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FINITE-ELEMENT MODEL FOR LAMINATED BEAM-PLATES COMPOSITE USING LAYERWISE DISPLACEMENT THEORY

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The paper uses the layerwise theory, i.e. the zigzag behaviour of the in-plane displacements through the thickness, and the Lagrange interpolation functions for finite element to compute the stresses and displacements in beams made by composite materials. The layerwise method can determine the interlaminar stresses and other localized effects with the same accuracy as 2D finite element method but less computer effort. We present as illustration two examples.

1. Introduction

Composite materials are those formed by combining two or more materials on a macroscopic scale such that they have better engineering properties than the conventional materials, for example metals. Some of the properties that can be improved by forming a composite material are stiffness, strength, weight reduction, corrosion resistance, thermal properties, fatigue life, and wear resistance. Fiber-reinforced composite materials, for example, consist of high strength and high elastic modulus fibers in a matrix material. The use of fiber-reinforced laminates in aerospace, civil buildings, automotive shipbuilding and other industry has increased tremendously during the past several years. This is largely due to the high strength-to-weight ratio of composites as well as their ability to be tailored to meet the design requirements of strength and stiffness. Coinciding with these new applications is the interest in the accurate prediction of the detailed response and failure characteristics of laminated plates.

Composite laminates are formed by stacking layers of different composite materials and/or fiber orientation. By construction, composite laminates have their planar dimensions by one or two orders of magnitude larger than their thickness. Therefore, composite laminates are treated as plate elements.

The layerwise theories can represent the zigzag behavior of the in-plane displacements through the thickness. The zigzag behavior is more pronounced for thick laminates where the transverse shear modulus changes abruptly trough the thickness and can be found in the exact 3-D elasticity solutions obtained by PAGANO [8], PAGANO and HATFIELD [9], SRINIVAS and RAO [16], NOOR [7], SAVOIA and REDDY [14] for bending of the rectangular laminated plates, and by VARADAN and BHASKAR [18] and REN [13] for bending of the laminated shells. In a series of papers, SWIFT and HELLER [17] studied laminated beams by assuming layerwise constant shear strains and a continuous transverse displacement trough the thickness. A similar approach was used by DUROCHER and SOLECKI [5] to study transversely isotropic plates with two or three layers. SEIDE [15] and CHAUDHRI and SEIDE [3] extended the work of Swift and Heller to laminated plates. DISCIUVA [4] proposed a generalized zigzag model by assuming a displacement field enabling a nonlinear variation of the in-plane displacements trough the laminate thickness and fulfils *a priori* the geometric and stress continuity conditions at the interfaces.

1.1. Overview on the laminate plate theories

Numerous displacement-based laminate theories have been proposed to describe the kinematics of laminated composites. Based on the assumed variation of the displacement field through the laminate thickness, these theories can be divided into the following approaches: two-dimensional theories, e.g. equivalent single-layer theories (ESL), three-dimensional elasticity theories (3-D): (traditional 3-D elasticity, layerwise theory (LWT)) and multiple models methods.

1.1.1. Equivalent single-layer theories (2-D). The simplest ELS laminate theory is the *classical laminated plate theory* (or CLPT), which is an extension of the Kirchhoff (classical) plate theory to laminated composite plates:

$$
\begin{aligned}
& u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_o}{\partial x}, \\
& v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_o}{\partial y}, \\
& w(x,y,z,t) = w_0(x,y,t), \\
& \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x} = 0,
\end{aligned} \tag{1.1}
$$

where (u_0, v_0, w_0) are the displacement components along the (x, y, z) coordinate directions, respectively, of a point on the midplane (i.e., $z = 0$).

The next theory in the hierarchy of ELS laminate theories is the *first-order shear deformation theory* (or FSDT) [11], which is based on the displacement field:

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t),\\
v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t),\\
w(x,y,z,t) &= w_0(x,y,t),
\end{aligned}
\tag{1.2}
$$

where ϕ_x and ϕ_y denote rotations about the y and x-axes, respectively. The first-order shear deformation theory requires shear correction factors [19, 20], that are difficult to determine for arbitrarily laminated composite plate structures. The shear correction factors depend not only on the lamination and geometric parameters, but also on the loading and boundary conditions.

Second and higher-order ELS laminated plate theories use higher-order polynomials in the expansion of the displacement components trough the thickness of the laminate [10], often difficult to be interpreted in physical terms. The second-order theory with transverse inextensibility is based on the displacement field:

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) + z^2\psi_x(x,y,t),\\
v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) + z^2\psi_y(x,y,t),\\
w(x,y,z,t) &= w_0(x,y,t).
\end{aligned}
\tag{1.3}
$$

Higher-order theories can better represent the kinematics, may not require the shear correction factors, and can yield more accurate interlaminar stress distributions. However, they involve considerably more computational effort.

The major deficiency of the ELS models in modeling composite laminates is that the transverse strain components are continuous across the interfaces between dissimilar materials; thus, the transverse stress components are discontinuous at the layer interfaces. This deficiency is most evident in relatively thick laminates, or in localized regions of complex loading and geometric and material discontinuities.

1.1.2. Multiple models methods. The analysis of composite laminates has provided the incentive for the development of many of the reported multiple model methods [1, 2, 6, 10]. In general, those models can be divided into two categories: the sequential or multi-step methods, the simultaneous methods. Most of the sequential multiple model methods are developed for global-local analysis. Typically the global region (i.e. the entire computational domain) is analyzed economically (often an ELS laminate model) to determine the displacement or force boundary conditions for a subsequent analysis of the local region (i.e. a small subregion of particular interest). The simultaneous multiple model methods are characterized by a simultaneous analysis of the entire computational domain where different subregions are modeled using different mathematical models and/or distinctly different levels of domain discretization.

2. Beam layerwise model

The layerwise finite element model studied in this work is the same as a conventional 2-D displacement finite element model in terms of interpolation capability and problem size for a 2-D body with parallel top and bottom surfaces.

A beam of variable thickness must be approximated as a constant-thickness beam in order to use the present element. In all practical cases, a laminated structure is made of constant-thickness laminate and therefore the present element can be used to model such structures. In contrast to the equivalent single-layer laminate theory, the layerwise theories are developed by assuming that the displacement field exhibits only C^0 – continuity through the laminate thickness. Thus, the displacement components are continuous through the laminate thickness but the derivatives of the displacements with respect to the thickness coordinate may be discontinuous at various points through the thickness, thereby allowing for the possibility of continuous transverse stresses at the interfaces separating dissimilar materials. Layerwise displacement fields provide a much more kinematically correct representation of the moderate to severe cross-sectional warping associated with the deformation of thick laminates.

The layerwise format maintains a 1-D type data structure. This provides several advantages over the conventional 2-D finite element models:

- First, the volume of input data is reduced.
- Secondly, the in-plane 1-D mesh and the transverse mesh can be refined independently of each other without having to reconstruct a 2-D finite element mesh.

The generalized laminate plate theory proposed by Reddy will be adapted to laminated beams.

The displacement field in the k-th layer is written as [12]:

$$u(x,z,t) = \sum_{J=1}^{N} U^J(x,t)\Phi^J(z), \tag{2.1}$$

$$w(x,z,t) = \sum_{I=1}^{M} W^I(x,t)\Psi^I(z), \tag{2.2}$$

where u, w represent the displacement components in the x and z directions, respectively, of a material point initially located at (x, z) in the undeformed laminate. N and M are the numbers of finite element subdivisions through the laminate thickness. The $U_j(x,t)$ and $W_j(x,t)$ represent the axial displacement and transverse displacements along lines of constant z in the undeformed beam corresponding to nodes $1, 2, ..., N$ trough the thickness of the beam. The $\Phi^J(z)$ and $\Psi^J(z)$ $(j = 1, 2, ..., N)$ are linear Lagrangian interpolation polynomials which are nonzero only between nodes $j-1$ and $j+1$ through the thickness. Typically N should be greater than or equal to the number of material layers in the laminate, so that transverse stresses can be accurately determined. In general, N and M do not have to be equal; however, the finite element formulation is simplified by making N and M equal, that implies that $\Phi^J(z)$ and $\Psi^J(z)$ are the same interpolation functions.

For a linear variation through each numerical layer, the shape functions are (see Fig. 1):

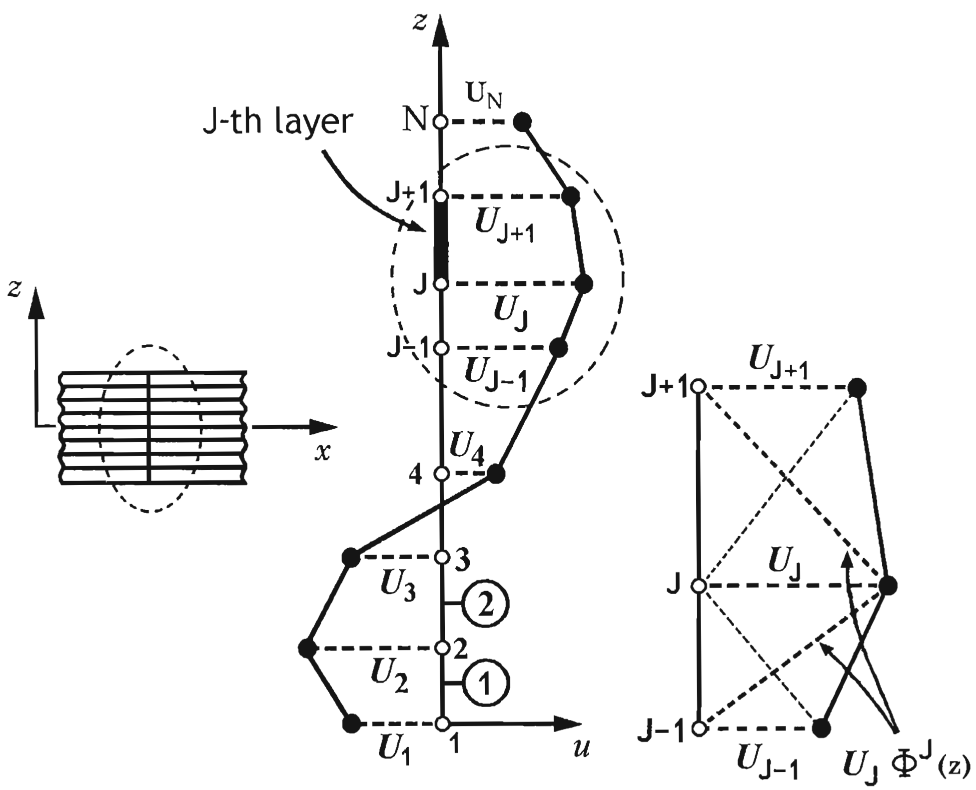

FIG. 1. Bending of a laminated beam and the linear approximation functions $\Phi^J(z)$ used in the layerwise theory.

$$\begin{aligned}
&\Phi^1(z) = \psi_1(\bar{z}), z_1 \leq \bar{z}_1 \leq z_2 ,\\
&\Phi^J(z) = \begin{cases} \psi_2(\bar{z}), z_{I-1} \leq \bar{z}_1 \leq z_I , \\ \psi_1(\bar{z}), z_I \leq \bar{z}_k \leq z_{I+1} , \end{cases} \\
&\Phi^N(z) = \psi_2(\bar{z}), z_{N-1} \leq \bar{z}_N \leq z_N ,
\end{aligned} \tag{2.3}$$

where:

$$\psi_1^{(k)} = 1 - \bar{z}, \psi_2^{(k)} = \bar{z} . \tag{2.4}$$

The strains associated with the displacement field (2.2) are:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \sum_{J=1}^{N} \frac{\partial U^J}{\partial x} \Phi^J ,\\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = \sum_{J=1}^{N} \frac{\partial \Phi^J}{\partial z} W^J ,\\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \sum_{J=1}^{N} \left(\frac{\partial \Phi^J}{\partial z} U^J + \frac{\partial W^J}{\partial x} \Phi^J \right) .
\end{aligned} \tag{2.5}$$

The finite element model corresponding to each of these theories is developed by applying the principle of virtual displacements to a representative physical element of the beam.

The governing equations of motion for the present layerwise theory can be derived using the principle of virtual displacements:

$$0 = \int_0^T (\delta U + \delta V - \delta K) dt . \tag{2.6}$$

The constitutive equation for the k-th orthotropic lamina with an arbitrary layer angle can be written as:

$$\begin{aligned}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix}_k &= \begin{bmatrix} \bar{C}_{11} & \bar{C}_{13} & 0 \\ \bar{C}_{13} & \bar{C}_{33} & 0 \\ 0 & 0 & \bar{C}_{55} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} \\
&= \begin{bmatrix} \bar{C}_{11} & \bar{C}_{13} & 0 \\ \bar{C}_{13} & \bar{C}_{33} & 0 \\ 0 & 0 & \bar{C}_{55} \end{bmatrix}_k \begin{Bmatrix} \sum_J^N \frac{\partial U^J}{\partial x} \Phi^J \\ \sum_J^N \frac{\partial \Phi_J}{\partial z} W^J \\ \sum_J^N \left(\frac{\partial \Phi^J}{\partial z} U^J + \frac{\partial W^J}{\partial x} \Phi^J \right) \end{Bmatrix}
\end{aligned} \tag{2.7}$$

where there are the transformed elastic coefficients in the (x, z) systems, which are related to the elastic coefficients in the material axes, C_{ij}.

Substitution of Eqs. (2.2), (2.5) in (2.6), followed by integration with respect to y and z, yields:

$$\tag{2.8} \int_L \left\{ \sum_J^N \left(N_{xx}^J \frac{\partial \delta U^J}{\partial x} + N_{xz}^J \delta U^J + N_{zz}^J \delta W^J + \tilde{N}_{xz}^J \frac{\partial \delta W^J}{\partial x} \right) \right\} - \int_L b \left(q_b \delta W_1 + q_t \delta W_N \right) dx - \sum_J^N \hat{N}_{xx}^J \delta U^J \bigg|_{x_A}^{x_B} - \sum_J^N \hat{N}_{xz}^J \delta W^J \bigg|_{z_1}^{z_2} = 0$$

where:

$$\tag{2.9} \begin{aligned} N_{xx}^J &= \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \sigma_{xx} \Phi^J dz & N_{xz}^J &= \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \sigma_{xz} \frac{\partial \Phi^J}{\partial z} dz\,, \\ N_{zz}^J &= \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \sigma_{zz} \frac{\partial \Phi^J}{\partial z} dz & \tilde{N}_{xz}^J &= \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \sigma_{xz} \Phi^J dz\,. \end{aligned}$$

Using the constitutive relations (2.7), the expressions of the resultant forces that require laminawise integration take now the forms:

$$\left\{ \begin{array}{c} N_{xx}^J \\ N_{zz}^J \end{array} \right\} = \left[\begin{array}{cc} A_{11}^{IJ} & \tilde{A}_{13}^{IJ} \\ \tilde{A}_{13}^{JI} & \hat{A}_{33}^{IJ} \end{array} \right] \left\{ \begin{array}{c} \dfrac{\partial U^J}{\partial x} \\ W^J \end{array} \right\} = \sum_J^N A^{IJ} e^J\,,$$

$$\left\{ \begin{array}{c} N_{xz}^J \\ \tilde{N}_{xz}^J \end{array} \right\} = \left[\begin{array}{cc} \bar{A}_{55}^{IJ} & B_{55}^{IJ} \\ \bar{B}_{55}^{JI} & D_{55}^{IJ} \end{array} \right] \left\{ \begin{array}{c} U^J \\ \dfrac{\partial W^J}{\partial x} \end{array} \right\} = \sum_J^N B^{IJ} e_S^J\,,$$

where:

$$A_{11}^{IJ} = \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \bar{C}_{11}^{(k)} \Phi^J \Phi^I dz\,, \qquad \tilde{A}_{13}^{IJ} = \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \bar{C}_{13}^{(k)} \frac{\partial \Phi^J}{\partial z} \Phi^I dz\,,$$

$$\tilde{A}_{13}^{JI} = \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \bar{C}_{13}^{(k)} \frac{\partial \Phi^I}{\partial z} \Phi^J dz\,, \qquad \hat{A}_{33}^{IJ} = \sum_k^{N_e} \int_{z_k}^{z_{k+1}} b \bar{C}_{33}^{(k)} \frac{\partial \Phi^I}{\partial z} \frac{\partial \Phi^J}{\partial z} dz\,,$$

$$\hat{A}_{55}^{IJ} = \sum_{k}^{N_e} \int_{z_k}^{z_{k+1}} b\bar{C}_{55}^{(k)} \frac{\partial \Phi^I}{\partial z} \frac{\partial \Phi^J}{\partial z} dz\,, \qquad \bar{B}_{55}^{IJ} = \sum_{k}^{N_e} \int_{z_k}^{z_{k+1}} b\bar{C}_{55}^{(k)} \frac{\partial \Phi^J}{\partial z} \Phi^I dz\,,$$

$$\bar{B}_{55}^{JI} = \sum_{k}^{N_e} \int_{z_k}^{z_{k+1}} b\bar{C}_{55}^{(k)} \frac{\partial \Phi^I}{\partial z} \Phi^J dz\,, \qquad \bar{D}_{55}^{IJ} = \sum_{k}^{N_e} \int_{z_k}^{z_{k+1}} b\bar{C}_{55}^{(k)} \Phi^I \Phi^J dz\,.$$

The subscript S indicates that the quantities are related to the transverse shear, and:

$$e^J = \left\{ \begin{array}{c} \dfrac{\partial U^J}{\partial x} \\ W^J \end{array} \right\}, \qquad e_S^J = \left\{ \begin{array}{c} U^J \\ \dfrac{\partial W^J}{\partial x} \end{array} \right\}. \tag{2.10}$$

Substitution of Eqs. (2.9) into Eq. (2.8) gives the compact form of Hamilton's principle as follows:

$$\int_L \left[\sum_J^N \sum_K^N \left(\partial e^{J^T} A^{JK} e^K + \partial e_S^{J^T} B^{JK} e_S^K \right) \right] = \int_L b \left(q_b \delta W_1 + q_t \delta W_N \right) dx\,. \tag{2.11}$$

3. Finite element approach

Over each finite element, the displacements (U^J, W^J) are expressed as a linear combination of shape functions $\hat{\Psi}_i$ and nodal values (U_i^J, W_i^J) as follows:

$$(U^J,\, W^J) = \sum_{i=1}^{NPE} \left(W_i^J,\, U_i^J \right) \hat{\Psi}_i\,, \tag{3.1}$$

where NPE is the number of nodes per element.

Substituting approximations (3.1) in (2.10) we obtain:

$$\begin{aligned} e^J &= \left\{ \begin{array}{c} \dfrac{\partial U_J}{\partial x} \\ W_j \end{array} \right\} = \begin{bmatrix} h^T & 0 \\ 0 & g^T \end{bmatrix} \left\{ \begin{array}{c} U_i^J \\ W_i^J \end{array} \right\} = H u_e^J\,, \\ e_S^J &= \left\{ \begin{array}{c} U_j \\ \dfrac{\partial W_J}{\partial x} \end{array} \right\} = \begin{bmatrix} g^T & 0 \\ 0 & h^T \end{bmatrix} \left\{ \begin{array}{c} U_i^J \\ W_i^J \end{array} \right\} = G u_e^J\,, \end{aligned} \tag{3.2}$$

where:

$$h = \left\{ \frac{\partial \hat{\Psi}_1}{\partial x} \; \frac{\partial \hat{\Psi}_2}{\partial x} \; \ldots\ldots \; \frac{\partial \hat{\Psi}_{NPE}}{\partial x} \right\}^T ,$$

$$g = \left\{ \hat{\Psi}_1 \; \hat{\Psi}_2 \; \ldots\ldots \; \hat{\Psi}_{NPE} \right\}^T , \tag{3.3}$$

and the displacements vector for each element:

$$u_e^J = \left\{ u^J \; w^J \right\}^T ,$$

$$u^J = \left\{ U_1^J \; U_2^J \; \ldots\ldots \; U_{NPE}^J \right\}^T , \tag{3.4}$$

$$w^J = \left\{ W_1^J \; W_2^J \; \ldots\ldots \; W_{NPE}^J \right\}^T .$$

The stiffness matrix K_e takes the form:

$$K_e = \int_L \begin{bmatrix} H^T A^{11} H + G B^{11} G & H^T A^{12} H + G B^{12} G & \ldots\ldots & H^T A^{1N} H + G B^{1N} G \\ H^T A^{21} H + G B^{21} G & H^T A^{22} H + G B^{22} G & \ldots\ldots & H^T A^{2N} H + G B^{2N} G \\ \vdots & \vdots & \ddots & \vdots \\ H^T A^{N1} H + G B^{11} G & H^T A^{N2} H + G B^{N2} G & \ldots\ldots & H^T A^{NN} H + G B^{NN} G \end{bmatrix} dx \tag{3.5}$$

and the loading vector P_e is:

$$P_e = P_{e1} \delta W^1 + \ldots + P_{eN} \delta W^N = \int_L \left[q_1(x) \cdot \delta W^1 + \ldots + q_N(x) \cdot \delta W^N \right] dx . \tag{3.6}$$

4. Results and discussion

A linear static analysis of the composite beam was performed. In the direction of the thickness coordinate we choose a linear piecewise Lagrange interpolation functions, and along the beam we use one-dimensional elements with quadratic Lagrange interpolation functions.

In the general case we should consider any number of layers, but in this numerical example we consider 8 layers and the reduced Gauss points correspond to the centroid of each layer of each finite element. We study two numerical examples concerning in each case simply supported beam. Beam length is 10 cm and width 2 cm and the beam is loaded by a concentrated force at the middle span equal with 1kN. The first one is a symmetric laminate with angle-ply $(0/45/-45/90)_s)(0°$ corresponds to outer layers), and in the second case $(90/\pm 45/0)_{s\cdot s}$ (90° corresponds to outer layers). Each layer, in both the examples has the same thickness $h_k = 0.1$. The following layer material properties are used $(E_T - 1$ msi,):

$$(4.1) \qquad \frac{E_1}{E_2} = 25, \qquad G_{12} = G_{13} = 0.5E_2, \qquad G_{23} = 0.2E_2, \qquad \nu = 0.25.$$

Here the subscript L denotes the direction parallel to the fibers, subscript T denotes the inplane direction perpendicular to the fibers, and the subscript z denotes the out-of-plane direction.

The displacements and stresses results are presented in the Fig. 2 and Fig. 3 and the comparison with the classical method in the Table 1.

Table 1. Dimensionless displacement of the point (x=L/2, z=H/2) and stresses.

Beam model	w(100) / H
CBT	-3.92
LWT	-3.90185

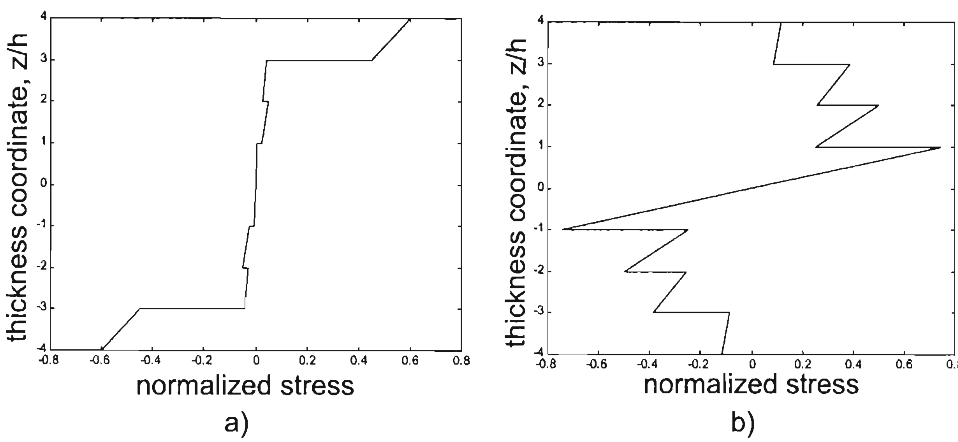

FIG. 2. Maximum normal stress, $\sigma_{xx}(a/2, z)$, distribution through the thickness of a symmetrically laminated (a), $(0/\pm 45/90)_s$ and (b) $(90/\pm 45/0)_s$ beams subject to three-point bending.

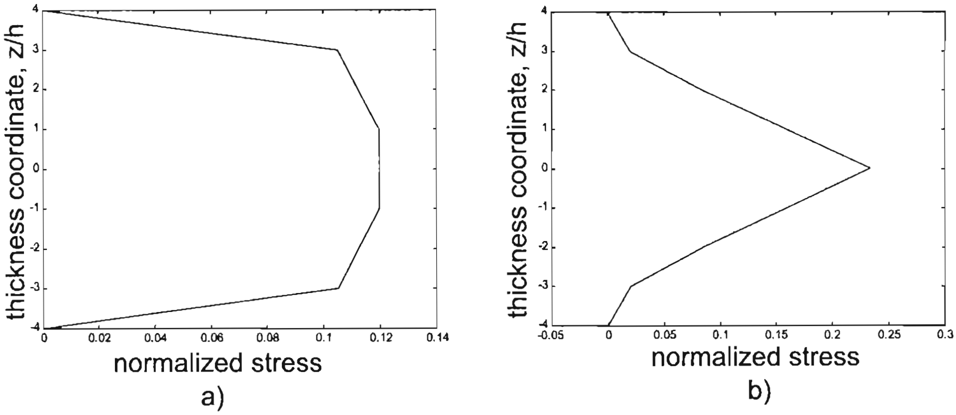

FIG. 3. Variation of transverse shear stress $\bar{\sigma}_{xz}(0,z) = \sigma_{xz}bh/F$ through the thickness of a symetrically laminated (a), $(0/\pm 45/90)_s$ and (b), $(90/\pm 45/0)_s$ beams subject to three-point bending.

CONCLUSIONS

The resulting layerwise finite element model is capable of computing interlaminar stresses and other localized effects with the same accuracy as a conventional 2D finite element model.

For these two examples considered, the results are near to the classical theory, in order to have a comparison.

The layerwise theory can be very useful for complex elements. Thus, we may use simultaneously on the structure conventional finite elements and in other subregions – the LWT elements.

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