ENGINEERING TRANSACTIONS • *Engng. Trans.* • **47**, 3–4, 299–319, 1999
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.634.1999

ON THE PROPAGATION OF GENERALIZED THERMOELASTIC VIBRATIONS IN PLATES

K. L. VERMA

DEPARTMENT OF MATHEMATICS,
GOVERNMENT POST-GRADUATE COLLEGE HAMIRPUR,
(H.P.) 177005 India

and

N. HASEBE

DEPARTMENT OF CIVIL ENGINEERING,
NAGOYA INSTITUTE OF TECHNOLOGY,
Gokiso-Cho, Showa-Ku, Nagoya 466, JAPAN

The heat conduction equation in the context of generalized theories of thermoelasticity is used to study the propagation of plane harmonic waves in a thin, flat, infinite, homogeneous, thermoelastic isotropic plate of finite width. The frequency equations corresponding to the symmetric and antisymmetric modes of vibration of the plate are obtained, and some limiting cases of the frequency equations are then discussed. The comparison of the results for the theories of generalized thermoelasicity have also been made. The results obtained have been verified numerically and are represented graphically for aluminum epoxy composite plate.

1. Introduction

The basic governing equations of thermoelasticity in the usual framework of linear coupled thermoelasticity consist of the wave-type (hyperbolic) equations of motion and the diffusion-type (parabolic) equation of heat conduction. It is observed that a part of the solution of the energy equation propagates with an infinite speeds. This implies that if an isotropic homogeneous elastic medium is subjected to thermal or mechanical disturbances, the effects in the temperature and displacement fields are felt instantaneously at an infinite distance from the source of disturbance. Therefore, a part of the solution has an infinite velocity of propagation, which is physically impossible.

To overcome this problem, some researchers such as [1 – 5], have tried to modify the Fourier law of heat conduction so as to get a hyperbolic differential equation of heat conduction. These works include the time needed for acceleration of the heat flow in the heat conduction equation along with the coupling between the temperature and strain fields. The paradox in the existing coupled theory of thermoelasticity has also been discussed by BOLEY [6]. This new theory that is named the "Generalized Theory of Thermoelasticity" eliminates the paradox of an infinite velocity of propagation and is based upon the more general linear functional relationship between the heat flow and the temperature gradients. LORD and SHULMAN [7] have formulated a generalized dynamical theory of thermoelasticity (here in after called LS theory) by using a form of the heat conduction equation that includes the time needed for acceleration of the heat flow. Some researchers such as ACKERMAN *et al.* [8], NAYFEH and NASSER [9] have investigated the Maxwell's surface waves propagating along a half-space consisting of linearly elastic materials that conduct heat. MONDAL[10] obtained the frequency equations, corresponding to a thermoelastic plane wave in an infinite thermoelastic plate immersed in an infinite liquid that is kept at uniform temperature, for symmetric and anti-symmetric vibrations about the vertical axis, taking into account the thermal relaxations.

Recently, the theory of thermoelasticity without energy dissipation, which provide sufficient basic modifications in the constitutive equation that permit treatment of much wider class of flow problems, is proposed by GREEN and NAGHDI [13] (here in after called GN theory). The discussion presented in [13] includes the derivation of a complete set of governing equations of the linearized version of the theory for homogeneous and isotropic materials in terms of displacement and temperature fields, and a proof of the uniqueness of the solution of the corresponding initial mixed boundary value problem. The uniqueness of the solution for an initial boundary value problem in this theory, formulated in terms of stress and energy-flux, has been established in [14]. CHANDRASEKHARAIAH [15] investigated the one-dimensional wave propagation in the context of the GN theory.

VERMA [23] studied the field equations of linear thermoelasticity in GN theory with the help of integral transforms. They have discussed the dynamic behaviour of an elastic half-space due to a thermal shock and a mechanical load on the boundary, and found that the disturbances consist of two coupled waves that propagate with finite speeds, without attenuation, and displacement is continuous at both the wavefronts while the temperature, strain, and stress are discontinuous.

In this paper, we investigate the propagation of plane harmonic waves in an infinite homogeneous isotropic plate of thickness $2d$ according to the generalized theories of thermoelasticity[7, 13]. The frequency equations corresponding to the symmetric and antisymmetric modes of vibration of the plate are obtained, and some limiting cases of the frequency equations are then discussed. The comparison of the results for LS and GN theories of generalized thermoelasticity have also been presented. We found that the in GN theory, coupled waves propagate with finite speeds, without attenuation. It has also been observed that, on the whole, the results obtained of the GN theory are qualitatively similar to those of the LS theory. The results have been verified numerically and are represented graphically for aluminum epoxy composite plate.

2. Formulation

We consider an infinite homogeneous isotropic, thermally conducting elastic plate at uniform temperature θ_0 in the undisturbed state having thickness $2d$. Let the faces of the plate be the planes $z = \pm d$, referred to a rectangular set of Cartesian axes $O(x, y, z)$. We choose x-axis in the direction of the propagation of waves so that all particles on a line parallel to y-axis are equally displaced. Therefore all the field quantities will be independent of y-coordinate. The motion is supposed to take place in two dimensions (x, z). Here u, w are the displacements in the x, z directions respectively. In linear generalized theory of thermoelasticity, the governing field equations for the temperature $\theta(x, z, t)$ and the displacement vector $\mathbf{u}(x, z, t) = (u, 0, w)$ in the absence of the body forces and heat sources are [7] given by

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla \text{div}\mathbf{u} - \beta\nabla\theta = \rho\ddot{\mathbf{u}}, \tag{2.1}$$

$$K\nabla^2\theta - \rho C_e(\dot{\theta} + \tau_0\ddot{\theta}) = \theta_0\beta \text{div}(\dot{\mathbf{u}} + \tau_0\ddot{\mathbf{u}}), \tag{2.2}$$

where

$$\beta = (3\lambda + 2\mu)\alpha_t, \tag{2.3}$$

λ, μ are Lamé's parameters; ρ is the density of the medium; C_e and τ_0 are the specific heat at constant strain and thermal relaxation time, respectively; K and α_t are, respectively, the coefficient thermal conductivity and linear thermal expansion, an overdot denotes the partial derivative with respect to the time variable. We define the following dimensionaless quantities:

$$x^* = \frac{v_1}{k_1}x, \quad z^* = \frac{\nu_1}{k_1}z, \quad t^* = \frac{\nu_1^2}{k_1}t, \quad u^* = \frac{\nu_1^3\rho}{k_1\beta T_0}u, \quad w^* = \frac{\nu_1^3\rho}{k_1\beta T_0}w, \tag{2.4}$$

$$\varepsilon_1 = \frac{\beta^2\theta_0}{\rho C_e\nu_1^2}, \quad \theta^* = \frac{\theta}{\theta_0}, \quad \tau_0^* = \frac{\nu_1^2}{k_1}\tau_0, c_2 = \frac{\mu}{2(\lambda+2\mu)}, \quad c_3 = 1 - c_2. \tag{2.5}$$

Here $\nu_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$ is the velocity of compressional waves and $k_1 = K/\rho C_e$ is the thermal diffusivity in the x-direction.

Moreover ε_1 is the thermoelastic coupling constant, and τ_0^* is the thermal relaxation constant. Introducing the above quantities (2.4) and (2.5) in Eqs. (2.1) – (2.2), after suppressing the *, we obtain

$$c_2\nabla^2\mathbf{u} + c_3\nabla\,\mathrm{div}\,\mathbf{u} - \nabla\theta = \ddot{\mathbf{u}}, \tag{2.6}$$

$$\nabla^2\theta - (\dot{\theta} + \tau_0\ddot{\theta}) = \varepsilon_1\mathrm{div}(\dot{\mathbf{u}} + \tau_0\ddot{\mathbf{u}}), \tag{2.7}$$

where $c_3 = 1 - c_2$.

The stresses, and temperature gradient relevant to our problem in the plate are

$$\tau_{zz} = \left[(1-2c_2)\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \theta\right]\beta\theta_0, \tag{2.8}$$

$$\tau_{zx} = \beta\theta_0 c_2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{2.9}$$

$$\theta_z = \frac{\partial\theta}{\partial z}. \tag{2.10}$$

For a plane harmonic wave travelling in the x-direction, the solutions u, w, and θ of Eqs. (2.6) – (2.7) take the form

$$u = f(z)\exp[i\xi(x-ct)], \tag{2.11}$$

$$w = g(z)\exp[i\xi(x-ct)], \tag{2.12}$$

$$\theta = h(z)\exp[i\xi(x-ct)], \tag{2.13}$$

where $c(=\omega/\xi)$ and ξ are phase velocity and wave number respectively; ω is the circular frequency.

Substituting u, w, and θ from Eqs. (2.11) – (2.13) into Eqs. (2.6) – (2.7), we get

$$(2.14)\qquad \begin{aligned} (c_2D^2-\xi^2+\xi^2c^2)f+i\xi c_3Dg-i\xi h &= 0,\\ i\xi c_3Df+(D^2-c_2\xi^2+\xi^2c^2)g-Dh &= 0,\\ i\xi^3\varepsilon_1\tau c^2f+\varepsilon_1\tau D\xi^2c^2g+(D^2-\xi^2+\tau\xi^2c^2)h &= 0,\end{aligned}$$

where

$$(2.15)\qquad \tau=\tau_0+i/c\xi.$$

The solution to Eqs. (2.14) is

$$(2.16)\qquad \begin{aligned} f(z) &= [P_1\exp(-\xi s_1z)+P_2\exp(-\xi s_2z)+P_3\exp(-\xi s_3z)\\ &\qquad +Q_1\exp(\xi s_1z)+Q_2\exp(\xi s_2z)+Q_3\exp(\xi s_3z)],\\ g(z) &= [m_1P_1\exp(-\xi s_1z)+m_2P_2\exp(-\xi s_2z)+m_3P_3\exp(-\xi s_3z)\\ &\qquad -m_1Q_1\exp(ks_1z)-m_2Q_2\exp(ks_2z)-m_3Q_3\exp(ks_3z)],\\ h(z) &= \xi[l_1P_1\exp(-\xi s_1z)+l_2P_2\exp(-\xi s_2z)+l_3P_3\exp(-\xi s_3z)\\ &\qquad +l_1Q_1\exp(\xi s_1z)+l_2Q_2\exp(\xi s_2z)+l_3Q_3\exp(\xi s_3z)],\end{aligned}$$

where

$$(2.17)\qquad \begin{aligned} m_j &= is_j, \qquad m_3=0,\\ l_j &= \frac{1}{i}\left[s_j^2-1+c^2\right], \quad l_3=0, \quad j=1,2.\end{aligned}$$

P_j, Q_j $(j=1,2,3)$ are arbitrary constants, and s_1^2, s_2^2 are the roots of the equation

$$(2.18)\qquad s^4+As^2+B=o,$$

where

$$(2.18)_1\qquad A=(c^2-2)+\tau c^2(1+\varepsilon_1)],$$

$$(2.18)_2\qquad B=[1-\tau c^2(1+\varepsilon_1)+c^4\tau-c^2)$$

and

$$(2.18)_3\qquad s_3^2=1-\frac{c^2}{c^2}.$$

s_1^2, s_2^2 correspond to the coupled longitudinal and thermal waves, whereas s_3^2 corresponds to the transverse wave. This is in agreement with the corresponding results obtained by NAYFEH and NASSER [9]. The displacements and temperature of the plate are thus

$$
\begin{aligned}
u &= [P_1\exp(-\xi s_1 z) + P_2\exp(-\xi s_2 z) + P_3\exp(-\xi s_3 z) \\
&\qquad + Q_1\exp(\xi s_1 z) + Q_2\exp(\xi s_2 z) + Q_3\exp(\xi s_3 z)]\exp[i(x - ct)], \\
w &= [m_1 P_1\exp(-\xi s_1 z) + m_2 P_2\exp(-\xi s_2 z) + m_3 P_3\exp(-\xi s_3 z) \\
&\qquad - m_1 Q_1\exp(\xi s_1 z) - m_2 Q_2\exp(\xi s_2 z) \\
&\qquad - m_3 Q_3\exp(\xi s_3 z)]\exp[i(x - ct)], \\
\theta &= \xi[l_1 P_1\exp(-\xi s_1 z) + l_2 P_2\exp(-\xi s_2 z) + l_3 P_3\exp(-\xi s_3 z) \\
&\qquad + l_1 Q_1\exp(\xi s_1 z) + l_2 Q_2\exp(\xi s_2 z) \\
&\qquad + l_3 Q_3\exp(\xi s_3 z)]\exp[i(x - ct)].
\end{aligned}
\tag{2.19}
$$

3. Boundary conditions

The boundary conditions demand that stresses and temperature gradient on the surfaces of the plate should vanish. Hence for all x and t,

$$
\begin{aligned}
\tau_{zz} &= \tau_{xz} = \theta_{,z} = 0 \quad \text{on } z = -d, \\
\tau_{zz} &= \tau_{xz} = \theta_{,z} = 0 \quad \text{on } z = d.
\end{aligned}
\tag{3.1}
$$

Substituting the expressions (2.19) for the displacement components and temperature into (2.8) – (2.10), and introducing the boundary conditions for the stresses and temperature gradient (3.1), we obtain the following six equations involving the arbitrary constants P_1, P_2, P_3, Q_1, Q_2, and Q_3:

$$
\begin{aligned}
&\sum_{j=1}^{3}(iF - c_1 m_j s_j - l_j(P_j e^{-\xi s_j d} + Q_j e^{\xi s_j d}) = 0, \\
&\sum_{j=1}^{3}(\iota m_j - s_j)(P_j e^{-\xi s_j d} - Q_j e^{\xi s_j d}) = 0,
\end{aligned}
\tag{3.2}
$$

$$\sum_{j=1}^{3}(-l_j s_j)(P_j e^{-\xi s_j d} - Q_j e^{\xi s_j d}) = 0,$$

$$\sum_{j=1}^{3}(iF - c_1 m_j s_j - l_j)(P_j e^{\xi s_j d} + Q_j e^{-\xi s_j d}) = 0,$$

$$\sum_{j=1}^{3}(\iota m_j - s_j)(P_j e^{\xi s_j d} - Q_j e^{-\xi s_j d}) = 0,$$

$$\sum_{j=1}^{3}(-l_j s_j)(P_j e^{\xi s_j d} - Q_j e^{-\xi s_j d}) = 0,$$

where $F = 1 - 2c_2, \;\; j = 1, 2, 3.$

4. Frequency equation

In order that the six boundary conditions could be satisfied simultaneously, the determinant of the coefficients of the arbitray constants must vanish.This gives an equation for the frequency of the plate oscillations. The frequency equation is found to split into two factors, each of which yields the equations

$$D_1 G_1 \coth(\xi s_1 d) - D_2 G_2 \coth(\xi s_2 d) + D_3 G_3 \coth(\xi s_3 d) = 0, \tag{4.1$_1$}$$

and

$$D_1 G_1 \tanh(\xi s_1 d) - D_2 G_2 \tanh(\xi s_2 d) + D_3 G_3 \tanh(\xi s_3 d) = 0, \tag{4.1$_2$}$$

where

$$D_j = iF - c_1 m_j s_j - l_j, \tag{4.2}$$

$$G_1 = -Y_3 Z_2, \tag{4.3}$$

$$G_2 = -Y_3 Z_1, \tag{4.4}$$

$$G_3 = Y_1 Z_2 - Y_2 Z_1, \tag{4.5}$$

$$Y_j = i m_j - s_j, \quad Z_j = -l_j s_j, \quad j = 1, 2, 3, \tag{4.6}$$

where m_j and l_j are given in Eq. (2.17).

These are the period equations which correspond to the symmetric and antisymmetric motion of the plate with respect to the medial plane $z = 0$. It can be shown that $(4.1)_1$ corresponds to the symmetric motion and $(4.1)_2$ corresponds to the antisymmetric motion .

The displacements and temperature in the symmetric motion are given by

$$
\begin{aligned}
u &= [H_1 \cosh(\xi s_1 d) + H_2 \cosh(\xi s_2 d) + H_3 \cosh(\xi s_3 d)] \exp[i\xi(x-ct)], \\
w &= [m_1 H_1 \sinh(\xi s_1 d) + m_2 H_2 \sinh(\xi s_2 d) \\
&\qquad + m_3 H_3 \sinh(\xi s_3 d)] \exp[i\xi(x-ct)], \\
\theta &= [l_1 H_1 \cosh(\xi s_1 d) + l_2 H_2 \cosh(\xi s_2 d)] \exp[i\xi(x-ct)],
\end{aligned} \tag{4.7}
$$

and in the antisymmetric motion by

$$
\begin{aligned}
u &= [H_1 \sinh(\xi s_1 d) + H_2 \sinh(\xi s_2 d) + H_3 \sinh(\xi s_3 d)] \exp[i\xi(x-ct)], \\
w &= -[m_1 H_1 \cosh(\xi s_1 d) + m_2 H_2 \cosh(\xi s_2 d) \\
&\qquad + m_3 H_3 \cosh(\xi s_3 d)]] \exp[i\xi(x-ct)], \\
\theta &= [l_1 H_1 \sinh(\xi s_1 d) + l_2 H_2 \sinh(\xi s_2 d)] \exp[i\xi(x-ct)],
\end{aligned} \tag{4.8}
$$

where $m_j (j = 1, 2, 3)$ and $l_k (k = 1, 2)$ are given in Eq. (2.17).

The discussion of transcendental Eq. (4.1) in general is difficult; we therefore, consider the results for some limiting cases.

5. Symmetric modes

For waves long compared with the thickness $2d$ of the plate, ξd is small and consequently $\xi d s_1$, $\xi d s_2$ and $\xi d s_3$ may be assumed to be small as long as c is finite. Hence the hyperbolic function can be replaced by their arguments and from Eq. (4.1) we then obtain

$$
(s_1^2 - s_2^2)][(1+s_3^2)^2 \left\{ s_1^2 + s_2^2 + c^2 - 1 \right\} - 4 s_1^2 s_2^2] = 0, \tag{5.1}
$$

where

$$
s_1^2 + s_2^2 = -[c^2 - 2 + c^2 \tau (1 + \varepsilon_1)], \tag{5.2}
$$

$$
s_1^2 s_2^2 = (c^2 \tau - 1)(c^2 - 1) - c^2 \tau \varepsilon_1. \tag{5.3}
$$

Hence either

$$
(s_1^2 - s_2^2) = 0, \tag{5.4}
$$

or

$$[(1 + s_3^2)\left\{s_1^2 + s_2^2 + c^2 - 1\right\} - 4s_1^2 s_2^2] = 0. \tag{5.5}$$

If

$$s_1^2 = s_2^2 \tag{5.6}$$

the form of the original solution assumed, (2.19), cannot satisfy the boundary conditions. Hence Eq. (5.5) holds. On using the Eqs. (5.3) – (5.4), Eq. (5.5) reduces to

$$\left[2 - \frac{c^2}{c_2}\right]^2 \left[1 - c^2\tau(1+\varepsilon_1)\right] = 4\left[(c^2\tau - 1)(c^2 - 1) - \varepsilon_1 c^2 \tau\right]. \tag{5.7}$$

This equation gives the phase velocity of long compressional or plate waves c_p in the generalized theory of thermoelasticity. For aluminum epoxy composite plate, for which the physical data will be given in Sec. 8, the velocity of plate waves is $c_p = 0.554$ (non-dimensional).

When the strain and thermal fields are uncoupled, the coupling constant ε_1 is identically zero, and Eq. (5.7) reduces to

$$c^2 = 4\beta^2\left(1 - \frac{\beta^2}{\alpha^2}\right), \tag{5.8}$$

which agrees with EWING *et al.* [22].

For very *short waves* and c such that s_1, s_2 and s_3 are real, ξd is large and the hyperbolic functions tend to unity. The Equation (4.1) becomes

$$(s_1 - s_2)[-(1 + s_3^2)^2\left\{s_1^2 + s_1^2 + s_1 s_2 + c^2 - 1\right\} + 4s_1 s_2 s_3(s_1 + s_2)] = 0. \tag{5.9}$$

Evidently $(s_1 - s_2)$ is a factor, factorizing (5.9), and we obtain

$$[-(1 + s_3^2)^2\left\{s_1^2 + s_1^2 + s_1 s_2 + c^2 - 1\right\} + 4s_1 s_2 s_3(s_1 + s_2)] = 0. \tag{5.10}$$

Equation (5.10) can be identified with the phase velocity equation for Rayleigh waves in isotropic half-space. This is in agreement with the corresponding result of NAYFEH and NASSER [9]. For aluminum epoxy composite plate for which the physical data are given in Sec. 8, Rayleigh waves speed have been found to be $c_R = 0.384$ (non-dimensional).

5.1. Classical case

When the strain and thermal fields are uncoupled to each other. The coupling constant ε_1 is identically zero, and Eq. (5.10) reduces to

$$\left[2 - \frac{c^2}{c_2}\right]^4 = 16(1 - c^2)\left(1 - \frac{c^2}{c_2}\right). \tag{5.11}$$

This is in agreement with the corresponding result of NAYFEH and NASSER [9].

5.2. *Case of coupled thermoelasticity*

This case corresponds to no thermal relaxation time, i.e. $\tau_0 = 0$ and hence for $\tau = i/\xi c$. Proceeding along the same lines as in the previous section, we again arrived at Eq. (5.10) with s_1, s_2 satisfying Eqs.

$$\begin{aligned} s_1^2 + s_2^2 &= -[c^2 - 2 + ci\xi^{-1}(1+\varepsilon_1)], \\ s_1^2 s_2^2 &= [(ci\xi^{-1} - 1)(c^2 - 1) - ci\xi^{-1}\varepsilon_1], \end{aligned} \tag{5.12}$$

and s_3 as in $(2.18)_4$.

In this case, the frequency equation after some algebraic manipulations and using the condition $\omega(=\xi c) \gg 1$, (5.10) reduces to

$$(1+\varepsilon_1)\left(2 - \frac{c^2}{c_2}\right)^4 = 16\left[(1+\varepsilon_1) - c^2\right]\left(1 - \frac{c^2}{c_2}\right), \tag{5.13}$$

which agrees with the results of LOCKETT [21].

Also when $\varepsilon_1 = 0$, the frequency equation in the coupled thermoelastic case reduces to $c^2 = 1/i\omega$ and (5.11) represents the classical Rayleigh waves .

6. ANTISYMMETRIC MODES

For waves long compared with the thickness of the plate, and for s_1, s_2 and s_3 real, we may replace the hyperbolic functions by the approximation

$$\tanh x \cong x - \frac{1}{3}x^3. \tag{6.1}$$

After some algebraic transformation and reductions, and neglecting the quantities of $O[\xi d]^3$, we obtain

$$\frac{c^2}{c_2} - \frac{4\xi^2 d^2}{3}\left[(c_2 - 1)\left(1 + \frac{c^2}{c_2}\right) - \frac{c^2}{4c_2}(c^2 - 1)\right]. \tag{6.2}$$

This is the dispersion equation for long flexural waves and it can be seen that the phase velocity tends to zero as the wave length increases to infinity.

For waves short compared with the thickness of the plate, that is for $\xi d \to \infty$, and c such that s_1, s_2 , and s_3 are real, Eq. $(4.1)_2$ reduces to Rayleigh Eq. (5.10), and the propagation degenerates to Rayleigh waves associated with both free surfaces of the plate in generalized thermoelasticity.

7. Thermoelasticity without energy dissipation

The fundamental equations for such a medium, with heat sources and body forces absent, in the context of generalized thermoelasticity developed by GREEN and NAGHDI [13], are given by

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} - \gamma\nabla\theta = \rho\ddot{\mathbf{u}}, \tag{7.1}$$

$$\rho C\ddot{\theta} + \gamma\theta_0\mathrm{div}\,\ddot{\mathbf{u}} = k^*\nabla^2\theta. \tag{7.2}$$

Here $\mathbf{u}(x,z,t) = (u,0,w)$ is the displacement vector; θ is the temperature change above the uniform reference temperature θ_0; ρ is the mass density; C is the specific heat at constant deformation; λ and μ are the Lamé's parameters; $\gamma = (3\lambda+2\mu)\beta^*$; β^* is the coefficient of volume expansion; and k^* is a material constant characteristic of the theory.

The strain tensor $\mathbf{E}$ and the stress tensor $\mathbf{T}$ associated with $\mathbf{u}$ and θ are given by the following geometrical and constitutive relations, respectively, as

$$\mathbf{E} = \frac{1}{2}[\nabla\mathbf{u} + \nabla\mathbf{u}^T], \tag{7.3}$$

$$\mathbf{T} = \lambda(\mathrm{div}\,\mathbf{u})I + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - \gamma\theta I. \tag{7.4}$$

In all the above equations, the direct vector/tensor notation [23] is employed; also, an overdot denotes the partial derivative with respect to the time variable t. Some of the symbols and the notations used here are slightly different from those employed in [13]. We suppose that the constants appearing in Eqs. (7.1) and (7.2) satisfy the inequalities

$$\mu > 0,\ (\lambda+2\mu) > 0, \quad \rho > 0, \quad \theta_0 > 0, \quad C > 0, \quad \rho > 0, \quad k^* > 0. \tag{7.5}$$

Then Eqs. (7.1) and (7.2) represent a fully hyperbolic system that permits finite speeds for both elastic and thermal disturbances, which are coupled together in general.

Define the dimensionless quantities

$$\begin{aligned} \mathbf{x}' &= \frac{1}{l}\mathbf{x} \quad t' = \frac{v}{l}t \quad \mathbf{u}' = \frac{1}{l}\frac{(\lambda+2\mu)}{\gamma\theta_0}\mathbf{u}, \\ \theta' &= \frac{\theta}{\theta_0} \quad \mathbf{E}' = \frac{(\lambda+2\mu)}{\gamma\theta_0}\mathbf{E} \quad \mathbf{T}' = \frac{1}{\gamma\theta_0}\mathbf{T}. \end{aligned} \tag{7.6}$$

Here l is a standard length and v is a standard speed. Introducing Eq. (7.6) into Eqs. (7.1) – (7.4) and suppressing primes, we obtain

$$C_2^2\nabla^2\mathbf{u} + (C_1^2 - C_2^2)\nabla\mathrm{div}\,\mathbf{u} - C_1^2\nabla\theta = \ddot{\mathbf{u}}, \tag{7.7}$$

$$C_3^2\nabla^2\theta = \ddot{\theta} + \in \mathrm{div}\,\mathbf{u}, \tag{7.8}$$

$$\mathbf{E} = \frac{1}{2}[\nabla\mathbf{u} + \nabla\mathbf{u}^T], \tag{7.9}$$

$$\mathbf{T} = \left(1 - 2\frac{C_2^2}{C_1^2}\right)(\mathrm{div}\,\mathbf{u})I + \frac{C_2^2}{C_1^2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - \theta I, \tag{7.10}$$

where

$$C_1^2 = \frac{\lambda + 2\mu}{\rho v^2} \quad C_2^2 = \frac{\mu}{\rho v^2} \quad C_3^2 = \frac{k^*}{\rho C v^2}, \quad \varepsilon_1 = \frac{\gamma^2\theta_0}{\rho C(\lambda + 2\mu)}. \tag{7.11}$$

We observe that C_1 and C_2, respectively, represent the non-dimensional speeds of purely elastic dilatation and shear waves and that C_3 represents the non-dimensional speed of purely thermal waves. Also, ε_1 is the usual thermoelastic coupling parameter. It is also seen that the expression for the non-dimensional speed C_3 of pure thermal waves in the GN theory differ from its counterparts in the LS theory. In the LS theory C_3 is determined by a relaxation time, while in the GN theory C_3 is determined principally by the material constant k^* [25].

Subtituting u, w, and θ from (2.11) – (2.13) into Eqs. (7.7) and (7.8), we obtain

$$(C_2^2 D^2 - C_1^2\xi^2 + \xi^2 c^2)f + i\xi(C_1^2 - C_2^2)Dg - iC_1^2\xi h = 0 \tag{7.12}$$

$$i\xi(C_1^2 - C_2^2)Df + (C_1^2 D^2 - C_2\xi^2 + \xi^2 c^2)g - C_1^2 Dh = 0, \tag{7.13}$$

$$i\xi^3\varepsilon_1 c^2 f + \varepsilon_1\xi^2 c^2 Dg + [C_3^2(D^2 - \xi^2) + \xi^2 c^2]h = 0. \tag{7.14}$$

The solution to Eqs. (7.12) – (7.14) is again of the form $(2.16)_{1,2,3}$ where

$$m_j = is_j, \qquad m_3 = 0, \ j = 1, 2. \tag{7.15}$$

$$l_j = \frac{1}{iC_1^2}\left[C_1^2 s_j^2 - C_1^2 + c^2\right], \quad l_3 = 0, \ j = 1, 2. \tag{7.16}$$

Here s_1^2 and s_2^2 are the roots of the equation

$$s^4 + As^2 + B = 0, \tag{7.17}$$

where

$$A = \frac{[\{(1+\varepsilon_1)C_1^2 + C_3^2\}c^2 - 2C_1^2C_3^2]}{\Delta}, \tag{7.18}$$

$$B = \frac{[c^2 - \{(1+\varepsilon_1)C_1^2 + C_3^2\}c^2 + C_1^2C_3^2]}{\Delta}, \tag{7.19}$$

where $\Delta = C_1^2C_3^2$, and

$$s_3^2 = 1 - \frac{c^2}{C_2^2}. \tag{7.20}$$

s_1^2, s_2^2 corresponds to the coupled longitudinal and thermal waves whereas s_3^2 corresponds to the transverse wave.

When there is no coupling i.e. $\varepsilon_1 = 0$, then

$$s_1^2 = \frac{c^2}{C_1^2} - 1 \quad s_2^2 = \frac{c^2}{C_3^2} - 1. \tag{7.21}$$

Thus we see that s_1^2, s_2^2 corresponds to elastic and thermal waves, respectively.

Stresses and temperature gradient in this theory are

$$\tau_{zz} = \left[\left(1 - 2\frac{C_2^2}{C_1^2}\right)\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \theta\right], \tag{7.22}$$

$$\tau_{zx} = \frac{C_2^2}{C_1^2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{7.23}$$

$$\theta_{,Z} = \frac{\partial \theta}{\partial z}. \tag{7.24}$$

Introducing (7.23) – (7.25), into the boundary conditions $(3.1)_{1,2}$ and proceeding as in the previous sections, we obtain the frequency equations of the form $(4.1)_{1,2}$ with $F = (C_1^2 - 2C_2^2)$.

The displacements and temperature in the symmetric and antisymmetric cases are given by $(4.7)_{1,2,3}$ and $(4.8)_{1,2,3}$ respectively, where m_j $(j = 1,2,3)$ and l_k $(k = 1,2)$ are given in Eqs. (7.15) and (7.16).

Limiting cases of the frequency equations in the context of linear theory of thermoelasticity without energy dissipation are now discussed.

7.1. Symmetric modes

For *waves long* compared with the thickness $2d$ of the plate, Eq. (5.7) reduces to

$$\left[2-\frac{c^2}{C_2^2}\right]^2\left[C_1^2C_3^2-c^2\{C_3^2+(1+\varepsilon_1)C_1^2\}+c^2C_1^2C_3^2\right] \\ =4\left[(c^4-c^2\{C_3^2+(1+\varepsilon_1)C_1^2\}+C_1^2C_3^2\right]. \tag{7.25}$$

This equation gives the phase velocity of long compressional or plate waves in linear theory of thermoelasticity without energy dissipation.

For very *short waves* and c such that s_1, s_2, s_3 are real, and ξd is large the hyperbolic function tends to unity, and we obtain the equations which are similar to (5.9) – (5.10) with s_1 and s_2 given in (7.18).

When the strain and thermal fields are uncoupled, the coupling constant ε_1 is identically zero, and Eq. (5.10) reduces to

$$\left[2-\frac{c^2}{C_2^2}\right]^4=16(1-c^2)\left(1-\frac{c^2}{C_2^2}\right), \tag{7.26}$$

which is of the same form as (5.11) in LS theory.

7.2. Antisymmetric modes

For waves long compared with thickness of the plate, and s_1, s_2, and s_3 real, we may replace the hyperbolic functions by the approximation (6.1), and (6.2) reduces to

$$\frac{c^2}{C_2^2}-\frac{4\xi^2d^2}{3}\left[(C_2^2-1)\left(1+\frac{c^2}{C_2^2}\right)-\frac{c^2}{4C_2^2}(c^2-1)\right], \tag{7.27}$$

which is of the same form as (6.2) in LS theory.

This is the dispersion equation for long flexural waves and it can be seen that the phase velocity tends to zero as the wave length increases to infinity in the linear theory of thermoelasticity without energy dissipation.

For waves short compared with the thickness of the plate, that is $\xi d \to \infty$, and c such that s_1, s_2, and s_3 are real, Eqs. $(4.1)_{1,2}$ reduces to Rayleigh Eq. (5.10) and the propagation degenerates to Rayleigh waves associated with free surfaces of the plate in this theory.

8. Numerical discussion and conclusions

In general the waves are dispersive; To discuss the long and short waves, we need to find numerical solution of the Eqs. $(4.1)_{1,2}$. For values of c which makes s_1, s_2, and s_3 imaginary, the hyperbolic functions become periodic and so an infinite number of higher modes exists. Computation for the symmetric and antisymmetric modes have been carried out for a aluminum epoxy composite plate whose physical data is given as

$$\lambda = 7.59 \times 10^{11}\text{dynes/cm}^2, \qquad \mu = 1.89 \times 10^{11}\text{dynes/cm}^2,$$

$$K^* = 0.6 \times 10^{-2}\text{cal/cm sec}^\circ\text{C} \qquad \rho = 2.19\ \text{gm/cm}^3,$$

$$C_e = 0.23\ \text{cal/C}^\circ, \qquad \varepsilon_1 = 0.073, \qquad \tau_0 = 6.131 \times 10^{-3}\ \text{s}.$$

The phase and group velocities, (c and $U = c + \xi \dfrac{dc}{d\xi}$, respectively) dispersion curves, are plotted as a function of the wavenumber assuming the thickness $2d$ of the plate is fixed. These curves have been calculated from expression based on the dispersion relation in Eqs. $(4.1)_{1,2}$, which are decoupled characteristic equations corresponding to symmetric and antisymmetric modes of vibrations in LS and GN theories of generalized thermoelasticity.

The additional new mode to those already observed in purely elastic materials is the quasi-thermal T-mode. Dispersion curves for symmetric and antisymmetric modes in LS theory of generalized thermoelasticity are shown in Fig. 1 and Fig. 2, the various modes get merged and then approach each other as wavenumber increases, where the phase and group velocities tend towards the Rayleigh surface wave speed. The wave modes are observed to be more effected at the zero wavenumber limit, due to the thermo-mechanical effects. This clearly demonstrates the difference between the coupled and generalized theory of thermoelasticity. In the first mode of symmetric vibration, the phase velocity decreases monotonically with increasing values of wavenumber from c_p (plate velocity) at $\xi = 0$ to c_R (Rayleigh surface wave speed) at $\xi = \infty$. The group velocity has the same asymptotic limits but has a minimum. In the second mode, the phase velocity is higher than the horizontal velocity of SV waves in the plate. Again, $c \to \infty$ and $U \to 0$ as $\xi \to 0$ and as $\xi \to \infty$, $c \to U \to$ horizontal velocity of SV waves in the plate. Both the maximum and minimum values of group velocity are associated with this mode at intermediate wavenumbers. Similar relations between phase and group velocity for higher modes are demonstrated in the dispersion curves in Fig. 1.

In the first mode antisymmetric vibration Fig. 2, the phase velocity increases monotonically with increasing wavenumber values ξ from $c = 0$ at $\xi = 0$ to $c = c_R$ at $\xi = \infty$. As $\xi \to 0$, $U \to 0$, which is characteristic of flexural waves, and as $\xi \to \infty$, $c \to U \to c_R$ in the plate. The maximum value of group velocity is equal to horizontal velocity of SV waves in the plate. The results obtained for flexural mode (first mode) are in agreement with the corresponding results obtained by EWING *et. al.* [22] (in Figs. 6–18). Dispersion curves for phase and

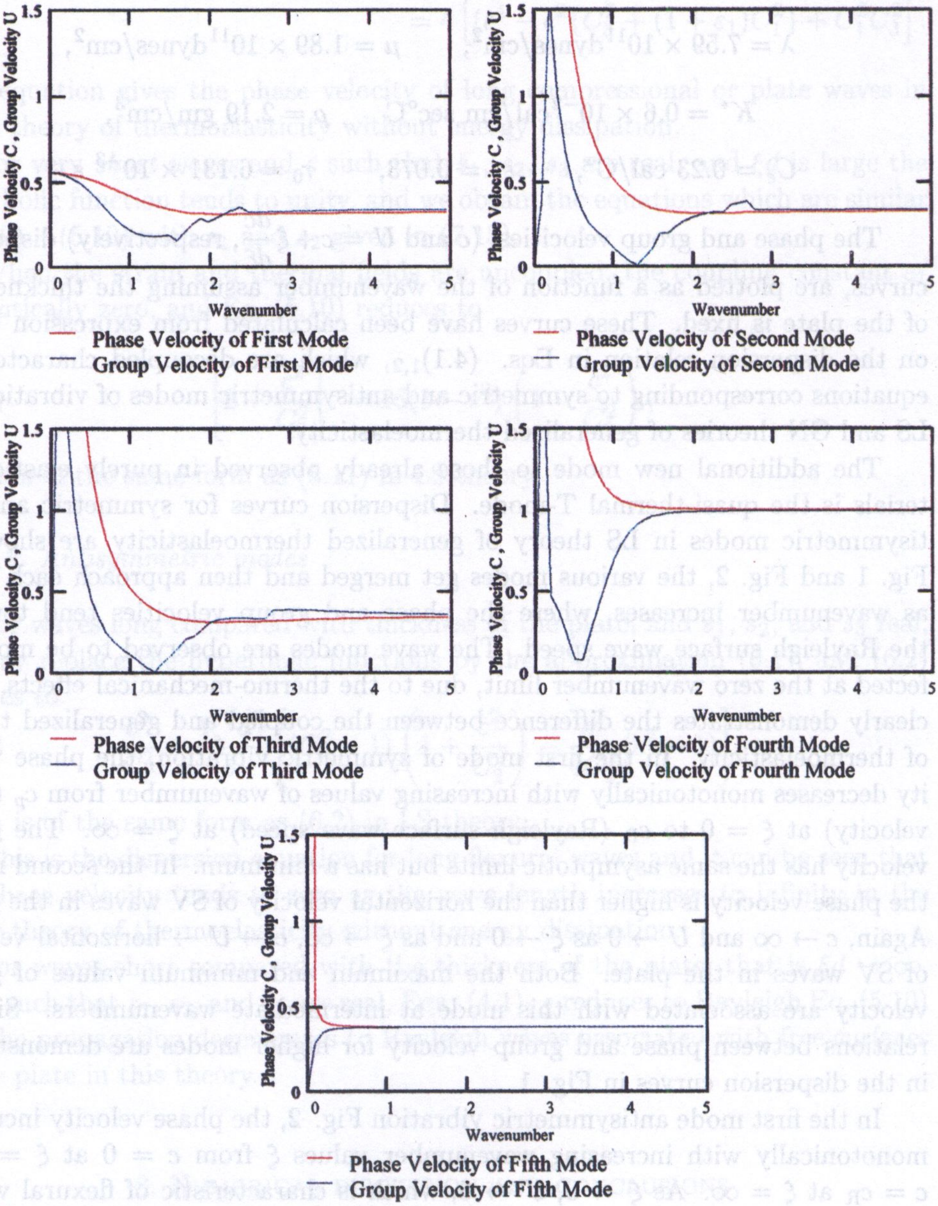

FIG. 1. Dispersion curves for symmetric modes.

group velocity for higher modes in LS theory are shown in Fig. 2. The turning of the phase and group velocity curves for fourth mode (antisymmetric), Fig. 2 and fifth mode (symmetric, Fig. 1) approach the c-axis at low wavenumber, at such a large values that these are multiplied by 10^{-4} to see them on the figures.

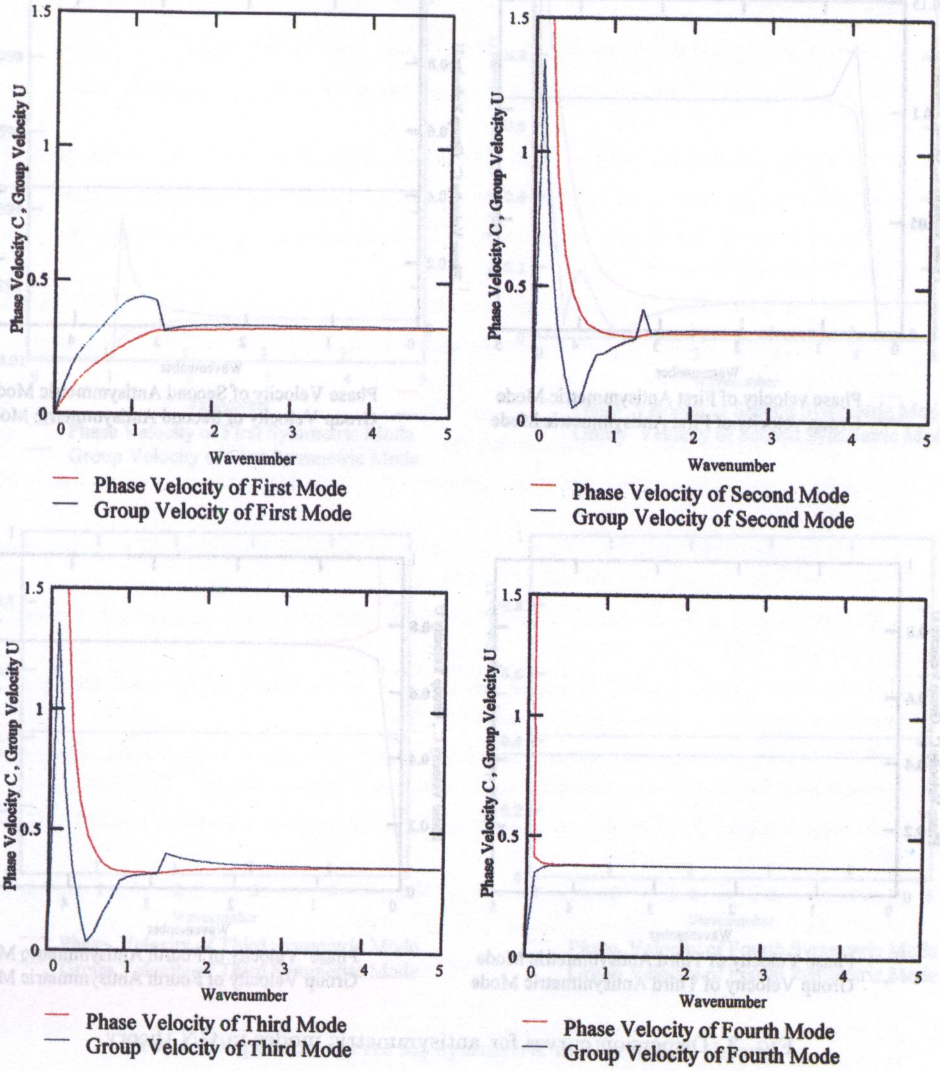

FIG. 2. Dispersion curves for antisymmetric modes.

Similar dispersion curves for antisymmetric and symmetric modes in GN theory of generalized thermoelasticity, for aluminum epoxy composite plate are shown in Figs. 3, 4. It has been found that phase velocity is equal to group velocity i.e., $c = U$ for second and third modes (antisymmetric), third and fourth modes (symmetric), and therefore these modes show no dispersion in the GN theory.

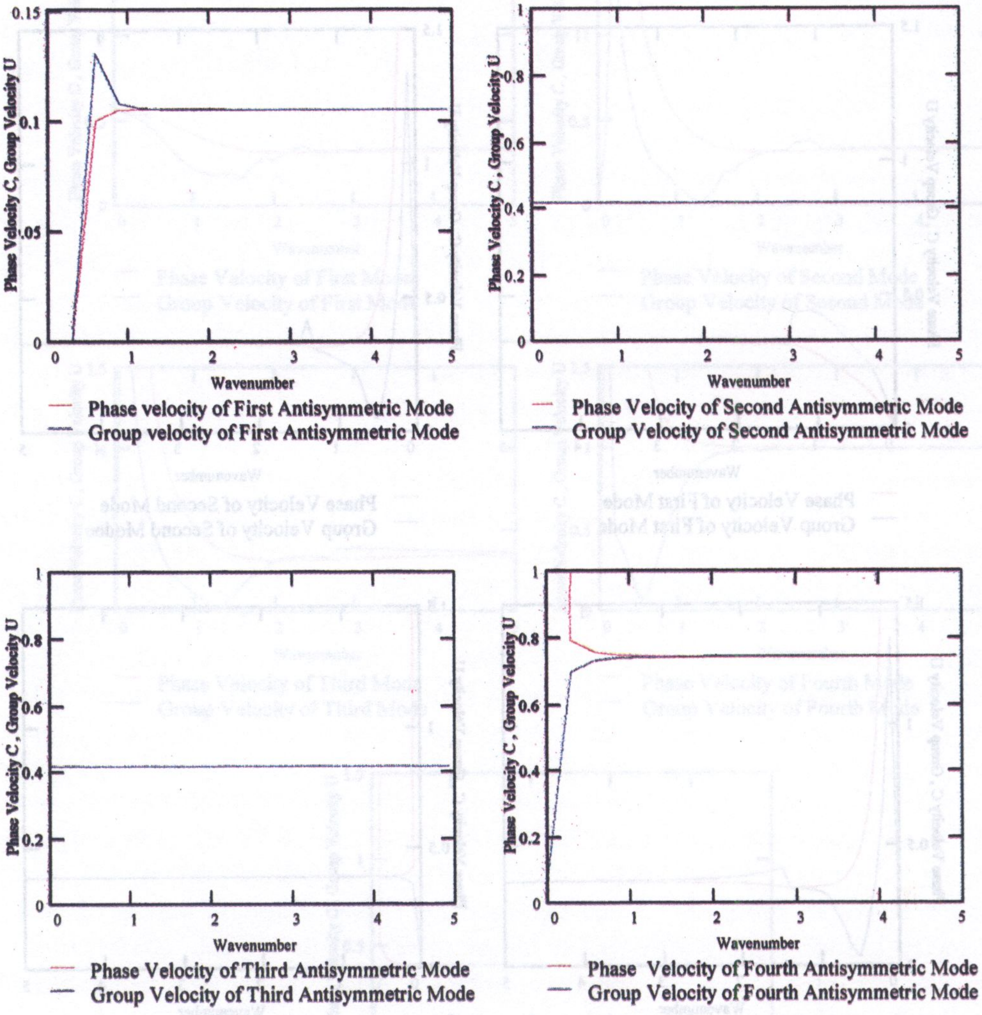

FIG. 3. Dispersion curves for antisymmetric modes in GN theory.

Further, once the solutions obtained, the GN theory shows that there exist symmetric and antisymmetric modes of coupled (thermal and elastic waves modes) waves, without any attenuation. The fact that, this is not the case in the LS theory, is an interesting feature inherent in GN theory, in LS theory the waves experience attenuation, and the attenuation factors decay exponentially [24, 25].

It has also been observed that predictions of the GN theory are qualitatively similar to those of the LS theory.

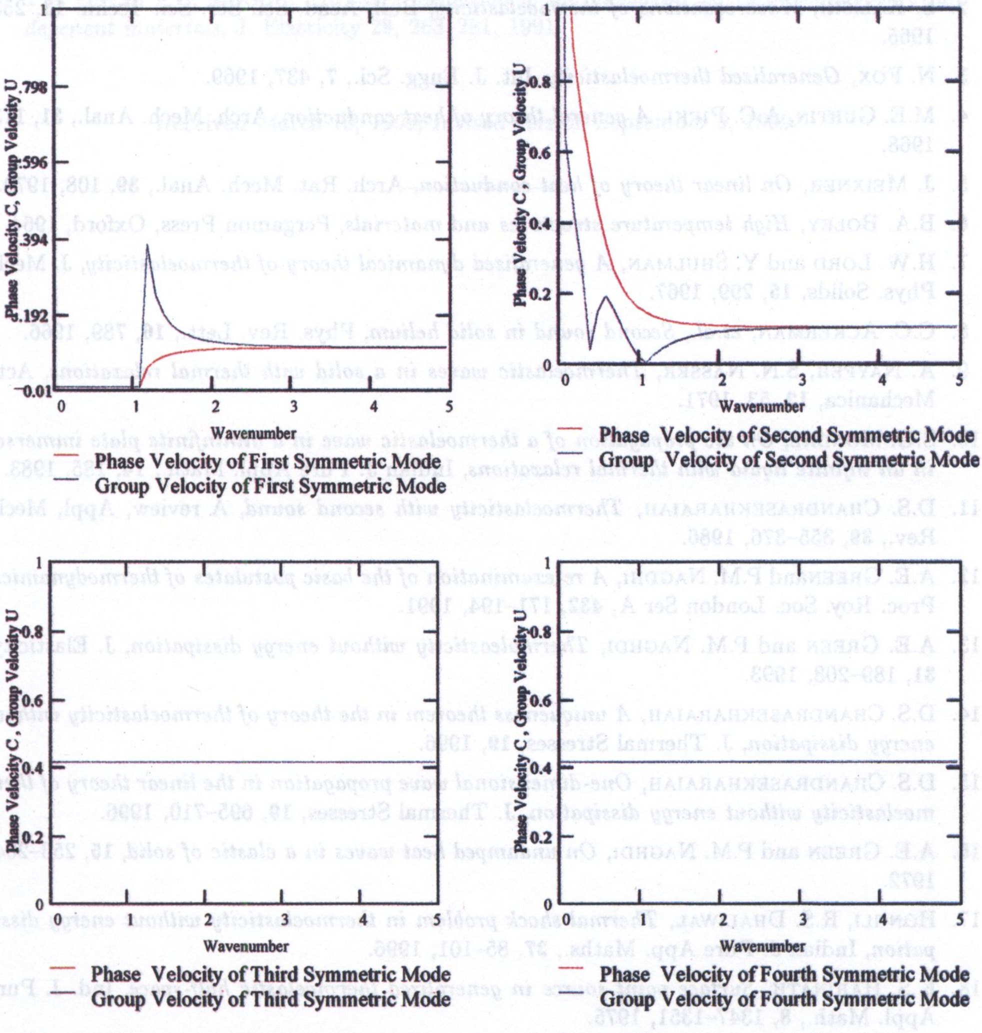

FIG. 4. Dispersion curves for symmetric modes in GN theory.

When the thermal relaxation time $\tau_0 \to 0$, then the results obtained in the analysis reduces to conventional coupled theory of thermoelasticity. When the coupling constant ε_1 is identically zero, the strain and thermal fields are uncoupled to each other. In this case the results can be obtained as in the uncoupled theory of thermoelasticity.

References

1. J. Ignaczak, *Generalized thermoelasticity and its applications*, [in:] R.B. Hetnarski [Ed.], Thermal Stresses III, Elsevier Science Publishers B.V., 1989.
2. S. Kaliski, *Wave equations of thermoelasticity*, Bull. Acad. Sci. Ser. Sci. Techn. **13**, 253, 1965.
3. N. Fox, *Generalized thermoelasticity*, Int. J. Engg. Sci., **7**, 437, 1969.
4. M.E. Gurtin, A.C. Pipki, *A general theory of heat conduction*, Arch. Mech. Anal., **31**, 113, 1968.
5. J. Meixner, *On linear theory of heat conduction*, Arch. Rat. Mech. Anal., **39**, 108, 1970.
6. B.A. Boley, *High temperature structures and materials*, Pergamon Press, Oxford, 1964.
7. H.W. Lord and Y. Shulman, *A generalized dynamical theory of thermoelasticity*, J. Mech. Phys. Solids, **15**, 299, 1967.
8. C.C. Ackerman, *et.al.*, *Second sound in solid helium*, Phys. Rev. Lett., **16**, 789, 1966.
9. A. Nayfeh, S.N. Nasser, *Thermoelastic waves in a solid with thermal relaxations*, Acta Mechanica, **12**, 53, 1971.
10. S.G. Mondal, *On the propagation of a thermoelastic wave in a thininfinite plate immersed in an infinite liquid with thermal relaxations*, Indian J. Pure Appl. Math., **14**, 185, 1983.
11. D.S. Chandrasekharaiah, *Thermoelasticity with second sound*, A review, Appl, Mech. Rev., **39**, 355–376, 1986.
12. A.E. Greenand P.M. Nagdhi, *A re-examination of the basic postulates of thermodynamics*, Proc. Roy. Soc. London Ser A, **432**, 171–194, 1991.
13. A.E. Green and P.M. Naghdi, *Thermoleasticity without energy dissipation*, J. Elasticity, **31**, 189–208, 1993.
14. D.S. Chandrasekharaiah, *A uniqueness theorem in the theory of thermoelasticity without energy dissipation*, J. Thermal Stresses, **19**, 1996.
15. D.S. Chandrasekharaiah, *One-dimensional wave propagation in the linear theory of thermoelasticity without energy dissipation*, J. Thermal Stresses, **19**, 695–710, 1996.
16. A.E. Green and P.M. Naghdi, *On undamped heat waves in a elastic of solid*, **15**, 253–264, 1972.
17. Hongli, R.S. Dhaliwal, *Thermal shock problem in thermoelasticity without energy dissipation*, Indian J. Pure App. Maths., **27**, 85–101, 1996.
18. K.S. Harinath, *Surface point source in generalized thermoelastic half-space*, Ind. J. Pure Appl. Math., **8**, 1347–1351, 1975.
19. A. Nayfehand S.N. Nasser, *Transient thermoelastic waves in a half-space with thermal relaxations*, J. Appl. Maths., Physics **23**, 50–67, 1972.
20. P. Chadwick, *Progress in solid mechanics*, [Eds. by R.Hill and I.N. Sneddon], **1**, North Holland Publishing Co., Amsterdam 1960.
21. F.J. Lockett, *Effect of thermal properties of a solid on the velocity of Rayleigh waves*, J. Mech. Phys. Solids **7**, 71, 1985.
22. W.M. Ewing, W.S. Jardetsky, F. Press, *Elastic waves in layered media*, McGraw-Hill, New York 1957.

23. K.L. VERMA, *Thermoelastic wave propagation problems in linear theory of thermoelasticity without energy dissipation*, (comm.)

24. T.S. ONCU and T.B. MOODIE, *Pade-extended Ray series expansions in generalized thermoelasticity*, J. Thermal Stresses, **14**, 85–99, 1991.

25. T.S. ONCU, T.B. MOODIE, *On the propagation of thermoelastic waves in temperature ratedepenent materials*, J. Elasticity **29**, 263–281, 1991.

Received March 10, 1999; revised version September 9, 1999.