0\sin(r_0 t + \varphi_0),\\
Q &= a\sin\gamma_0 - b\cos\gamma_0 + E_0\sin\theta_0\cos(r_0 t + \varphi_0),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= P_0 - E_0\sin\theta_0\sin\varphi_0,\\
b &= -Q_0 + E_0\sin\theta_0\cos\varphi_0,
\end{aligned}
$$

$$
(3.6)\qquad
\begin{aligned}
&E_0 = z_0 k_0/(y_0^2 - r_0^2), \quad y_0 = n_0 + A^{-1}\ell_3 \neq 0,\\
&n_0 = (C-A)A^{-1} r_0, \quad z_0 = (n_0 - r_0)A^{-1} + A^{-2}\ell_3,\\
&\gamma_0 = y_0 t, \qquad k_0 = K_0, \qquad \left|\frac{y_0}{r_0}\right| \leqslant 1.
\end{aligned}
$$

The last condition of (3.6) gives the following two cases:

(1) $A = B = \frac{1}{2}C$ gives a Lagrange's gyrostat rapidly spinning about its symmetry axis ($x_G = y_G = 0$).

(2) $A = B < C, A \neq \frac{1}{2}C$ leads to the initial fast spin r_0 of the body about the minor axis of the ellipsoid of inertia.

Here P_0 and Q_0 are the initial values of the variables P and Q defined in (2.5), while the variable $\gamma = \gamma_0$ has the meaning of the oscillation phase of the generating system. System (3.2) is essentially nonlinear, and therefore we introduce an additional variable γ, defined by the equation

$$\frac{d\gamma}{dt} = y, \qquad \gamma(0) = 0. \tag{3.7}$$

For $\varepsilon = 0$ we have $\gamma = \gamma_0 = y_0 t$, in accordance with (3.6).

Equations (3.3) and (3.5,6) give the general solution of the system (3.2) and (3.7) when $\varepsilon = 0$. Eliminating the constants with allowance for (3.3), it is possible to rewrite equations (3.5) as

$$\begin{aligned} P &= a\cos\gamma + b\sin\gamma + E\sin\theta\sin\varphi, \\ Q &= a\sin\gamma - b\cos\gamma + E\sin\theta\cos\varphi. \end{aligned} \tag{3.8}$$

Solving these equations, we obtain the results

$$\begin{aligned} a &= P\cos\gamma + Q\sin\gamma - E\sin\theta\sin(\gamma+\varphi), \\ b &= P\sin\gamma - Q\cos\gamma + E\sin\theta\cos(\gamma+\varphi), \end{aligned} \tag{3.9}$$

which define a change of variables P and Q to variables a and b of the Van der Pol type [15] and *vice versa*.

Using (3.2) and (3.7), we change the variables $P, Q, r, \psi, \theta, \varphi, \gamma$, to the new variables a, b, r, ψ, θ, φ, γ, α, $\dot{\gamma}$, where

$$\alpha = \gamma + \varphi. \tag{3.10}$$

After reduction, we obtain the following system:

$$\dot{a} = \varepsilon A^{-1}[M_1^0 \cos\gamma + M_2^0 \sin\gamma] - 2\varepsilon k^{-1} z^{-1} E^2 C^{-1} r M_3^0 \sin\theta \sin\alpha$$
$$- \varepsilon E \cos\theta(b - E\sin\theta\cos\alpha) - \varepsilon k^{-1} E \sin\theta \sin\alpha e H \ell' C^{-1} M_3^0$$
$$\left[1 + \frac{1}{2} r^{-2} (p\sin\varphi\sin\theta + q\cos\varphi\sin\theta + r\cos\theta)^2\right]$$
$$+ k^{-1} E\sin\theta\sin\alpha e H\ell' r^{-1}(p\sin\varphi\sin\theta + q\cos\varphi\sin\theta + r\cos\theta)$$
$$\{\varepsilon(p\sin\varphi\cos\theta + q\cos\varphi\cos\theta - r\sin\theta)(a\cos\alpha + b\sin\alpha)$$
$$+ (p\sin\theta\cos\varphi - q\sin\theta\sin\varphi)[r - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta]$$
$$+ \varepsilon\sin\varphi\sin\theta(-yQ + \varepsilon A^{-1} M_1^0) + \varepsilon\cos\varphi\sin\theta(yP + \varepsilon A^{-1} M_2^0)$$
$$+ \varepsilon C^{-1} M_3^0 \cos\theta\},$$

(3.11) $$\dot{b} = \varepsilon A^{-1}[M_1^0 \sin\gamma - M_2^0 \cos\gamma] + 2\varepsilon k^{-1} z^{-1} E^2 C^{-1} r M_3^0 \sin\theta\cos\alpha$$
$$+ \varepsilon E\cos\theta(a + E\sin\theta\sin\alpha) + \varepsilon k^{-1} E\sin\theta\cos\alpha e H\ell' C^{-1} M_3^0$$
$$\left[1 + \frac{1}{2} r^{-2} (p\sin\varphi\sin\theta + q\cos\varphi\sin\theta + r\cos\theta)^2\right]$$
$$- k^{-1} E\sin\theta\cos\alpha e H\ell' r^{-1}(p\sin\varphi\sin\theta + q\cos\varphi\sin\theta + r\cos\theta)$$
$$\{\varepsilon(p\sin\varphi\cos\theta + q\cos\varphi\cos\theta - r\sin\theta)(a\cos\alpha + b\sin\alpha)$$
$$+ (p\sin\theta\cos\varphi - q\sin\theta\sin\varphi)[r - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta]$$
$$+ \varepsilon\sin\varphi\sin\theta(-yQ + \varepsilon A^{-1} M_1^0) + \varepsilon\cos\varphi\sin\theta(yP + \varepsilon A^{-1} M_2^0)$$
$$+ \varepsilon C^{-1} M_3^0 \cos\theta\},$$
$$\dot{r} = \varepsilon C^{-1} M_3^0,$$
$$\dot{\psi} = \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\operatorname{cosec}\theta,$$
$$\dot{\theta} = \varepsilon(a\cos\alpha + b\sin\alpha),$$
$$\dot{\alpha} = CA^{-1} r + A^{-1}\ell_3 - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta,$$
$$\dot{\gamma} = (C - A)A^{-1} r + A^{-1}\ell_3,$$

where M_i^0 denote functions obtained from M_i^* as a result of substitution of (3.8) to (3.11), i.e.,

(3.12) $$M_i^0(a, b, r, \psi, \theta, \alpha, \gamma, t) = M_i^*(P, Q, r, \psi, \theta, \varphi, t), \qquad i = 1, 2, 3.$$

Note that the change from the two variables P and Q to the three variables a, b and γ is made for the sake of convenience: for $\varepsilon = 0$, the system for P and Q has the form of a linear system, while subsitution (3.8) is non-singular for a and b.

We consider a vector – valued function X whose components are provided with the slow variables a, b, r, ψ and θ of system (3.11). Thus, this system can be written in the form

$$\begin{aligned} \dot{X} &= \varepsilon X(x,\alpha,\gamma,t), \qquad \dot{\alpha} = CA^{-1}r + A^{-1}\ell_3 + \varepsilon Y(x,\alpha), \\ \dot{\gamma} &= (C-A)A^{-1}r + A^{-1}\ell_3, \quad x(0) = x_0, \quad \alpha(0) = \alpha_0, \quad \gamma(0) = 0. \end{aligned} \tag{3.13}$$

Here the vector-valued function X and the scalar function Y are defined by the right-hand sides of (3.11) whose initial values can be obtained in accordance with Eqs. (3.3) to (3.7) and (3.10).

Consider both systems (3.11) or (3.13) from the standpoint of employing the averaging method of [8, 15]. System (3.11) contains the slow variables a, b, r, ψ, and θ and fast variables represented by the phases α, γ and time t. This system is essentially nonlinear and it is extremely difficult to employ the averaging method directly [16]. Let us assume, for the sake of simplicity that the perturbing moments M_i^* are independent of t.

Since $M_i^*(i = 1,2,3)$ are 2π-periodic in φ, it follows in accordance with (3.8) to (3.11), that functions M_i^* of (3.12) will be 2π-periodic functions of α and γ. Then system (3.13) contains two rotation phases α and γ, and the corresponding frequencies $CA^{-1}r + A^{-1}\ell_3$ and $(C-A)A^{-1}r + A^{-1}\ell_3$ are variables.

In the averaging systems (3.11) or (3.13), two cases should be distinguished:

(1) The nonresonant case, when frequencies $CA^{-1}r + A^{-1}\ell_3$ and $(C-A)A^{-1}r + A^{-1}\ell_3$ are non-commensurable.

(2) The resonant case, when these frequencies are commensurable.

A very important feature of system (3.13) is the fact that the ratio of the frequencies is constant $[(C-A)A^{-1}r + A^{-1}\ell_3]/[CA^{-1}r + A^{-1}\ell_3] = 1 - [Ar/(Cr+\ell_3)]$, and the resonant case occurs for

$$(Cr + \ell_3)/Ar = i/j \leqslant 2, \tag{3.14}$$

where i and j are relatively prime natural numbers while in the nonresonant case $(Cr + \ell_3)/Ar$ is an irrational number.

As a result of (3.14), averaging of the nonlinear system (3.13), in which X is independent of t, is equivalent to averaging of a quasilinear system with constant frequencies; this can be achieved by introducing the independent variable γ. In the nonresonant case $[(Cr + \ell_3)/Ar] \neq i/j$, we obtain the first approximation of the averaged system by averaging the right-hand sides of the system (3.11) with respect to the fast variables α and γ. As a result, we obtain the following equations for the slow variables:

$$\begin{aligned} \dot{a} &= \varepsilon A^{-1}\mu_1 - \varepsilon Eb\cos\theta - 2\varepsilon k^{-1}E^2z^{-1}rC^{-1}\sin\theta\mu_3^d + \mu_1^k, \\ \dot{b} &= \varepsilon A^{-1}\mu_2 + \varepsilon Ea\cos\theta + 2\varepsilon k^{-1}E^2z^{-1}rC^{-1}\sin\theta\mu_3^0 + \mu_2^k, \\ \dot{r} &= \varepsilon C^{-1}\mu_3, \qquad \dot{\psi} = \varepsilon E, \qquad \dot{\theta} = 0, \end{aligned} \tag{3.15}$$

where

$$\mu_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[M_1^0\cos\gamma + M_2^0\sin\gamma]d\alpha d\gamma,$$

$$\mu_2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[M_1^0\sin\gamma - M_2^0\cos\gamma]d\alpha d\gamma,$$

$$\mu_3 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0 d\alpha d\gamma,$$

$$\mu_3^d = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\sin\alpha d\alpha d\gamma,$$

$$\mu_3^{\circ} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}M_3^0\cos\gamma d\alpha d\gamma,$$

$$\mu_1^k = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\dot{K}_1 d\alpha d\gamma,$$

$$\mu_2^k = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\dot{K}_2 d\alpha d\gamma,$$

$$\begin{aligned} \dot{K}_1 = {} & -\varepsilon K^{-1}E\sin\theta\sin\alpha eH\ell'C^{-1}M_3^0\Big[1 + \frac{1}{2}r^{-2}(p\sin\varphi\sin\theta + q\cos\varphi\sin\theta \\ & + r\cos\theta)^2\Big] + K^{-1}E\sin\theta\sin\alpha eH\ell' r^{-1}(p\sin\varphi sin\theta + q\cos\varphi\sin\theta + r\cos\theta) \\ & \{\varepsilon(p\sin\varphi\cos\theta + q\cos\varphi\cos\theta - r\sin\theta)(a\cos\alpha + b\sin\alpha) + (p\sin\theta\cos\varphi \\ & -q\sin\theta\sin\varphi\,[r - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta] + \varepsilon\sin\varphi\sin\theta \\ & (-\,yQ + \varepsilon A^{-1}M_1^0) + \varepsilon\cos\varphi\sin\theta(yP + \varepsilon A^{-1}M_2^0) + \varepsilon C^{-1}M_3^0\cos\theta\}, \end{aligned}$$

$$\dot{K}_2 = \varepsilon K^{-1} E \sin\theta \cos\alpha eH\ell' C^{-1} M_3^0 \left[1 + \frac{1}{2} r^{-2} (p \sin\varphi \sin\theta + q \cos\varphi \sin\theta + r\cos\theta)^2\right] - K^{-1} E \sin\theta \cos\alpha eH\ell' r^{-1} (p\sin\varphi\sin\theta + q\cos\varphi\sin\theta + r\cos\theta)$$
$$\{\varepsilon(p\sin\varphi\cos\theta + q\cos\varphi\cos\theta - r\sin\theta)(a\cos\alpha + b\sin\alpha) + (p\sin\theta\cos\varphi - q\sin\theta\sin\varphi)[r - \varepsilon(a\sin\alpha - b\cos\alpha + E\sin\theta)\cot\theta] + \varepsilon\sin\varphi\sin\theta (-yQ + \varepsilon A^{-1} M_1^0) + \varepsilon\cos\varphi\sin\theta(yP + \varepsilon A^{-1} M_2^0) + \varepsilon C^{-1} M_3^0 \cos\theta\}\,.$$

Solving the averaged system (3.15) for perturbing moments of specific form, we can determine the motion of the gyrostat in the non-resonant case with an error of order ε on an interval of time variation of order ε^{-1}.

The last equation in the system (3.15) can be integrated to yield $\theta = \theta_0 =$ const.

The above system is equivalent to a two-frequency system with constant frequencies, since both frequencies are proportional to the axial component r of the angular velocity vector. Therefore, the applicability of the averaging method can be substantiated in the same way as for a quasi-linear system. The principal assertion involves the following assumption.

Assume that the function X is sufficiently smooth with respect to α and γ, and that it satisfies a Lipschitz condition [16]. Then on the plane of permissible values of the parameters C and A, there exists a set L of measure zero such that if $(C, A) \in L$, then for the solutions of system (3.13) and (3.15) we have the bound $|x(t,\varepsilon) - \xi(\varepsilon,t)| \leqslant R\varepsilon, t \in [0, \theta\varepsilon^{-1}]$, in which $\xi(\varepsilon, t)$ is the solution of system (3.15) averaged with respect to the phases α and γ; $\xi = (a, b, r, \psi, \theta)$ and $R =$ const. The proof is obvious by using Gronwall's lemma, on the basis of the standard change of variable procedure of the averaging method [17], as well as the arithmetic lemma used to estimate the "small denominators" [16].

System (3.13) is a single frequency system in the resonant case (3.14). Instead of α we introduce a new slow variable, namely a lineary combination of the phases with coefficients

$$\lambda = \alpha - i\gamma(i-j)^{-1}, \qquad i/j \neq 1, \qquad i/j \leqslant 2; \qquad i, j > 0. \tag{3.16}$$

System (3.13) gives the following form of a standard system with rotating phase

$$\begin{aligned} \dot{X} &= \varepsilon X(x, i\gamma(i-j)^{-1} + \lambda, \gamma), \\ \dot{\lambda} &= \varepsilon Y(x, i\gamma(i-j)^{-1} + \lambda), \\ \dot{\gamma} &= (C - A)A^{-1} r + A^{-1}\ell_3, \end{aligned} \tag{3.17}$$

its right sides being $(2|i-j|\pi)$-periodic in γ. We set up the first approximation system by averaging the right-hand sides of system (3.17) with respect to the above period of variation of the argument γ. As a result, we obtain the following system of equations for the slow variables:

$$\begin{aligned}
\dot{a} &= \varepsilon A^{-1}\mu_1^* - \varepsilon Eb\cos\theta - 2\varepsilon K^{-1}E^2Z^{-1}C^{-1}r\sin\theta\mu_3^{*d} + \mu_1^{*k},\\
\dot{b} &= \varepsilon A^{-1}\mu_2^* + \varepsilon Ea\cos\theta + 2\varepsilon K^{-1}E^2Z^{-1}C^{-1}r\sin\theta\mu_3^{*0} + \mu_2^{*k},\\
\dot{r} &= \varepsilon C^{-1}\mu_3^*, \qquad \dot{\psi} = \varepsilon E, \qquad \dot{\theta} = 0, \qquad \dot{\lambda} = -\varepsilon E\cos\theta,
\end{aligned} \tag{3.18}$$

where

$$\mu_1^*(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} [M_1^0\cos\gamma + M_2^0\sin\gamma]d\gamma,$$

$$\mu_2^*(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} [M_1^0\sin\gamma - M_2^0\cos\gamma]d\gamma,$$

$$\mu_3^*(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} M_3^0 d\gamma,$$

$$\mu_3^{*d}(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} M_3^0\sin\gamma d\gamma,$$

$$\mu_3^{*0}(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} M_3^0\cos\gamma d\gamma,$$

$$\mu_1^{*k}(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} \dot{K}_1 d\gamma,$$

$$\mu_2^{*k}(a,b,r,\psi,\theta,\lambda) = \frac{1}{2\pi|i-j|}\int_0^{2\pi|i-j|} \dot{K}_2 d\gamma.$$

Therefore, the motion of the gyrostat in the resonant case can be established by following the reasoning presented in [15, 17] for a heavy solid.

4. The case of linear dissipative perturbed moments

We consider the perturbed motion of a gyrostat analogous to that of the Lagrangian case with allowance for the moments acting on our gyrostat from the external medium. We assume that the perturbing moments $M_i (i = 1, 2, 3)$ have the form of [18]

$$M_1 = -\varepsilon I_1 p, \qquad M_2 = -\varepsilon I_1 q, \qquad M_3 = -\varepsilon I_3 r; \qquad I_1, I_3 > 0, \tag{4.1}$$

where I_1 and I_3 are certain constants of proportionality which depend on the properties of the medium and the shape of the gyrostat. Also, we assume another case of a small moment that is constant along the axis of symmetry; for this case, the perturbed moments $M_i (i = 1, 2, 3)$ take the form

$$M_1 = M_2 = 0, \qquad M_3 = \varepsilon M_3^* = \text{const.} \tag{4.2}$$

Each of the Eqs. (4.1) and (4.2) has been studied separately in [6, 7] when the restoring moment depends on the nutation angle only.

In this case, we consider the motion of the gyrostat acted upon by the sum of the two cases together, i.e., the perturbed vector moment takes the form:

$$M_1 = -\varepsilon I_1 p, \quad M_2 = -\varepsilon I_1 q, \quad M_3 = -\varepsilon I_3 r + \varepsilon M_3^*; \quad I_1, I_3 > 0. \tag{4.3}$$

Let us write the perturbing moments using expressions (2.5) for p and q,

$$M_1 = -\varepsilon^2 I_1 P, \quad M_2 = -\varepsilon^2 I_1 Q, \quad M_3 = -\varepsilon I_3 r + \varepsilon M_3^*; \quad I_1, I_3 > 0. \tag{4.4}$$

For the fundamental oscillations (non-resonant case), we introduce the new slow variables a, b, r, ψ and θ, so that the averaged system (3.15) takes the form

$$\begin{aligned}
\dot{a} = {} & -\varepsilon a I_1 A^{-1} - \varepsilon E b \cos\theta + \frac{1}{2}\varepsilon K^{-1} b E \sin\theta e H \ell' r^{-1} (p \sin\varphi \sin\theta \\
& + q \cos\varphi \sin\theta + r \cos\theta)(p \sin\varphi \cos\theta + q \cos\varphi \cos\theta - r \sin\theta) \\
& - \frac{1}{2}\varepsilon K^{-1} a E \cos\theta e H \ell' r^{-1} (p \sin\varphi \sin\theta + q \cos\varphi \sin\theta + r \cos\theta) \\
& \cdot (p \sin\theta \cos\varphi - q \sin\theta \sin\varphi),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\dot{b} = {} & -\varepsilon b I_1 A^{-1} + \varepsilon E a \cos\theta - \frac{1}{2}\varepsilon K^{-1} a E \sin\theta e H \ell' r^{-1} (p \sin\varphi \sin\theta \\
& + q \cos\varphi \sin\theta + r \cos\theta)(p \sin\varphi \cos\theta + q \cos\varphi \cos\theta - r \sin\theta) \\
& - \frac{1}{2}\varepsilon K^{-1} b E \cos\theta e H \ell' r^{-1} (p \sin\varphi \sin\theta + q \cos\varphi \sin\theta + r \cos\theta) \\
& \cdot (p \sin\theta \cos\varphi - q \sin\theta \sin\varphi),
\end{aligned}$$

$$\begin{aligned} \dot{r} &= -\varepsilon C^{-1}(I_3 r - M_3^*), \\ \dot{\psi} &= \varepsilon E, \\ \dot{\theta} &= 0\,. \end{aligned}$$

Integrating the third equation in (4.5), we obtain

$$r = (r_0 - I_3^{-1} M_3^*) e^{-\varepsilon I_3 C^{-1} t} + M_3^* I_3^{-1}. \tag{4.6}$$

Equation (4.5) for $\dot{\psi}$ can be integrated using (4.6), to yield

$$\psi = \psi_0 + \frac{\varepsilon z k}{y^2}\left[t - \frac{C}{2\varepsilon I_3}\left(\frac{r_0 - I_3^{-1} M_3^*}{y}\right)^2 \left(e^{-2\varepsilon I_3 C^{-1} t} - 1\right)\right]. \tag{4.7}$$

From the last equation of (4.5), it is easy to see that the nutation angle is a constant value, that is

$$\theta = \theta_0. \tag{4.8}$$

Making use of (4.6), (4.8) and the first two equations of (4.5), one obtains

$$\begin{aligned} a &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \cos\eta + Q_0 \sin\eta - E_0 \sin\theta_0 \sin(\eta + \varphi_0)], \\ b &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \sin\eta - Q_0 \cos\eta + E_0 \sin\theta_0 \cos(\eta + \varphi_0)], \\ \eta &= E_0 C I_3^{-1} \cos\theta_0 [e^{\varepsilon I_3 C^{-1} t} - 1]. \end{aligned} \tag{4.9}$$

As the results of substitution into expressions (3.2) and (3.7) of p and q for P, Q, of a and b from (4.9) and of r from (4.6), we obtain

$$\begin{aligned} P &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \cos(\gamma - \eta) - Q_0 \sin(\gamma - \eta) \\ &\qquad + E_0 \sin\theta_0 \sin(\gamma - \eta - \varphi_0)] + E_0 \sin\theta_0 \sin\varphi_0, \\ Q &= e^{-\varepsilon I_1 A^{-1} t}[P_0 \sin(\gamma - \eta) + Q_0 \cos(\gamma - \eta) \\ &\qquad - E_0 \sin\theta_0 \cos(\gamma - \eta - \varphi_0)] + E_0 \sin\theta_0 \cos\varphi_0, \\ \gamma &= \frac{(C - A)}{A I_3} M_3^* \left[r_0 I_3 (M_3^*)^{-1} - 1\right] t e^{-\varepsilon I_3 C^{-1} t} + \frac{1}{A}[(C - A) M_3^* + \ell_3]\, t, \\ p_0 &= \varepsilon P_0, \qquad q_0 = \varepsilon Q_0, \qquad k = \varepsilon K. \end{aligned} \tag{4.10}$$

Thus we have constructed the solution of the first approximation system for the slow variables in the case of dissipative moment (4.3). If the resonance relation

(3.14) is satisfied, the averaging should be performed in accordance with the scheme (3.18). In this case, all the integrals μ_i^* from (3.18) coincide with the corresponding integrals μ_i of (3.15). Therefore, the resonance in effect does not appear and the resultant solution is suitable for describing the motion for any ratio $[(Cr + \ell_3)/Ar] \neq 1$.

We conclude from (4.7) and (4.8) that the nutation angle θ remains constant during the motion while the precession angle ψ depends on time t. For the zero-order approximation of ε, we note that

$$\dot{\theta} = 0, \qquad \dot{\psi} = 0 \qquad \text{and} \qquad \dot{\varphi} = r_0, \tag{4.11}$$

that is, the case of rotation with fast spin r_0 about the symmetry axis is then obtained.

Figures 2 to 10 show the behaviour of the angular velocity v of the gyrostat for different values of the point charge e and the gyrostatic momentum ℓ_3, with initial values of the nutation angle $\theta_0 = \pi/6, \pi/4$, and $\pi/3$.

It is obvious that the angular velocity v increases when the point charge e increases and also when the gyrostatic momentum ℓ_3 increases.

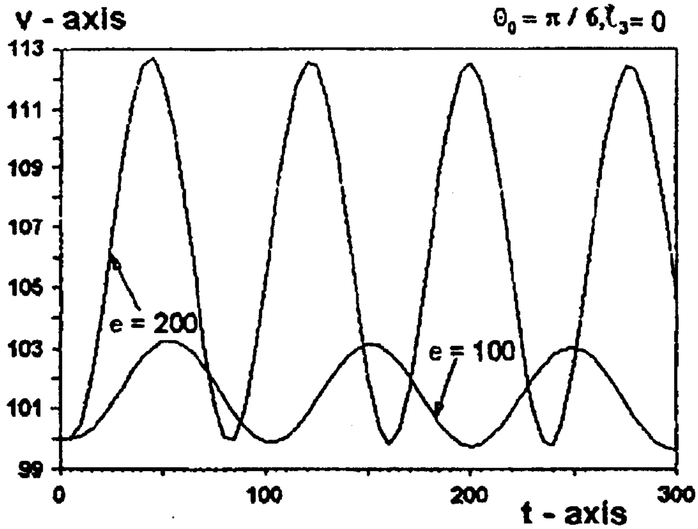

FIG. 2.

Figures 11, 12 and 13 describe the behaviour of the angular velocity of the gyrostat when the point charge $e = 200$ gauss and the gyrostatic momentum $\ell_3 = 0$, 50, and 150, with different initial values of the nutation angle, $\theta_0 = \pi/6$ and $\pi/3$. It is clear that the angular velocity v increases with increasing nutation angle θ_0. We note also that the angular velocity v oscillates in a similar manner with increases of the gyrostatic momentum ℓ_3.

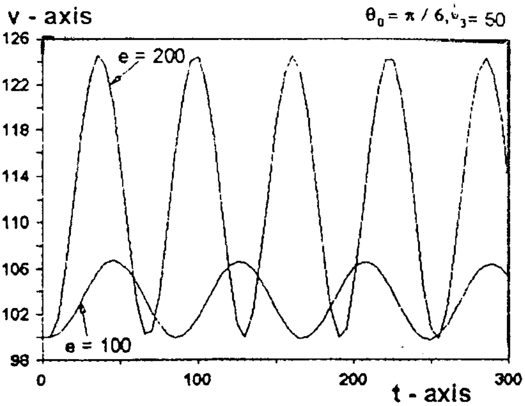

FIG. 3.

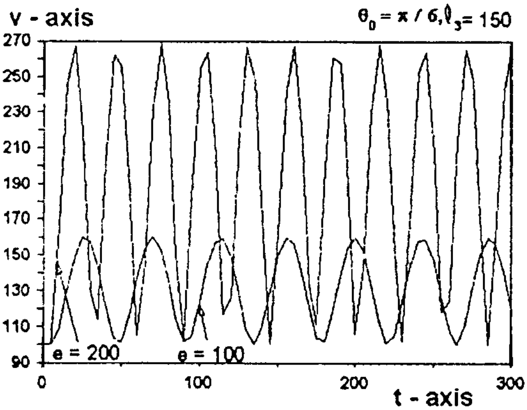

FIG. 4.

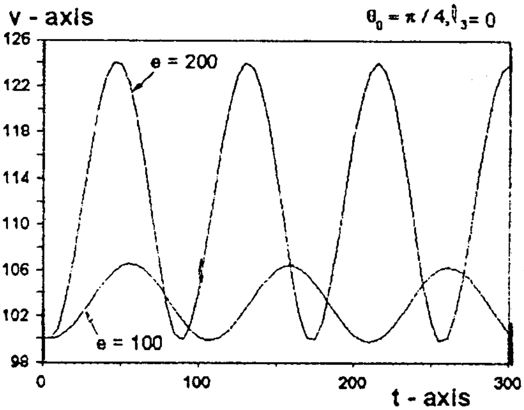

FIG. 5.

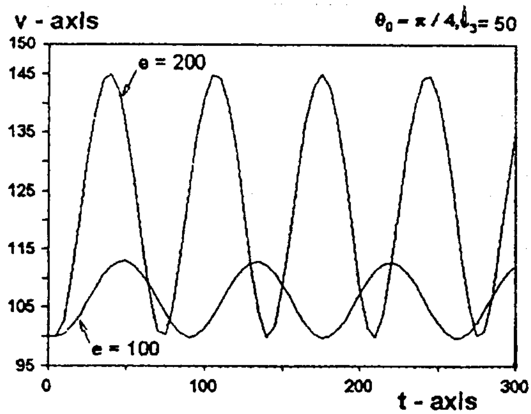

FIG. 6.

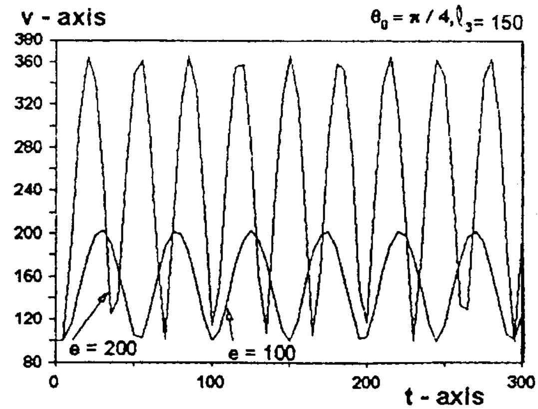

FIG. 7.

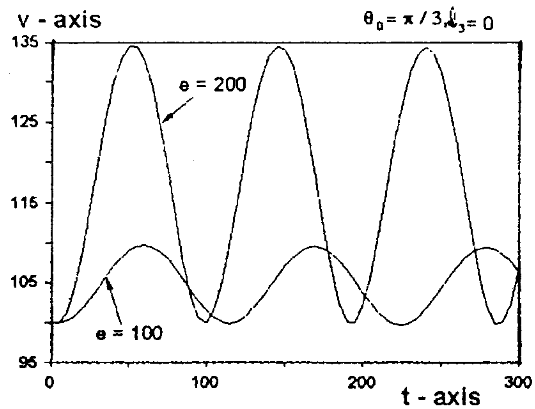

FIG. 8.

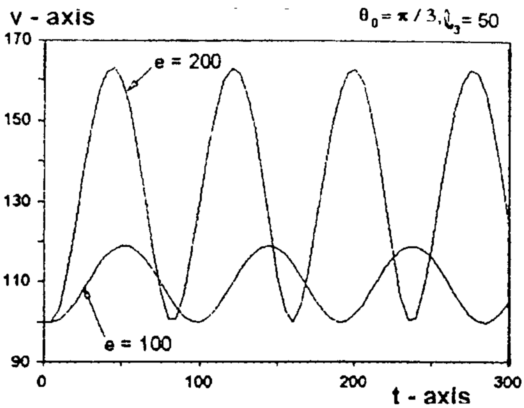

FIG. 9.

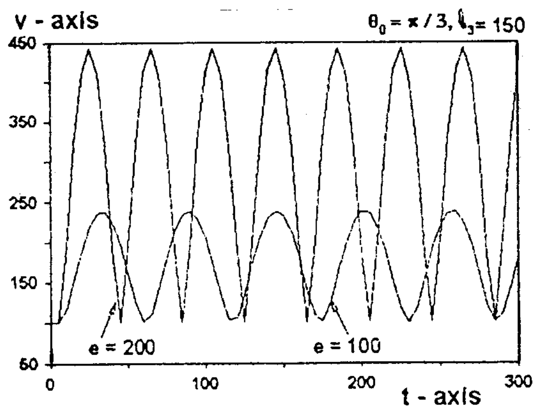

FIG. 10.

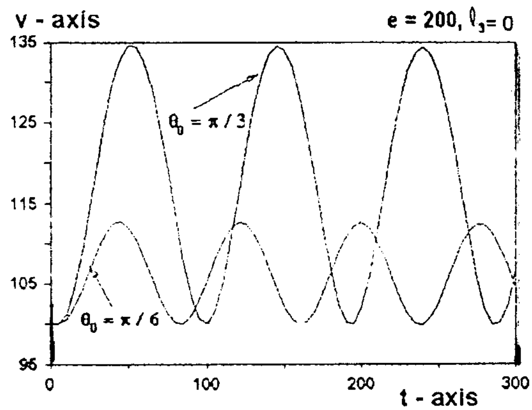

FIG. 11.

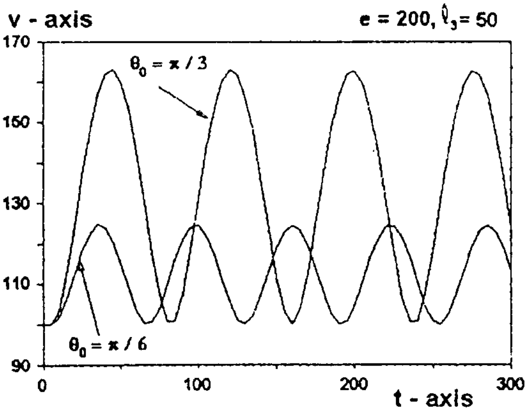

FIG. 12.

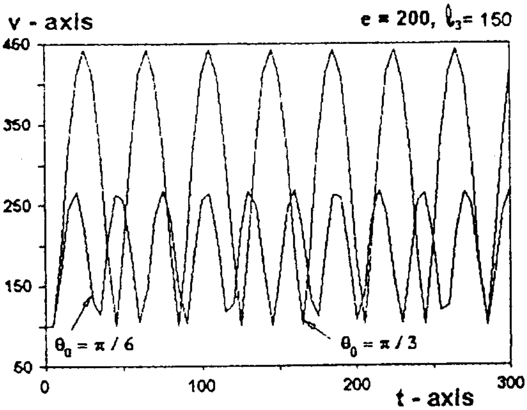

FIG. 13.

REFERENCES

1. I.G. MALKIN, *Some problems in the theory of nonlinear oscillations*, United States Atomic energy Commission, Technical Information Service, ABC 3766, 1959.

2. R. CID and A. VIGUERAS, *The analytical theory of the earth's rotation using a symmetrical gyrostat as a model*, Acad. Ciencias Zaragoza, **45**, pp. 83–93, 1990.

3. R. MOLINA and V. VIGUERAS, *Analytical integration of a generalized Euler-Poisson problem: applications*, IAV Symposium, No. 172, Paris 1995.

4. F.A. EI-BARKI and A.I. ISMAIL, *Limiting case for the motion of a rigid body about a fixed point in the Newtonian force field*, ZAMM, **75**, 11, pp. 821–829, 1995.

5. D.D. LECHCHENKO, S.N. SALLAM, *Perturbed rotational motions of a rigid body with mass distribution near to the case of Lagrange*, Dep. Vokr NUUNTU, No 1656 ok 88, 28-06, 22, 1988.

6. D.D. Lechchenko, S.N. Sallam, *Perturbed rotational motions of a rigid body relative to fixed point* IZV. ANUSS R., Mekh. Tvjord. Tela, **5**, pp. 16–23, 1990.

7. D.D. Lechchenko, S.N. Sallam, *Perturbed rotational motions of a rigid body that are close to regular precession* PMM, **54**, 2, pp. 224–232, 1990.

8. L.D. Akulenko, D.D. Lechchenko and F.L. Chernoushko, *Perturbed motions of a rigid body close to the Lagrange case*, PMM, **43**, 5, pp. 771-778, 1979.

9. A.H. Nayfeh, *Introduction to perturbation techniques*, Wiley-Interscience, New York 1981.

10. A.H. Nayfeh, *Perturbation methods*, Wiley-Interscience, New York 1973.

11. A.H. Nayfeh, *Problems in perturbation*, Wiley-Interscience, New York 1985

12. R. Cid and A. Vigueras, *Actas IV Asamblea Nacional de Astronomia y Astrofisica*, Santiago de Compostela, pp. 912, 1983 (in Spanish).

13. R.F. Grant, *Analytical mechanics*, Univ. of Utah, 1962.

14. V.S. Sergeev, *Periodic motions of a heavy rigid body with a fixed point, that is close to being dynamically symmetrical*, PMM, **47**, **1**, pp. 163–166, 1983.

15. N.N. Bogoliubov and U.A. Mitropolski, *Asymptotic method in the theory of nonlinear oscillations* [in Russian], Nauka 1974.

16. V.I. Arnol'd, *Supplementary chapters in the theory of ordinary differential equations* [in Russian], Nauka, Moscow 1970.

17. V.M. Volosov and B.T. Morgunov, *Method of averaging in the theory of nonlinear oscillatory systems* [in Russian], Moscow, IZV-VOMGU, 1971.

18. V.N. Koshlyakov, *Some particular cases of integration of the dynamic Euler equations associated with motion of a gyroscope in a resistant medium*, PMM **17**, 2, pp. 137–148, 1953.

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