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STABILITY OF SLIGHTLY WRINKLED PLATES INTERACTING WITH AN ELASTIC SUBSOIL

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The first aim of the contribution is to formulate an engineering theory describing the linear stability of periodically waved shallow shell-like structures (slightly wrinkled plates, cf. Fig. 1) interacting with a Winkler foundation. On the basis of the proposed theory, the effect of elastic foundation on the value of a critical force is investigated. The second aim is to compare the proposed model of wrinkled plates resting on elastic medium with the known theories of orthotropic plates. It is shown that the obtained solutions depend on the shell wavelength parameter l. The general results are illustrated by an example.

Key words: shell, modelling, periodic structure, stability, elastic foundation.

1. Introduction

The subject of the paper is a thin periodic shallow shell-like structure, shown in Fig. 1 and referred to as a slightly wrinkled plate. The plate is made of a linear-elastic homogeneous material. The structure interacts with an elastic medium modelled by the Winkler foundation. It is assumed that the wrinkled plate wavelengths l_1, l_2 are small enough compared to the minimum characteristic length dimension L of the projection of the structure on a reference plane Ox_1x_2. At the same time, the thickness δ of the shell under consideration is supposed to be constant and small compared both to the structure length parameter l (where $l := \sqrt{(l_1)^2 + (l_2)^2}$) and to the midsurface minimum curvature radius R. Therefore $\delta \ll l \ll L$, and l will be called the mezostructure length parameter.

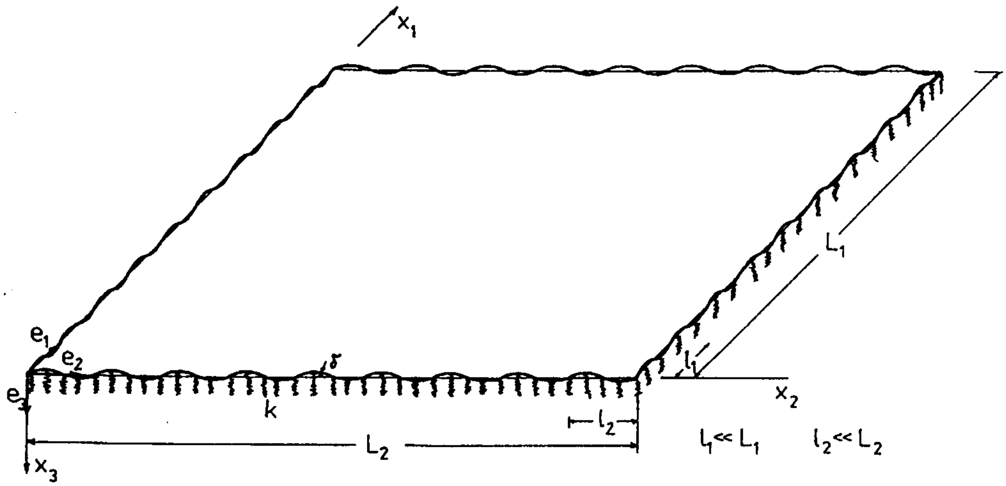

FIG. 1. A scheme of the wrinkled plate on an elastic foundation.

From a formal point of view the structure under consideration can be described in the framework of the well-known theory of thin elastic shallow shells. However, due to the mezo-periodic shape of the wrinkled plate midsurface, this direct description leads to the shell equations with periodic highly-oscillating coefficients, which are too complicated to be used in the analysis of stability problems and numerical calculations. That is why various approximate models of this problem were used. For example, the wrinkled plate can be modelled as an orthotropic plate (TROITSKY [4]; SEYDEL [5], where the plate mean flexural stiffness has to be determined. Moreover, the orthotropic plate investigated in [4] was periodically waved in one direction only. The structure investigated in this paper can be periodically folded in both directions. The first aim of this research is to formulate a simplified mathematical model of the wrinkled plates resting on the elastic medium, which can be applied as a useful tool for investigations of stability problems. The second aim is to show the influence of foundation on the values of critical forces for a wrinkled plate interacting with this foundation.

The paper is a continuation of previous investigations given in (MICHALAK, WOŹNIAK and WOŹNIAK [1]) and its main thesis is that the effect of the structure length dimensions l_1, l_2 play a crucial role in the analysis of stability of wrinkled plates in an elastic medium. Throughout the paper indices i, $j, k, ...$ run over $1, 2, 3$, being related to the orthogonal Cartesian coordinates x_1, x_2, x_3, with the vector basis $\mathbf{e}_i$ shown in Fig. 1. Indices $\alpha, \beta, \gamma, ...$ run over 1, 2 and are related to the shell midsurface parameters θ^1, θ^2. We also introduce non-tensorial indices $a, b, c, ...$ which run over the sequence $1, ..., n$. The summation convention holds for all aforementioned sub- and super-scripts. The time coordinate is denoted by t and the overdot stands for a time derivative.

2. Direct description of a wrinkled plate interacting with an elastic medium

Let the midsurface of the undeformed wrinkled plate be given by the parametric representation $x^i = R^i(\theta^1, \theta^2)$, where θ^1, θ^2 are surface parameters. In the sequel, the above parametrisation is given by $x^1 = \theta^1$, $x^2 = \theta^2$ and $x^3 = z(\theta^1, \theta^2) \in \Pi$, where Π is a regular region on the plane Ox_1x_2. Moreover, the plate is periodically folded in x^1- and x^2-directions, conditions $x^3 = z(\theta^1, \theta^2) = z(\theta^1 + l_1, \theta^2) = z(\theta^1, \theta^2 + l_2)$ hold in the whole domain of the definition of $z(\cdot)$. The above shell-like structure is said to be shallow when values of function $z(\cdot)/l$ and all its derivatives up to the second order are small as compared to unity. Under notation $\Delta \equiv (0, l_1) \times (0, l_2)$, function $z(\cdot)$ is referred to as the Δ-periodic functions, Δ being called the representative plane element of a wrinkled plate. Using the known notations: $G^i_\alpha \equiv R^i_{,\alpha}$, $\mathbf{g}_\alpha \equiv G^i_\alpha \mathbf{e}_i$, $\mathbf{n} = \mathbf{g}_1 \times \mathbf{g}_2 / |\mathbf{g}_1 \times \mathbf{g}_2|$, we obtain the first and second metric tensor of the undeformed midsurface $g_{\alpha\beta} = G^i_\alpha G_{i\beta}$, $b_{\alpha\beta} = n_i G^i_{\alpha|\beta}$, respectively. The Ricci tensor $\varepsilon_{\alpha\beta}$ together with $g_{\alpha\beta}$, $b_{\alpha\beta}$, $g = \det g_{\alpha\beta}$ are Δ-periodic functions of x_1, x_2. For shallow shells we assume, (Green, Zerna [2]); $g_{\alpha\beta} \cong \delta_{\alpha\beta}$, $g \cong e = \det(e_{ab}) = 1$, $b_{\alpha\beta} \cong z_{,\alpha\beta}$ and the Christoffel symbols related to $g_{\alpha\beta}$ are negligibly small: $\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} \cong 0$.

The displacement vector field of the wrinkled plate midsurface is denoted by $\mathbf{u} = u^i(\mathbf{x}, t)\mathbf{e}_i$, external (surface) loadings by $\hat{\mathbf{p}} = \hat{p}^i(\mathbf{x}, t)\mathbf{e}_i$, $\mathbf{x} \in \Pi$, external boundary forces by $\mathbf{p} = p^i(\mathbf{x}, t)\mathbf{e}_i$, $\mathbf{x} \in \partial\Pi$ and ρ is the mass density averaged over the shell thickness and related to the midsurface unit area.

In the framework of the technical non-linear theory for thin elastic shallow-shells (Woźniak [3]), we obtain:

(i) Strain-displacement relations in which non-linear terms involving gradients of u_α are neglected:

$$\varepsilon_{\alpha\beta} = G^i_{(\alpha} u_{i,\beta)} + \frac{1}{2} u^3_{,\alpha} u_{3,\beta}, \qquad \kappa_{\alpha\beta} = n^i u_{i,\alpha\beta}, \tag{2.1}$$

(ii) Constitutive equations:

$$n^{\alpha\beta} = D H^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}, \qquad m^{\alpha\beta} = B H^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}, \tag{2.2}$$

where

$$H^{\alpha\beta\gamma\mu} = \frac{1}{2} \left\{ g^{\alpha\mu} g^{\beta\gamma} + g^{\alpha\gamma} g^{\beta\mu} + \nu (\in^{\alpha\gamma} \in^{\beta\mu} + \in^{\alpha\mu} \in^{\beta\gamma}) \right\},$$

$$D \equiv E\delta / 12(1 - \nu^2), \qquad B \equiv E\delta^3 / 12(1 - \nu^2)$$

E, ν, being the Young modulus and Poissons ratio, respectively.

(iii) Equations of motion:

$$\int_{\Pi} (n^{\alpha\beta}\delta\varepsilon_{\alpha\beta} + m^{\alpha\beta}\delta\kappa_{\alpha\beta})d\Pi + \frac{d}{dt}\int_{\Pi} \rho\dot{u}^i d\Pi = \int_{\Pi} (\hat{p}^i\delta u_i - ku^3\delta u^3)d\Pi + \int_{\partial\Pi} p^i\delta u_i d\partial\Pi, \tag{2.3}$$

$$\delta\varepsilon_{\alpha\beta} = G^i{}_{(\alpha}\delta u_{i,\beta)} + u^3_{,\alpha}\delta u_{3,\beta}, \qquad \delta\kappa_{\alpha\beta} = n^i\delta u_{i,\alpha\beta},$$

which have to hold for any virtual displacement field δu_i. Since $e_{\alpha\beta}$, $\kappa_{\alpha\beta}$, $n^{\alpha\beta}$, $m^{\alpha\beta}$ are highly-oscillating Δ-periodic functions, Eqs. (2.1) – (2.3), are too complicated to be used in engineering calculations. The first aim of this research is to formulate from Eqs. (2.1) – (2.3), a simplified model of a stability of the wrinkled plates in elastic medium, using the modelling procedures initiated in [1].

3. Modelling approach

The modelling approach to the waved plates was presented in (Michalak, Woźniak and Woźniak [1]). In this research, the stability of a wrinkled plate interacting with an elastic foundation will be investigated. Bearing in mind that the wrinkled plate has Δ-periodic structure, we introduce the notation $\Delta(\mathbf{x}) = \Delta + \mathbf{x}$, for every $\mathbf{x} = (x^1, x^2)$ such that $\Delta(\mathbf{x}) \subset \Pi$; $\Delta(\mathbf{x})$ will be called a periodic cell, assigned at a point $\mathbf{x}$. The main concepts of the proposed approach are:

1° *Averaging operator* $< \cdot > (\mathbf{x})$. Let $f(\mathbf{z})$ be an integrable function defined on Π. We shall use the notation

$$< f > (x) = \frac{1}{l_1 l_2}\int_{\Delta(x)} f(z)dz_1dz_2, \tag{3.1}$$

if $f(\cdot)$ is a Δ – periodic function then $< f >$ is constant.

2° *Long wave approximation.* A differentiable function $F(\mathbf{x}, t)$, $\mathbf{x} \in \Pi$, will be called the regular macro-function if for an arbitrary $z \in \Pi$ and every $\mathbf{x} \in \Delta(z)$, the following formula holds:

$$(\forall f)\left[< f\hat{F} > (x) \cong < f > \hat{F}(x)\right] \qquad \text{where} \quad \hat{F} \in \{F, \nabla F, \dot{F}, ...\}. \tag{3.2}$$

3° Let $h^a(\cdot)$, $a = 1, ..., n$, be a system of n Δ-periodic eigenfunctions which are obtained as approximate solutions to the eigenvalue problem on a periodicity cell Δ with periodic boundary conditions. The choice of these functions depends on the form of free vibrations of the cell Δ. These functions are linear independent continuous functions defined on R^2, having continuous first and second order derivatives. Moreover, values $h^a(\mathbf{x})$ have to satisfy conditions $h^a(\mathbf{x}) \in O(l^2)$, $h^a_{,\alpha}(\mathbf{x}) \in O(l)$, $h^a_{,\alpha\beta}(x) \in O(l)$, $<\rho h^a>= 0$. Functions h^a will be referred to as *the micro-shape functions.* Generally speaking, every linear combination of micro-shape functions describe disturbances of the plate displacements $u_i(\mathbf{x}, t)$, caused by the periodic plate mezostructure. Hence, the choice of these functions depends on the problem under consideration (the form of the periodicity cell Δ, the class of micromotions, which we want to investigate).

The main assumption of the proposed modelling approach is the *Basic Kinematic Hypothesis* which restricts consideration to a class of motions given by

$$u_i(\mathbf{x}, t) \cong U_i(\mathbf{x}, t) + h^a(\mathbf{x}) \quad V_i^a(\mathbf{x}, t), \qquad \mathbf{x} = (x^1, x^2) \in \Pi, \quad t \geq 0, \tag{3.3}$$

where $U_i(\cdot, t), V_i^a(\cdot, t)$ are arbitrary regular macro-functions called macrodisplacements and oscillation correctors, respectively; and $h^a(\cdot)$ are the micro-shape functions, the choice of which is postulated in every problem under consideration. The macrodisplacements $U_i(\mathbf{x}, t) =< u_i > (\mathbf{x}, t)$ describe the averaged motion of the wrinkled plates and functions $h^a(\cdot)V_i^a(\cdot, t)$ describe the local displacement oscillations due to a periodic structure of these plates.

We also assume that gradients a local displacement oscillations are small compared to gradients of macrodisplacements U^3, i.e.

$$h^a_{,\alpha}(\mathbf{x})V_i^a(\mathbf{x}, t) \ll U^3_{,\alpha}(\mathbf{x}). \tag{3.4}$$

Hence Eqs. (2.1) can be approximated by

$$\varepsilon_{\alpha\beta} = G^i{}_{(\alpha}[U_{i,\beta)} + h^a_{,\beta)}V_i^a] + \frac{1}{2}U^3_{,\alpha}U_{3,\beta}, \qquad \kappa_{\alpha\beta} = n^i[U_{i,\alpha\beta} + h^a_{,\alpha\beta}V_i^a]. \tag{3.5}$$

It was shown in [1] that the aforementioned modelling hypotheses lead from Eqs. (2.1) – (2.3), (3.1) – (3.5) to a certain averaged model of a wrinkled plate resting on the elastic medium. Let us define $\Pi^0 := \{\mathbf{x} = (x_1, x_2) \in \Pi : \mathbf{x} + \Delta \subset \Pi\}$. Following [1], for every $\mathbf{x} \in \Pi^0$ and every instant t, let us introduce the system of averaged stresses given by

$$\begin{aligned} \mathbf{N}^\alpha(\mathbf{x}, t) &=< n^{\alpha\beta}\mathbf{g}_\beta > (\mathbf{x}, t), \qquad \mathbf{M}^{\alpha\beta}(\mathbf{x}, t) =< m^{\alpha\beta}\mathbf{n} > (\mathbf{x}, t), \\ \mathbf{N}^a(\mathbf{x}, t) &=< n^{\alpha\beta}\mathbf{g}_\alpha h^a_{,\beta} > (\mathbf{x}, t), \qquad \mathbf{M}^a(\mathbf{x}, t) =< m^{\alpha\beta}h^a_{|\alpha\beta}\mathbf{n} > (\mathbf{x}, t), \\ & \qquad \mathbf{x} \in \Pi^0. \end{aligned} \tag{3.6}$$

Substituting the right-hand sides of Eqs. (3.3), (3.5) into Eqs. (2.3), bearing in mind formulae (3.6), and applying the aforementioned approximation formulae, after assuming that $\hat{p}^i$ and p^i are macrofunctions, we obtain the condition

$$\int_{\Pi} \Big[\mathbf{M}^{\alpha\beta} \cdot \delta\mathbf{U}_{,\alpha\beta} + \mathbf{N}^{\alpha} \cdot \delta\mathbf{U}_{,\alpha} + \left\langle n^{\alpha\beta}\right\rangle U^3_{,\alpha} \cdot \delta U_{3,\beta} + < k > U^3 \delta U_3 \tag{3.7}$$

$$+(\mathbf{N}^a + \mathbf{M}^a) \cdot \delta\mathbf{V}^a + \left\langle kh^a h^b\right\rangle V_3^b \delta V_3^b\Big]\, d\Pi + \frac{d}{dt}\int_{\Pi} \Big(\left\langle \rho \right\rangle \dot{\mathbf{U}} \cdot \delta\mathbf{U}$$

$$+\left\langle \rho h^a h^b\right\rangle \dot{\mathbf{V}}^b \cdot \delta\mathbf{V}^a\Big)\, d\Pi - \int_{\Pi} \hat{\mathbf{p}} \cdot \delta\mathbf{U} d\Pi + \int_{\partial\Pi} p^i \delta U_i d\partial\Pi,$$

which has to hold for every macro-function $\delta\mathbf{U}$, $\delta\mathbf{V}^a$, and where coefficients $< k >$, $\left\langle kh^a h^b\right\rangle$ describe an influence of elastic foundation on the averaged (global) behaviour of the structure.

4. The averaged model for stability analysis of a wrinkled plate

After applying the divergence theorem and du Bois-Reymond lemma to Eqs. (3.7) and using (3.6), we arrive at the system of equations in macrodisplacements U_i and correctors V_i^a constituting the governing equations of the averaged theory of wrinkled plates on an elastic foundation. The equations of motion written down in the coordinate form are

$$\begin{aligned}
&M^{\gamma\alpha\beta}_{,\alpha\beta} - N^{\gamma\alpha}_{,\alpha} + < \tilde{\rho} > \ddot{U}^{\gamma} = p^{\gamma},\\
&M^{3\alpha\beta}_{,\alpha\beta} - N^{3\alpha}_{,\alpha} - (< n^{\alpha\beta} > U^3_{,\alpha})_{,\beta} + KU^3 + < \rho > \ddot{U}^3 = p^3,\\
&N^{\gamma|a} + M^{\gamma|a} + < \rho h^a h^b > \ddot{V}^{b\gamma} = 0,\\
&N^{3|a} + M^{3|a} + K^{ab} V^{b3} + < \rho h^a h^b > \ddot{V}^{b3} = 0,
\end{aligned} \tag{4.1}$$

Constitutive equations have the form

$$\begin{aligned}
\mathbf{N}^{\alpha} &= \mathbf{D}^{\alpha\beta}\mathbf{U}_{,\beta} + \mathbf{D}^{a\alpha}\mathbf{V}^a,\\
\mathbf{N}^{a} &= \mathbf{D}^{ab}\mathbf{V}^b + \mathbf{D}^{a\beta}\mathbf{U}_{,\beta},\\
\mathbf{M}^{\alpha\beta} &= \mathbf{B}^{\alpha\beta\gamma\delta}\mathbf{U}_{,\gamma\delta} + \mathbf{B}^{a\alpha\beta}\mathbf{V}^a,\\
\mathbf{M}^{a} &= \mathbf{B}^{ab}\mathbf{V}^b + \mathbf{B}^{a\alpha\beta}\mathbf{U}_{,\alpha\beta},
\end{aligned} \tag{4.2}$$

where we have denoted

$$
\begin{aligned}
\mathbf{D}^{\alpha\beta} &\equiv D\left\langle H^{\delta\alpha\gamma\beta}\mathbf{g}_\delta \otimes \mathbf{g}_\gamma \right\rangle,\\
\mathbf{D}^{a\alpha} &\equiv D\left\langle H^{\beta\alpha\gamma\delta}\mathbf{g}_\beta \otimes \mathbf{g}_\gamma h^a_{,\delta} \right\rangle,\\
\mathbf{D}^{ab} &\equiv D\left\langle H^{\alpha\beta\gamma\delta}\mathbf{g}_\alpha \otimes \mathbf{g}_\gamma h^a_{,\beta} h^b_{,\delta} \right\rangle,\\
\mathbf{B}^{\alpha\beta\gamma\delta} &\equiv B\left\langle H^{\alpha\beta\gamma\delta}\mathbf{n} \otimes \mathbf{n} \right\rangle,\\
\mathbf{B}^{a\alpha\beta} &\equiv B\left\langle H^{\alpha\beta\gamma\delta} h^a_{|\gamma\delta}\mathbf{n} \otimes \mathbf{n} \right\rangle,\\
\mathbf{B}^{ab} &\equiv B\left\langle H^{\alpha\beta\gamma\delta} h^a_{|\alpha\beta} h^b_{|\gamma\delta}\mathbf{n} \otimes \mathbf{n} \right\rangle,\\
K &\equiv \langle k \rangle,\\
K^{ab} &\equiv \left\langle k h^a h^b \right\rangle.
\end{aligned}
\tag{4.3}
$$

We assume that on the edges of middle plane of the wrinkled plate, forces $P^{\alpha\beta}$ are acting. From Eqs. (3.7) we obtain the boundary condition

$$
N^{\gamma\beta} - M^{\gamma\alpha\beta}_{,\alpha} = P^{\gamma\beta}, \qquad x \in \partial\Pi, \tag{4.4}
$$

where $N^{\gamma\beta}$ and $M^{\gamma\alpha\beta}$ are defined by Eqs. (4.2).

In order to calculate *critical values* of forces $P^{\alpha\beta}$ which are applied to the middle plane of a wrinkled plate we assume that macro-displacements of the middle plane of the wrinkled plate satisfy $U^\gamma = 0$, and that there are no external loads and body forces. Then from Eqs. (4.1), (4.4) we obtain the equilibrium equations

$$
\begin{aligned}
M^{3\alpha\beta}_{,\alpha\beta} - N^{3\alpha}_{,\alpha} + KU^3 - P^{\alpha\beta}U^3_{,\alpha\beta} &= 0,\\
N^{\gamma|a} + M^{\gamma|a} &= 0,\\
N^{3|a} + M^{3|a} + K^{ab}V^{b3} &= 0.
\end{aligned}
\tag{4.5}
$$

Solving Eq. (4.5) for the given boundary conditions for U_3, we will find that the assumed buckling of the wrinkled plate in an elastic medium is possible only for certain values of $P^{\alpha\beta}$. The smallest of these values determine the desired critical values of $P^{\alpha\beta}$.

Let us observe that since $h^a \in O(l^2)$, the coefficients $\mathbf{D}^{a\alpha}$, $\mathbf{D}^{ab}$, K^{ab} depend on the mesostructure length parameter l, and hence, the aforementioned equations describe the mesostructure length-scale effect on the critical values of forces $P^{\alpha\beta}$. Moreover, coefficients K and K^{ab} describe the influence of elastic foundation on the values of these forces.

5. Applications

We shall investigate the stability of a rectangular slightly wrinkled plate resting on an elastic foundation of the Winkler type. We assume that the wavelengths of a midsurface are mezo-periodic and equal $l_1 = l_2 = l$. Using Eqs. (4.2), (4.3), (4.5) we obtain the system of equations for $U_3 = U_3(\mathbf{x}, t)$ and $V_i^a = V_i^a(\mathbf{x}, t)$. For the sake of simplicity we restrict the function describing disturbances in the plate displacements to the first term in series $h^a(\cdot)V_i^a(\cdot, t)$. Introducing only one meso-shape function $h^a(\cdot) = h^1(\cdot) = h(\cdot)$, after notation $V_i^a(\mathbf{x}, t) = V_i(\mathbf{x}, t)$, the system equations describing stability of wrinkled plate on elastic medium will take the form

$$\begin{aligned}
(5.1)\qquad & B\Big\langle H^{1111}N^3N^3\Big\rangle U_{3,1111} + 4B\Big\langle H^{1212}N^3N^3\Big\rangle U_{3,1212} \\
& + 2B\Big\langle H^{1122}N^3N^3\Big\rangle U_{3,2211} + B\Big\langle H^{2222}N^3N^3\Big\rangle U_{3,2222} \\
& -D^{3311}U_{3,11} - D^{3322}U_{3,22} - D^{311}V_{1,1} - D^{322}V_{2,2} + KU_3 \\
& -P^{11}U_{3,11} - 2P^{12}U_{3,12} - P^{22}U_{3,22} = 0, \\
& D^{311}U_{3,1} + D^{11}V_1 = 0, \\
& D^{322}U_{3,2} + D^{22}V_2 = 0,
\end{aligned}$$

where we have denoted

$$\begin{aligned}
(5.2)\qquad D^{11} \equiv\ & D\Big\langle H^{1111}(G_1^1 h_{,1})^2\Big\rangle + D\Big\langle H^{1212}(G_1^1 h_{,2})^2\Big\rangle \\
& +B\Big\langle H^{1111}(n^1 h_{,11})^2\Big\rangle + 2B\Big\langle H^{1122}(n^1)^2 h_{,11}h_{,22}\Big\rangle \\
& +2B\Big\langle H^{1212}(n^1 h_{,12})^2\Big\rangle + 2B\Big\langle H^{1221}(n^1 h_{,21})^2\Big\rangle B\Big\langle H^{2222}(n^1 h_{,22})^2\Big\rangle
\end{aligned}$$

(5.2) [cont.]

$$D^{22} \equiv D\left\langle H^{2121}(G_2^2 h_{,1})^2\right\rangle + D\left\langle H^{2222}(G_2^2 h_{,2})^2\right\rangle$$

$$+B\left\langle H^{1111}(n^2 h_{,11})^2\right\rangle + 2B\left\langle H^{1122}(n^2)^2 h_{,11}h_{,22}\right\rangle$$

$$+2B\left\langle H^{1212}(n^2 h_{,12})^2\right\rangle + 2B\left\langle H^{1221}(n^2 h_{,21})^2\right\rangle + B\left\langle H^{2222}(n^2 h_{,22})^2\right\rangle$$

$$D^{3311} \equiv D\left\langle H^{1111}(G_1^3)^2\right\rangle + D\left\langle H^{2121}(G_2^3)^2\right\rangle$$

$$D^{3322} \equiv D\left\langle H^{1212}(G_1^3)^2\right\rangle + D\left\langle H^{2222}(G_2^3)^2\right\rangle,$$

$$D^{311} \equiv D\left\langle H^{1111}G_1^1 G_1^3 h_{,1}\right\rangle + D\left\langle H^{2112}G_1^1 G_2^3 h_{,2}\right\rangle,$$

$$D^{322} \equiv D\left\langle H^{1221}G_2^2 G_1^3 h_{,1}\right\rangle + D\left\langle H^{2222}G_2^2 G_2^3 h_{,2}\right\rangle,$$

$$K = \langle k \rangle .$$

Eliminating correctors V_i by means of Eqs. $(5.1)_2$ and $(5.1)_3$ we obtain the following equation describing the stability condition of a wrinkled plate resting on Winkler elastic foundation:

$$\begin{aligned}(5.3)\qquad & B\left\langle H^{1111}N^3N^3\right\rangle U_{3,1111} + 4B\left\langle H^{1212}N^3N^3\right\rangle U_{3,1212} \\ & +2B\left\langle H^{1122}N^3N^3\right\rangle U_{3,2211} + B\left\langle H^{2222}N^3N^3\right\rangle U_{3,2222} \\ & -\left[D^{3311} - \frac{(D^{311})^2}{D^{11}}\right] U_{3,11} - \left[D^{3322} - \frac{(D^{322})^2}{D^{22}}\right] U_{3,22} + KU_3 \\ & -P^{11}U_{3,11} - 2P^{12}U_{3,12} - P^{22}U_{3,22} = 0.\end{aligned}$$

In order to compare both the microstructural and orthotropic plate models in the analysis of stability of wrinkled plates on an elastic foundation, we shall investigate the simple problem of a stability of a long rectangular plate resting on the Winkler elastic foundation simply supported and uniformly compressed along the edges in direction of x^1 (Fig. 2). In this case we shall look for solution of Eqs. (5.3) in the form

$$U_3 = \sum_{m=1}^{\infty} w_m \sin \alpha_m x_1, \tag{5.4}$$

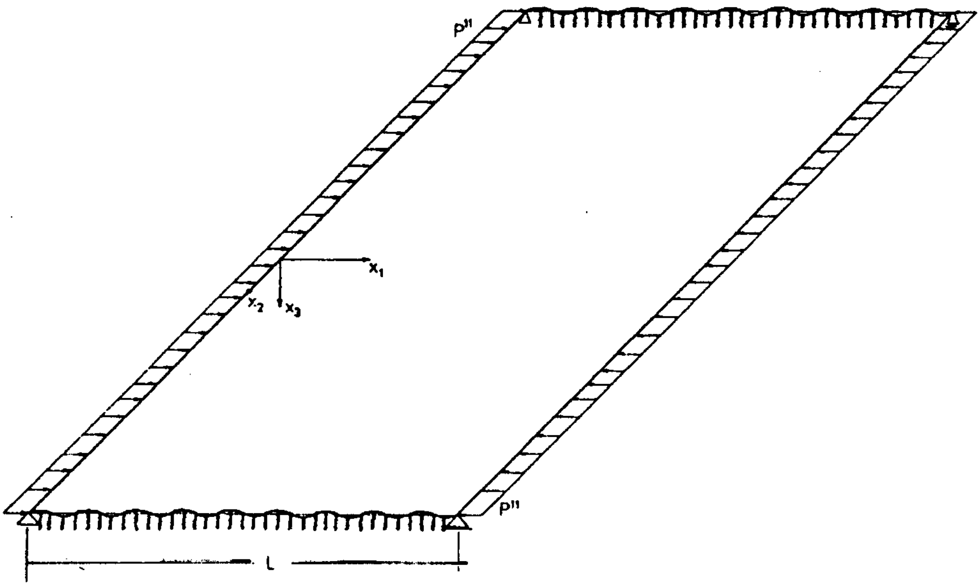

FIG. 2. A scheme of a long rectangular plate compressed along the edges in direction of x^1.

where $\alpha_m := m\pi/L$, L being the span of the plate ($L \gg l$). Substituting the right-hand sides of Eqs. (5.4) into (5.3) we obtain the following critical value of a compressive force:

$$(P^{11})_{\mathrm{cr}} = B(\alpha_m)^2 + \frac{K}{(\alpha_m)^2} + \left[D^{3311} - \frac{(D^{311})^2}{D^{11}}\right]. \tag{5.5}$$

Now consider a slightly wrinkled plate which will be treated as an orthotropic plate. In this case, the equilibrium equations have a form (TROITSKY [4])

$$B_{11}U_{3,1111} + kU_3 = P^{11}U_{3,11}, \tag{5.6}$$

where [4, 5]

$$B_{11} = \frac{1}{1 + (\pi f/l)^2} \frac{E\delta^3}{12(1-\nu^2)}. \tag{5.7}$$

Substituting the right-hand sides of Eqs. (5.4) into (5.6) we obtain the critical value of a compressive force

$$(P^{11})_{\mathrm{cr}} = B_{11}(\alpha_m)^2 + \frac{k}{(\alpha_m)^2}. \tag{5.8}$$

Let the shell midsurface be given by $z = \frac{l}{40}\sin\left(\frac{2\pi}{l}x^1\right)$. Due to the form of the periodicity cell and conditions from Sec. 3 which have to be satisfied by a micro-shape function, we assume the micro-shape function in the form $h = l^2 \sin\left(\frac{2\pi}{l}x^1\right)$. In this case, formulae (5.5) and (5.8) for the critical forces, yield

a) microstructural model

$$(P^{11})_{\text{cr}} = D\frac{\pi^2 m^2}{12}\left(\frac{\delta}{L}\right)^2 + \frac{KL^2}{m^2\pi^2} + D\left[2\pi^2(f/l)^2 - \frac{2\pi^2(f/l)^2}{1+\frac{4\pi^4}{12}(f/l)^2(\delta/L)^2(L/l)^2}\right], \tag{5.9}$$

b) orthotropic plate model

$$(P^{11})_{\text{cr}} = D\frac{\pi^2 m^2}{12}\left(\frac{\delta}{L}\right)^2 \frac{1}{1+\pi^2(f/l)^2} + \frac{kL^2}{m^2\pi^2}. \tag{5.10}$$

The smallest value of $(P^{11})_{\text{cr}}$ will be obtained, for the Winkler coefficient $k = 10^{-12}D/\delta^2$, by taking $m = 1$. In Fig. 3 for a wrinkled plate with thickness $\delta = L/1000$, the diagrams are shown of the smallest value of critical forces $(P^{11})_{\text{cr}}$ versus ratio l/L, where the continuous line describes the critical forces for a microstructural models, while the dotted line is related to critical forces for the orthotropic plate models. The value of critical forces for orthotropic plate model is independent of the ratio l/L being equal to $(P^{11})_{\text{cr}} = 9.187 \cdot 10^{-7}D$. For the microstructural model, the value of critical force is a function of the ratio l/L, and for $l/L = 0.10$ is equal to $(P^{11})_{\text{cr}} = 9.488 \cdot 10^{-7}D$.

Now let us consider the stability of simply supported rectangular wrinkled plate in an elastic medium, which is uniformly compressed in direction of x^1. In this case we look for solutions of Eqs. (5.3) in the form

$$U_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\alpha_m x^1 \sin\beta_n x^2, \tag{5.11}$$

where $\alpha_m := m\pi/L_1$ and $\beta_n := n\pi/L_2$, L_1, L_2 are the lengths of the wrinkled plate where $L_1 \gg l$ and $L_2 \gg l$. Substituting the right-hand sides of Eqs. (5.11) into (5.3) we obtain the critical value of compressive force

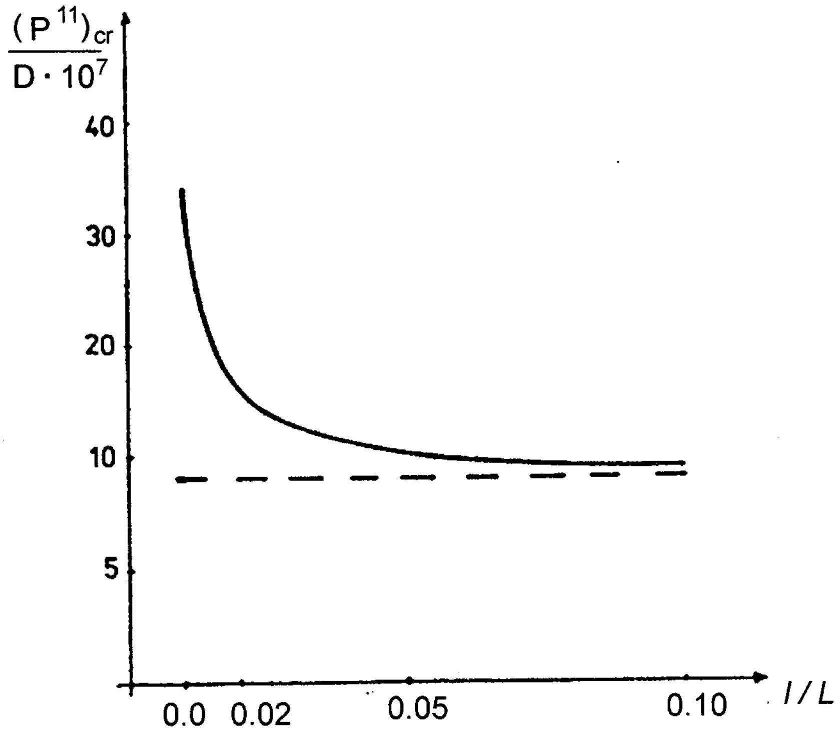

FIG. 3. The smallest value of a critical forces $(P^{11})_{\mathrm{cr}}$ for a long rectangular plate resting on the Winkler elastic foundation versus ratio l/L, for a microstructural and a orthotropic models.

$$(P^{11})_{\mathrm{cr}} = \frac{B\left((\alpha_m)^2 + (\beta_n)^2\right)^2}{(\alpha_m)^2} + \frac{K}{(\alpha_m)^2} + \left[D^{3311} - \frac{(D^{311})^2}{D^{11}}\right] + \left[D^{3322} - \frac{(D^{322})^2}{D^{22}}\right]\frac{(\beta_n)^2}{(\alpha_m)^2}. \tag{5.12}$$

Let the shell midsurface be given by $z = f \sin\left(\frac{2\pi}{l}x^1\right) \sin\left(\frac{2\pi}{l}x^2\right)$ and the shape function be assumed in the form $h = l^2 \sin\left(\frac{2\pi}{l}x^1\right) \sin\left(\frac{2\pi}{l}x^2\right)$. In this case, for a constant thickness δ, formulae (5.2) yield

$$\begin{aligned} D^{11} &= D^{22} = D\frac{3-\nu}{2}\pi^2 l^2 + B12\pi^6\left(\frac{f}{l}\right)^2, \\ D^{3311} &= D^{3322} = D\frac{3\nu}{2}\pi^2\left(\frac{f}{l}\right)^2, \quad D^{311} = D^{322} = D\frac{3-\nu}{2}\pi^2 l\left(\frac{f}{l}\right). \end{aligned} \tag{5.13}$$

The smallest value of $(P^{11})_{\text{cr}}$ will be obtained by taking n equal 1. Thus, the expression for the critical force becomes

$$(5.14)\qquad (P^{11})_{\text{cr}} = D\frac{\pi^2}{12}\left(\frac{\delta}{L_1}\right)^2\left(m+\frac{1(L_1)^2}{m(L_2)^2}\right)^2 + \frac{K(L_1)^2}{m^2\pi^2}$$
$$+D\pi^2\frac{3-\nu}{2}\left(\frac{f}{l}\right)^2\left[1-\frac{1}{1+\dfrac{2\pi^4}{3-\nu}\left(\dfrac{\delta}{L_1}\right)^2\left(\dfrac{L_1}{l}\right)^2\left(\dfrac{f}{l}\right)^2}\right]\left(1+\frac{1(L_1)^2}{m^2(L_2)^2}\right).$$

For a square plate ($L_1 = L_2 = L$) from Eqs. (5.14) it is obvious that the smallest value of $(P^{11})_{\text{cr}}$ and the number of half-waves m depend on the modulus of foundation k and mesostructure length parameter l.

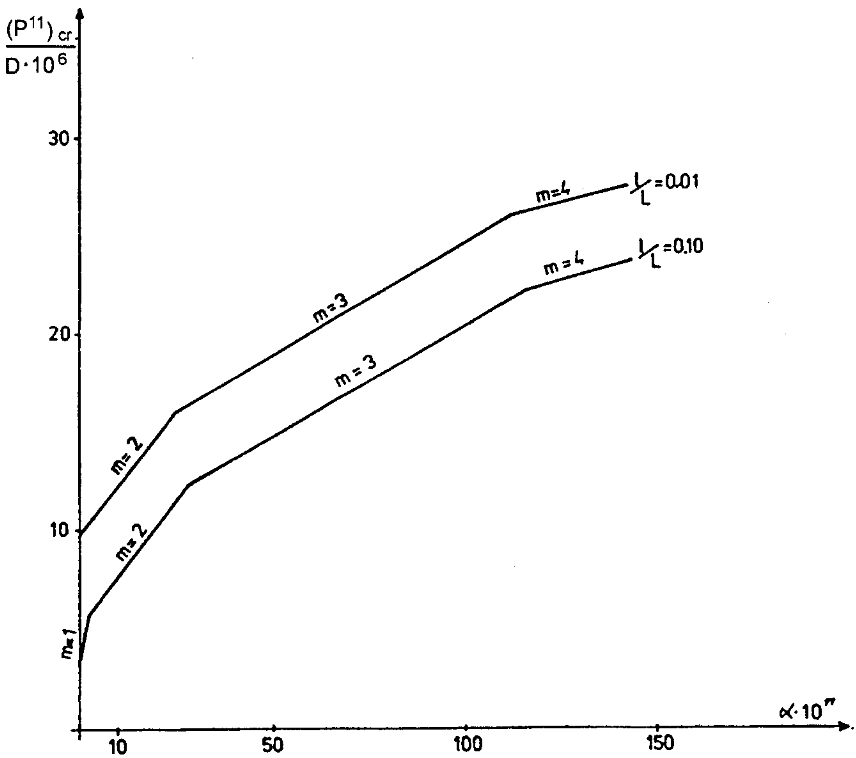

FIG. 4. The smallest value of a critical forces $(P^{11})_{\text{cr}}$ versus parameter α of the modulus of foundation; the number of half-waves m and ratio l/L are used as parameters. Here it assumed that $k = \alpha\dfrac{D}{\delta^2}$.

Let us assume that the wrinkled plate with thickness $\delta = L/1000$ is interacting with an elastic foundation in which elastic modulus has the value $k = \alpha \dfrac{D}{\delta^2}$. The diagram in Fig. 4 there presents the smallest value of critical force $(P^{11})_{\mathrm{cr}}$ for numbers of half-waves equal to $m = 1, 2, 3, 4$ versus the parameter α from the modulus of foundation k, where ratios $l/L = 0.1$ and $l/L = 0.01$ are used as a parameter.

6. Conclusions

From the above example it follows that the proposed averaged theory of wrinkled plates resting on an elastic medium can be successfully applied to the stability analysis. The above examples lead to the following conclusions:

(i) The smallest value of a critical forces and the number of half-waves m depend on the modulus of foundation k and the mesostructure length dimension l.

(ii) The effect of the mesostructure length dimension l plays an important role in the stability analysis.

In order to compare the results related to the microstructural models of wrinkled plates in an elastic medium with the known theories of orthotropic plates, we have restricted ourselves to an illustrative example. Nevertheless, this example leads to the following conclusion:

(i) The value of critical force for orthotropic plate models is independent of the mesostructure length dimension l. On the contrary, in the microstructural models it depends on the mesostructure length dimension l.

(ii) The difference between the value of critical forces calculated in the framework of orthotropic plate theories and using the microstructural models for the ratio $L/l = 10$, is very small and equal 3.2% but for the ratio $L/l = 100$ this difference is equal to 168%.

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