ENGINEERING TRANSACTIONS • *Engng. Trans.* • **53**, 3, 335–343, 2005
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.445.2005

STABILITY OF AN ELASTIC COLUMN SUBJECTED TO NON-STATIONARY COMPRESSIVE LOADS

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The problem of dynamic stability of an elastic column with pinned ends subjected to non-stationary compressive axial loads is considered. The method of optimal Lyapunov functions for differential inclusions is applied to obtain sufficient conditions of stability of the column in the case of bounded loads. The obtained results, improving and generalising the classical solutions to the dynamic Euler problem, may be useful in designing civil engineering structures and mechanical systems consisting of compressed columns. The possibility of optimisation of the column characteristics with respect to its stability properties (e.g. stability margins in the space of parameters) is pointed out.

Key words: elastic column, non-stationary loads, column stability, Lyapunov function, differential inclusion.

1. Introduction

Elastic columns subjected to axial loads are basic components of many civil engineering structures and mechanical systems. There exists the crucial problem of stability of such columns. It is well known that if the equilibrium of a column subjected to a constant compressive axial load F_0 is disturbed, then it will return to its original position unless F_0 is not greater than the critical load F_{cr} of the column. However, the column becomes unstable if its load is greater than the critical load. In such a case, a sudden change in shape and size (called buckling) takes place. Since the column buckling is one of the major causes of failures of the structures, the possibility of buckling is usually taken into account in designing. However, elastic columns utilised in civil engineering structures such as bridges and towers, working in changing environmental conditions, are usually subjected to non-stationary, unpredictable (e.g. moving) loads. In such a case, the column instability may occur also for sub-critical static loads. Therefore, many complex problems of stability of an elastic column subjected to non-stationary axial forces have been considered since many years [1, 2, 3].

Although certain partial results in this subject (e.g. for the so-called dynamic Euler problem of stability of a column under sinusoidal load [1, 2]) are known, no estimates of the stability region of the column under general non-stationary loads are available [3]. However, real loads rarely are exactly harmonic. Even the varying loads of the bridge columns caused by a moving train may not be harmonic loads. That is why the particular results concerning the column stability are usually not applicable in engineering practice.

In this work the problem of stability of the column as an Euler beam loaded by a general non-stationary compressive force is considered. The compressive force is supposed to be bounded and no other properties of the load, such as: periodicity or its stochastic characteristics are assumed. Therefore the obtained results can be applied to any practical situation. In order to obtain the stability estimates, the method of optimal Lyapunov functions is applied [4–5]. Improvements and generalisations of the classical results in this subject are presented.

2. Mathematical model of the loaded column

Let us consider an axially symmetric, elastic and homogeneous column of length L and with pinned ends. Applying the classical Euler theory of elastic beams, the column dynamics can be described by the following partial differential equation:

$$
\text{(2.1)} \qquad EJ\left[1+2c\frac{\partial}{\partial t}\right]\frac{\partial^4 w(x,t)}{\partial x^4}+[F_0+F_1(t)]\frac{\partial^2 w(x,t)}{\partial x^2} + 2D\frac{\partial w(x,t)}{\partial t}+M\frac{\partial^2 w(x,t)}{\partial t^2}=0,
$$

where EJ is the bending rigidity of the column, c, D are the coefficients of intrinsic and external attenuation, respectively, $M=\rho S$ is the mass coefficient, F_0 is the constant component of the external axial column load, whereas $F_1(t)$ is its time-dependent component. It is logical to assume that the non-stationary force $F_1(t)$ is bounded i.e. there is a constant $\alpha>0$ such that $|F_1(t)|\leq\alpha\leq F_0$ for all $t\geq t_0$. This condition ensures that the loading forces are always compressive.

If the non-stationary loads were fixed, i.e. described by a determined time-dependent function $F_1(t)$, then Eq. (2.1) with a variable coefficient would be an appropriate model of the column dynamics. However, if the function $F_1(t)$ is not uniquely determined and only its boundness is assumed then, in order to determine the column stability for time $t\geq t_0$, we have to study the stability of the following differential inclusion:

$$\frac{\partial^4 w(x,t)}{\partial x^4} + f_0 \frac{\partial^2 w(x,t)}{\partial x^2} + 2c \frac{\partial}{\partial t}\frac{\partial^4 w(x,t)}{\partial x^4} + 2d \frac{\partial w(x,t)}{\partial t} + \mu \frac{\partial^2 w(x,t)}{\partial t^2} \in \left\{ f_1 \frac{\partial^2 w(x,t)}{\partial x^2} : |f_1| \le \beta \right\}, \tag{2.2}$$

where $f_0 = F_0/EJ$, $f_1(t) = F_1(t)/EJ$, $\mu = \rho S/EJ = M/EJ$, $d = D/EJ$. The normalised force $f_1(t)$ satisfies the bounding condition $|f_1(t)| \le \alpha/EJ = \beta$ for all $t \ge t_0$. The assumption concerning pinned ends of the column can be expressed by the following boundary conditions:

$$w(0,t) = w(L,t) = 0, \qquad \partial^2 w(0,t)/\partial x^2 = \partial^2 w(L,t)/\partial x^2 = 0, \tag{2.3}$$

that should be taken into account in the stability analysis.

3. Discretisation of the problem

It is convenient to study the column stability by using the second method of Lyapunov. To do this, let us transform the continuous model (2.2) of the beam into a discrete model represented by an infinite system of ordinary differential inclusions. This can be done by variables separation. Indeed, looking for partial solutions of the form $\Phi(x) \cdot g(t)$ and putting it into inclusion (2.2) one obtains the following inclusion:

$$\frac{\Phi^{IV}(x)}{\Phi(x)} + \frac{\Phi^{II}(x)}{\Phi(x)} \cdot \frac{f_0 \cdot g(t)}{g(t) + 2c \cdot \dot{g}(t)} + \frac{2d \cdot \dot{g}(t)}{g(t) + 2c \cdot \dot{g}(t)} + \frac{\mu \cdot \ddot{g}(t)}{g(t) + 2c \cdot \dot{g}(t)} \in \left\{ \frac{\Phi^{II}(x)}{\Phi(x)} \cdot \frac{f_1 \cdot g(t)}{g(t) + 2c \cdot \dot{g}(t)} : |f_1| \le \beta \right\}. \tag{3.1}$$

It is easy to see that only the sinusoidal modal functions:

$$\Phi_n(x) = \sin[\omega_n x], \qquad \omega_n = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots, \tag{3.2}$$

where ω_n – the spatial frequency of the n-th mode, satisfy boundary conditions (2.3) and enable us to separate variables x, t. In fact, putting (3.2) into inclusion (3.1) one obtains the following set of ordinary differential inclusions for the corresponding time-dependent functions $g_n(t)$:

$$\ddot{g}_n(t) + \frac{2[d + c\omega_n^4]}{\mu} \cdot \dot{g}_n(t) + \frac{\omega_n^2}{\mu}\left[\omega_n^2 - f_0\right] \cdot g_n(t) \in \left\{ \frac{\omega_n^2}{\mu} \cdot f_1 \cdot g(t) : |f_1| \le \beta \right\}, \qquad n = 1, 2, \ldots \tag{3.3}$$

The above system, together with the set of modal functions (3.2), is equivalent to the original problem described by differential inclusion (2.2). Thus the problem of stability of the column under non-stationary loads has been transformed to the stability problem of the infinite family of linear oscillators with non-stationary stiffness excitations.

4. Stability of the compressed column with pinned ends

It is obvious that the stability analysis of the beam should be performed for sub-critical constant loads $f_0 < \omega_1^2 = \pi^2/L^2$, i.e. for the constant component F_0 of the column load below the critical value $\pi^2 \cdot EJ/L^2$.

In order to prove stability of system (3.3) it is necessary to do this for each non-stationary oscillator (inclusion) numbered by $n = 1, 2, \ldots$ Since, the differential inclusions describing the oscillators have the unique standard form considered in the Appendix; it is possible to apply directly stability estimates (A.3) for the following sets of the corresponding parameters:

$$\begin{aligned} p &= p_n = \frac{d + c\omega_n^4}{\mu}, \quad q = q_n = \frac{\omega_n^2}{\mu}\left[\omega_n^2 - f_0\right], \\ r &= r_n = \frac{\omega_n^2}{\mu} \cdot \beta, \qquad n = 1, 2, \ldots \end{aligned} \tag{4.1}$$

Hence, the stability region of the column in the space of parameters is determined by the following inequalities:

$$\begin{aligned} &\left(p_n \geq (q_n/2)^{1/2} \quad \text{and} \quad r_n < q_n\right) \\ \text{or} \quad &\left(p_n < (q_n/2)^{1/2} \quad \text{and} \quad r_n < 2p_n\sqrt{q_n - p_n^2}\ \right), \qquad n = 1, 2, \ldots \end{aligned} \tag{4.2}$$

It is easy to deduce from (4.1) that the conditions $r_n < q_n$, $r_n < 2p_n\sqrt{q_n - p_n^2}$ provide certain bounds on the non-stationary part of the column load while the remaining conditions in (4.2) – certain constraints on the parameters of the column and the stationary part f_0 of the load.

The subsequent conditions in (4.2) ensure stability of the corresponding independent vibration modes $\Phi_n(x) \cdot g_n(t)$, for $n = 1, 2, \ldots$. Thus, in the general case, the stability conditions of the column are determined by the infinite set of inequalities (4.2). The question is how to check all the stability conditions and describe them in a finite closed form easily applicable in practice. In scientific papers usually stability of only a few first vibration modes of the column with a harmonic axial excitation are studied (e.g. [1, 2]). It is explained here below why such an approach is justified in the general case of non-stationary compressive loads.

To prove this, let us divide the set $\Omega = \{\omega_n : n = 1, 2, \ldots\}$ into two disjoint subsets Ω^+, Ω^- containing eigen-frequencies satisfying the condition $p_n \geq (q_n/2)^{1/2}$ and the opposite condition, respectively. It is easy to prove that the conditions $p_n \geq (q_n/2)^{1/2}, n = 1, 2, \ldots$ are equivalent to the following inequalities:

$$G\left(\omega_n^2\right) \geq 0, \qquad n = 1, 2, \ldots, \tag{4.3}$$

where

$$G(z) = c^2 z^4 - \left(\frac{\mu}{2} - 2cd\right) z^2 + \frac{\mu f_0}{2} z + d^2, \qquad z > 0. \tag{4.4}$$

Since the intrinsic attenuation in real beams is non-vanishing, the parameter c is positive, i.e. $c > 0$. Thus the polynomial $G(z)$ of order four takes positive values for sufficiently large z. Hence, the set Ω^- must be either empty or finite, contrary to the set Ω^+.

Similarly, since each condition $r_n < q_n$ determines the following simple bound

$$\beta < \omega_n^2 - f_0 = n^2 \pi^2 / L^2 - f_0 \tag{4.5}$$

on the non-stationary part of the load, only the minimum eigen-frequency $\omega_m \in \Omega^+$ is essential for the column stability. Thus, in the general case, in order to determine the beam stability one should check the unique condition $r_m < q_m$ for $\omega_m \in \Omega^+$ and, at most, a finite number of conditions $r_n < 2p_n\sqrt{q_n - p_n^2}$ for all $\omega_n \in \Omega^-$. This can be easily done for any fixed loads and parameters of the column. However, as it is seen, the vibration modes taken into account in the stability analysis cannot be chosen arbitrary but should be deduced from the values of the system parameters and the positive roots of the polynomial $G(z)$.

5. Sufficient conditions of the column stability

In theory and practice it is usually not satisfactory to prove merely the stability of a system. For example, if one wants to determine stability margins of the column under study or perform its parametric optimisation at the design stage, then it is necessary to know stability conditions for unfixed parameters of the column. However, in the general case, it is difficult to solve inequality (4.4) exactly and obtain simple practical results. Therefore, in order to obtain general analytical results one can apply certain approximations.

There are a few methods for approximate solving of the infinite system of inequalities (4.2). For example, in order to obtain sufficient conditions of stability of the column, one can replace the conditions $p_n < \sqrt{q_n/2}$, $r_n < 2p_n\sqrt{q_n - p_n^2}$, $i = 1, 2 \ldots$ by the following simplified inequalities:

$$p_n < \sqrt{q_n/2}, \qquad r_n < p_n\sqrt{2q_n}, \qquad i = 1, 2, \ldots. \tag{5.1}$$

Since, by definitions (3.2), (4.1), both $\{p_n\}$ and $\{q_n\}$ are increasing sequences, it suffices to take into account only the first condition (for $n = 1$) of the form (5.1). Hence, putting (4.1) into (5.1) for $n = 1$ and combining it with the condition (4.5) (for $n = 1$), one finally obtains the following sufficient condition for stability of the excited column:

$$\beta < \min\left[\omega_1^2 - f_0,\ \left(d + c\cdot\omega_1^4\right)\cdot\sqrt{2[1-(f_0/\omega_1^2)]/\mu}\right]. \tag{5.2}$$

Thus in the case of the general compressive loads it is possible to deduce the column stability from the stability of the first vibration mode. The estimate (5.2) is valid for any values of the column parameters and loads. It is however important to note that condition (5.2) is an approximate result and it may not be the necessary condition for the column stability.

Another estimate of the stability region follows from the fact that, independently of the attenuation coefficients c, d, there is an upper bound for the non-stationary part of the column load, namely $\beta < \beta_{\max} = \omega_1^2 - f_0 = \pi^2/L^2 - f_0$. Therefore, it is necessary to find such constraints on the column parameters that ensure the stability for the $|f_1(t)| < \beta_{\max}$. In order to do this, let us apply the following approximate conditions $G_4(\omega_n^2) \geq 0,\quad n = 1, 2, \ldots$, where

$$G_4(z) = c^2 z^4 - \left(\frac{\mu}{2} - 2cd\right) z^2 + \frac{\mu\cdot f_0}{2}\omega_1^2 + d^2 . \tag{5.3}$$

It is clear that the condition $G_4(z) \geq 0$ implies $G(z) \geq 0$ for any $z \geq \omega_1^2$. Moreover, the bi-quadratic inequality $G_4(z) \geq 0$ can be exactly solved. Indeed, one can easily prove that the conditions $G_4(\omega_n^2) \geq 0,\ n \in N$ are valid if $\mu \leq 8c\left(d + c\cdot f_0\omega_1^2\right)$, i.e. for sufficiently large attenuation of the column. Thus one obtains the following sufficient stability conditions of the column:

$$\beta < \omega_1^2 - f_0 = \pi^2/L^2 - f_0, \qquad \mu \leq 8c\left(d + c\cdot f_0\omega_1^2\right). \tag{5.4}$$

It is also possible to obtain interesting estimates of the stability region of the column in certain particular cases with vanishing intrinsic or external attenuation, namely: Case 1 ($c > 0,\quad d = 0$), Case 2 ($c = 0,\quad d > 0$). Despite the fact that in real systems attenuation is never vanishing, the above cases are interesting from the formal as well as the theoretical viewpoint.

CASE 1. ($c \geq 0$, $d = 0$) (neglected external attenuation of the column)

In this particular case the function $G(z)$ is a homogenous polynomial i.e.

$$G(z) = c^2 z^4 - \frac{\mu}{2} z^2 + \frac{\mu f_0}{2} z. \tag{5.5}$$

Hence, the condition $G(z) \geq 0$ for $z > 0$ is equivalent to the following inequality of order three:

$$G_3(z) = c^2 z^3 - \frac{\mu}{2} z + \frac{\mu \cdot f_0}{2} \geq 0, \tag{5.6}$$

which can be solved exactly so that the necessary and sufficient conditions of stability can be found in this case. However, in order to obtain simple practical results, let us consider here the problem of stability of the column with the maximal bound on the load $\beta < \beta_{\max} = \omega_1^2 - f_0$.

It can be easily proved that $G_3(z) \geq 0$ for all $z > 0$ if and only if $\mu \geq (27/2) \cdot c^2 \cdot f_0^2$. Hence, one can finally obtain the following sufficient conditions of the column stability:

$$\beta < \beta_{\max} = \omega_1^2 - f_0, \qquad \mu \leq \frac{27}{2} c^2 f_0^2. \tag{5.7}$$

CASE 2. ($c = 0$, $d > 0$) (neglected internal attenuation in the column)

In this case the polynomial $G(z)$ is of order two, i.e.

$$G(z) = G_2(z) = -\frac{\mu}{2} \cdot z^2 + \frac{\mu \cdot f_0}{2} \cdot z + d^2. \tag{5.8}$$

Hence, the condition $G(z) \geq 0$ reduces to the simple quadratic inequality. However, the function $G_2(z)$ has always the unique positive root

$$z_0 = \frac{1}{2}\left[f_0 + \sqrt{f_0^2 + \frac{8d^2}{\mu}}\right] \tag{5.9}$$

and $G_2(z) < 0$ (> 0) for $z > z_0$ ($0 < z < z_0$). That is why, in this case, it is not possible to study the column stability only under the assumption $|f_1(t)| < \beta_{\max} = \omega_1^2 - f_0$ and the condition $r_n < 2p_n\sqrt{q_n - p_n^2}$ has to be taken into account.

If $\omega_1^2 \leq z_0$ then there is $k > 1$ so that $G_2(\omega_i^2) \geq 0$ for $i = 1, ..., k-1$ and $G_2(\omega_i^2) < 0$ for $i \geq k$. Then, combining (4.1), (4.2), one can conclude that the inequality

$$\beta < \min\left[\omega_1^2 - f_0,\ \frac{2d}{\mu}\sqrt{\mu - \frac{\mu \cdot f_0}{\omega_k^2} - \frac{d^2}{\omega_k^4}}\right] \tag{5.10}$$

is the necessary and sufficient condition for the column stability.

Similarly, if $\omega_1^2 > z_0$ then $\forall(k \in N)\ G_2\left(\omega_k^2\right) < 0$ and the necessary and sufficient condition for the column stability takes the following form

$$\beta < \frac{2d}{\mu}\sqrt{\mu \frac{\mu}{2} - \frac{\mu \cdot f_0}{\omega_1^2} - \frac{d^2}{\omega_1^4}}. \tag{5.11}$$

6. Conclusion

The obtained estimates of the stability region of the excited columns not only improve the analogue results (obtained for periodic loads) available in the literature but also provide new analytical formulas that are valid for any non-periodic compressive load of the column. Applying the obtained formulas it is possible to determine the stability margins for any real column and perform parametric optimisation of a column at the design stage.

The results have confirmed the hypothesis that it suffices to check the stability of a finite number of vibration modes of the column in order to decide on its stability. However, the vibration modes that have to be taken into account in the stability analysis are not arbitrary but they depend on the column parameters and characteristics of the column load.

Appendix A.

In the analysis of many practical cases, the problem of stability of the linear oscillator

$$\ddot{y} + 2p \cdot \dot{y} + (q + z(t)) \cdot y = 0, \tag{A.1}$$

with a non-stationary bounded excitation $z(t)$, $|z(t)| \le r$ of the stiffness parameter has to be considered. There are known particular results in this subject, for example in the case of a harmonic excitation of the stiffness [1, 2]. However, estimates of the stability region in the general case of a non-stationary stiffness excitation are not so well known. Therefore we have presented here the general results obtained by using the method of Lyapunov functions.

It is obvious that equation (A.1) with undetermined perturbations $z(t)$ is equivalent to the following differential inclusion

$$\ddot{y} + 2p \cdot \dot{y} + q \cdot y \in \{z(t)) \cdot y : |z(t))| \le r\}. \tag{A.2}$$

According to the Lyapunov approach to stability of differential inclusions there is a critical perturbation $z_{\mathrm{cr}}(t)$ which enables to reduce the problem to the stability analysis of differential equation (A.1) with the determined perturbation $z(t) = z_{\mathrm{cr}}(t)$.

The existence of the critical perturbation for inclusion (A.2) is proved for example in [4, 5]. In the result the following inequalities:

$$(A.3)\qquad \left(p \geq (q/2)^{1/2} \quad \text{and} \quad r < q\right) \quad \text{or} \quad \left(p < (q/2)^{1/2} \quad \text{and} \quad r < 2p\sqrt{q - p^2}\right)$$

are found as conditions determining the stability region of the oscillator in the space of parameters (p, q, r).

Acknowledgment

Supported by MNiI grant 4T12B04829.

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Received July 30, 2004; revised version May 12, 2005.