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AN ANALYSIS OF VARIOUS DESCRIPTIONS OF STATE OF STRAIN IN THE LINEAR KIRCHHOFF–LOVE TYPE SHELL THEORY

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In the first part of the paper several methods of describing the state of strain in the thin shell with Kirchhoff-Love constraints are discussed. Among others Kilchevsky's idea of the description of the state of strain in an arbitrary point of the shell is presented by means of the strain tensor parallely shifted to the base on the middle surface. Attention has been called to the physical meaning of Kilchevsky's tensor. Two forms of solving the parallel shift problem, i.e. "operator" equations and generalized Taylor series, are described and analysed. The second part of the work deals with the geometric and physical consequences of the first approximation assumption $h/R \ll 1$. Some versions of the equations describing the state of strain, in particular the tensors of flexible deformation, have been discussed.

Introduction

The linear first approximation theory of thin shells has been the subject of consideration since the pioneer treatise of A. E. Love (1892) was published. The fundamental purpose of this investigation was the formulation of a theory interiorly coherent and possibly consistent with the general principles of classical field theories in the mechanics of deformable solids. As a consequence of the discussion on the choice of the best version of the theory of thin shells, a great number of concepts have been developed, often slightly different one from other (see e.g. [9, 10]). Strain and stress measures of a shell have been defined in various ways, still remaining compatible with the first approximation assumptions. Many works include, in particular, an analysis of the equations describing deformations, see e.g. [1, 2, 4, 7, 9, 12].

In the present paper an attempt is made to compare some of the known and most important descriptions of a state of strain in a shell with Kirchhoff-Love constraints. The purpose of the work is to discuss relationships between the quantities defined by various investigators as strain measures and to call attention to the geometric interpretation of approximations used by these writers.

This work will often utilizes the parallel shift idea. It is thought appropriate, despite the simple structure of Euclidean space, to prove that operators of the shift (shifters) which are known from the literature (see [12, 15, 18]) are resolvents of the differential equations of parallel transfer. The method of generalized Taylor series employed by N. A. Kilchevsky [3, 4] and W. T. Koiter [9] refers to the parallel shift techniques. This work will prove validity of the latter method and point to its geometric substance.

As a result of parallel displacement of the tensor of the state of strain in the neighbourhood of any point $S(u^1, u^2, u^3)$ of the shell to the base in the point $P(u^1, u^2, 0)$ on the middle surface, the tensor $\bar{\gamma}_{ij}$ is obtained, which describes extensions and a change of the angle between material fibres of the shell parallel to lines u^1, u^2 on the middle surface and to the normal u^3, respectively. Introducing the latter tensor to geometric considerations makes it possible to analyse the state of strain independent of changes in the normal coordinate system across the thickness of the shell. Comments known about this problem (see e.g. [4]) are modest, despite its basic charakter and essential importance in formulating equation which describe deformations of the shell.

In the present paper a further attempt has also been undertaken to analyse the influence of the first approximation assumption $h/R \ll 1$ (h—the thickness of the shell, R—the smallest radius of curvature of the middle surface of the shell) on the form of equations defining strain measures. The works of V. Z. Vlasov, V. V. Novozhilov, W. T. Koiter, A. E. Green and W. Zerna are examples of results obtained by utilizing the above assumption. The physical consequences of the assumption $h/R \ll 1$ which appear in the form of the tensor of changes of curvature have been emphasised.

1. Notation, geometric relations

The convected notation is used throughout the text. The configuration of the shell is referred to the normal coordinate system u^k ($k=1, 2, 3$). Quantities referred to arbitrary points of the shell are distinguished by capital letters from quantities

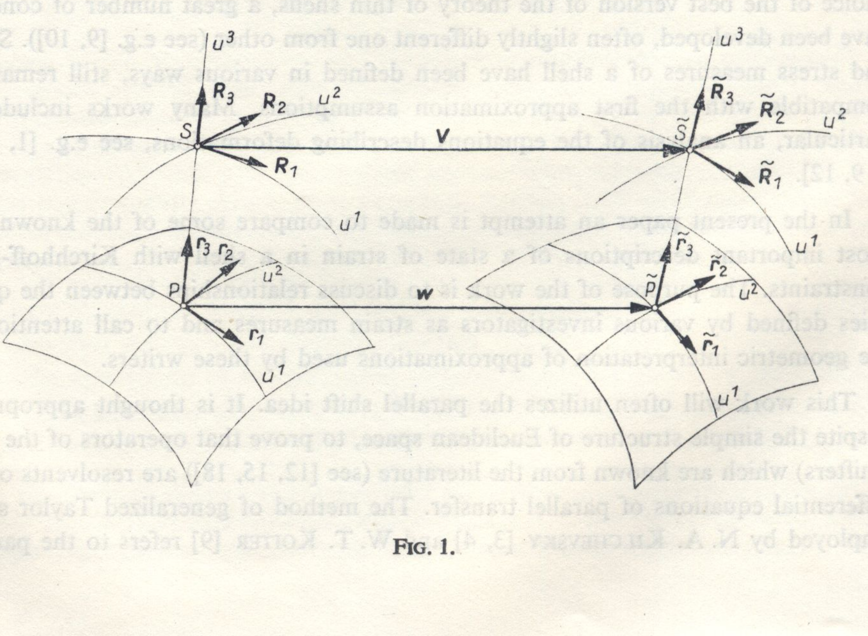

FIG. 1.

connected with the base $\mathbf{r}_i$ $(i=1, 2, 3)$ in the point $P(u^1, u^2, 0)$ on the middle surface. Therefore, the base in the point $S(u^1, u^2, u^3)$ is denoted by $\mathbf{R}_k$. Objects which characterize an actual configuration are distinguished by "$(\tilde{\ })$" from objects "( )" related to the initial, undeformed configuration. In the paper we denote by: A^σ_α — shifters [12, 15]; $g_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$—components of the first, second and third fundamental form of a middle surface; $e_{\alpha\beta}$—Ricci tensor related to the base $\mathbf{r}_\alpha$, $\alpha=1, 2$, Γ^i_{jk}, $\hat{\Gamma}^i_{jk}$—Christoffel symbols of the sec ond kind related to the bases r_k and R_k, respectively; $()\|_\alpha$, $()|_k$, $()\uparrow_k$—covariant derivatives related to the bases $\mathbf{r}_\alpha$ $(\alpha=1, 2)$ $\mathbf{r}_k$ and $\mathbf{R}_k$, $k=1, 2, 3$, $\mathbf{V}(u^1, u^2, u^3)$, $\mathbf{w}(u^1, u^2)$—displacement vectors of the points $S(u^1, u^2, u^3)$ and $P(u^1, u^2, 0)$, (Fig. 1); $()^*$—physical components of vectors and tensors. The infinitesimal rotation vector of the neighbourhood of the point $P(u^1, u^2, 0)$ is denoted by $\mathbf{\Omega}$

$$\mathbf{\Omega}=\frac{1}{2}(\mathrm{Rot}\,\mathbf{V}(u^1, u^2, u^3))_{u^3=0}.$$

We shall use the following known [7, 12] geometric relations which hold true in a normal coordinate system:

$$(1.1)\qquad \mathbf{R}_\alpha=A^\sigma_\alpha\,\mathbf{r}_\sigma,\qquad \mathbf{R}_3=\mathbf{r}_3,\qquad A^\sigma_\alpha=\delta^\sigma_\alpha-u^3\,b^\sigma_\alpha,\qquad g_{\alpha3}=0,\qquad g_{33}=1;$$

$$(1.2)\qquad \begin{aligned} &G_{\alpha\beta}=A^\sigma_\alpha A^\delta_\beta g_{\sigma\delta},\qquad G_{\alpha3}=0,\qquad G_{33}=1,\qquad \alpha,\beta,\sigma\delta=1,2,\\ &G^{\alpha\beta}=(A^{-1})^\alpha_\sigma (A^{-1})^\beta_\delta g^{\sigma\delta},\qquad A^\alpha_\sigma (A^{-1})^\sigma_\beta=\delta^\alpha_\beta,\qquad g=\det g_{\alpha\beta};\end{aligned}$$

$$(1.3)\qquad (A^{-1})^\alpha_\beta=(\delta^\alpha_\beta+u^3\,d^\alpha_\beta)/A,\qquad d^\alpha_\beta=b^\alpha_\beta-b^\sigma_\sigma\,\delta^\alpha_\beta,\qquad A=\det A^\alpha_\beta;$$

$$(1.4)\qquad \begin{aligned} &\hat{\Gamma}^\gamma_{\alpha\beta}=\Gamma^\gamma_{\alpha\beta}-u^3\,(A^{-1})^\gamma_\sigma\, b^\sigma_\alpha\|_\beta,\qquad \hat{\Gamma}^3_{3k}=0,\qquad k=1,2,3,\\ &\hat{\Gamma}^3_{\alpha\beta}=A^\sigma_\alpha\, b_{\sigma\beta},\qquad \hat{\Gamma}^\alpha_{3\beta}=-(A^{-1})^\alpha_\sigma\, b^\sigma_\beta,\qquad \alpha,\beta,\sigma,\gamma=1,2;\end{aligned}$$

$$(1.5)\qquad B_{\alpha\beta}=A^\sigma_\alpha\, b_{\sigma\beta}.$$

Mainardi–Codazzi equations assume the form

$$(1.6)\qquad b^\gamma_{[\alpha}\|_{\beta]}=0,\qquad \alpha,\beta,\gamma=1,2,$$

where "[ ]" denotes skew symmetry.

2. State of strain in an arbitrary point of the shell

As the starting point for the analysis of the problem, we shall assume the derivations of strain measures of the shell given below.

2.1. Derivation of A. E. Love

The first derivation of equations determining the state of strain for an arbitrary shell in lines of curvature coordinates was given by A. E. Love (1888, see [1] as well).

We proceed to describe the derivation of A. E. Love in a tensor notation for an arbitrary curvilinear coordinate system on the middle surface of the shell (see e.g. [15] or [16] Section 4).

The Green–Saint Venant strain tensor is defined by the known formula (see e.g. [7])

$$\gamma_{ij}=\frac{1}{2}(\tilde{G}_{ij}-G_{ij}), \qquad i, j=1, 2, 3. \tag{2.1}$$

The Kirchhoff–Love kinematic assumptions lead to the following relations for the actual configuration of the middle surface and parallel surfaces (u^3=const), respectively:

$$\tilde{g}_{\alpha 3}=0, \qquad \tilde{g}_{33}=1, \qquad \tilde{G}_{\alpha\beta}=\tilde{A}^{\sigma}_{\delta}\,\tilde{A}^{\delta}_{\beta}\,\tilde{g}_{\sigma\delta}, \qquad \tilde{G}_{33}=1. \tag{2.2}$$

The formula (2.1) with the aid of Eqs. (1.1) and (1.2) takes the form

$$\gamma_{\alpha\beta}=\mathring{\gamma}_{\alpha\beta}+u^3\rho_{\alpha\beta}+(u^3)^2\mu_{\alpha\beta}, \qquad \gamma_{k3}=0, \qquad \alpha, \beta=1, 2 \qquad k=1, 2, 3, \tag{2.3}$$

where

$$\mathring{\gamma}_{\alpha\beta}=\frac{1}{2}(\tilde{g}_{\alpha\beta}-g_{\alpha\beta}), \qquad \rho_{\alpha\beta}=b_{\alpha\beta}-\tilde{b}_{\alpha\beta}, \qquad \mu_{\alpha\beta}=\frac{1}{2}(\tilde{c}_{\alpha\beta}-c_{\alpha\beta}). \tag{2.4}$$

Formulae (2.3) have a two-point character, what discloses appearently by the functions $f_\alpha(u^1, u^2, u^3)$ when the physical components are computed:

$$\gamma^{*}_{\alpha\beta}=(f_\alpha f_\beta)^{-1}\,[\mathring{\gamma}^{*}_{\alpha\beta}+u^3\rho^{*}_{\alpha\beta}+(u^3)^2\mu^{*}_{\alpha\beta}], \tag{2.5}$$

where

$$f_\alpha(u^1, u^2, u^3)=\sqrt{G_{\alpha\alpha}/g_{\alpha\alpha}}.$$

Formulae (2.5), after linearization in respect to displacements, are consistent with the equations of A. E. Love given in the lines of curvature coordinates. In point 3.2 of the present work we shall discuss a linearizaiton of the formula (2.5), given in the monograph [1], with respect to the variable u^3.

2.2. Derivation of V. Z. Vlasov

The linear part of the Green–Saint Venant tensor (2.1) can be expressed in terms of components of the displacement vector **V** with the aid of the known formula

$$\gamma_{ij}=V_{(i\uparrow j)}, \tag{2.6}$$

equivalent to the following equations:

$$\begin{aligned}\gamma_{\alpha\beta}&=A^{\sigma}_{(\alpha}\,v_{\sigma}|_{\beta)}, \qquad \gamma_{33}=v_{3,3},\\ \gamma_{\alpha 3}&=v_{\alpha,3}+\frac{1}{2}b^{\sigma}_{\alpha}(v_\sigma-u^3 v_{\sigma,3})\end{aligned} \tag{2.7}$$

derived in the case of the curvature coordinate system by V. Z. Vlasov [2] and in the general form given above—by P. M. Naghdi [12]

Equations (2.7) take the form (2.3) when the linear distribution for shifted displacements

$$v_k = w_k(u^1, u^2) + u^3 \beta_k(u^1, u^2), \quad k=1, 2, 3 \tag{2.8}$$

and the relations between the vectors $\mathbf{w}$ and β

$$\beta_\alpha = -w_3|_\alpha, \quad \beta_3 = 0, \quad \alpha = 1, 2 \tag{2.9}$$

compatible with the Kirchhoff–Love hypothesis are assumed.

Two derivations leading to the formulae (2.3) have been presented above. The derivation of Vlasov–Naghdi is more general than the previous one since the Kirchhoff–Love assumptions have been used at the end of the procedure. Equations (2.7), due to their generality, can constitute the starting point for the theory including the effect of transverse shear stresses σ_{k3} ($k=1, 2, 3$).

2.3. Parallel transfer method

Equations (2.3) determine the strain components related to the base $\mathbf{R}_k$ in an arbitrary point $S(u^1, u^2, u_0^3)$ of the shell.

In order to analyse the state of strain along the line u^3, one should refer the strain tensor to the certain base constant for all these points, for example to the base $\mathbf{r}_k$ in the point $P(u^1, u^2, 0)$

$$\boldsymbol{\gamma}(u^1, u^2, u^3) = \bar{\gamma}_{ij}\, \mathbf{r}^i \otimes \mathbf{r}^j, \tag{2.10}$$

where a tensor product of vectors has been denoted by the symbol "$\otimes$".

The parallel transfer of the tensor $\boldsymbol{\gamma}$ from the point $P(u^1, u^2, 0)$ to $S(u^1, u^2, u_0^3)$ along the line u^3 is described by the following set of differential equations:

$$d\gamma_{ij} - (\hat{\Gamma}^r_{js}\, \gamma_{ir} + \hat{\Gamma}^r_{is}\, \gamma_{rj})\, du^s = 0, \quad i, j, r, s = 1, 2, 3 \tag{2.11}$$

with the initial conditions

$$\gamma_{ij}(u^1, u^2, 0) = \bar{\gamma}_{ij}(u^1, u^2, u_0^3). \tag{2.12}$$

A solution of the problem (2.11) and (2.12) will be introduced briefly.

Some special properties of a normal coordinate system along the line u^3 make it possible to obtain an exact and simultaneously simple solution of the problem stated above. Considering that $du^1 = du^2 = 0$ along the line u^3 and utilizing the formulae (1.4) of Christoffel symbols of the second kind in a normal coordinate system, the set of equations (2.11) can be written as follows:

$$\begin{gathered}\frac{d\gamma_{\alpha\beta}}{du^3} + (A^{-1})^\delta_\sigma\, b^\sigma_\alpha\, \gamma_{\delta\beta} + (A^{-1})^\delta_\sigma\, b^\sigma_\beta\, \gamma_{\delta\alpha} = 0,\\ \frac{d\gamma_{3\alpha}}{du^3} + (A^{-1})^\delta_\sigma\, b^\sigma_\alpha\, \gamma_{\delta 3} = 0, \quad \frac{d\gamma_{33}}{du^3} = 0.\end{gathered} \tag{2.13}$$

The homogeneous equations (2.13) are satisfied by the functions

$$\gamma_{\alpha\beta}=A_\alpha^\sigma(u^1,u^2,u^3)\,A_\beta^\delta(u^1,u^2,u^3)\,\pi_{\sigma\delta}(u^1,u^2),$$

(2.14)

$$\gamma_{\alpha3}=A_\alpha^\sigma(u^1,u^2,u^3)\,\pi_\sigma(u^1,u^2),\qquad \gamma_{33}=\pi(u^1,u^2).$$

Considering the initial conditions (2.12) one obtains the solution

$$\gamma_{\alpha\beta}=A_\alpha^\sigma(u^1,u^2,u_0^3)\,A_\beta^\delta(u^1,u^2,u_0^3)\,\bar{\gamma}_{\sigma\delta}(u^1,u^2,u_0^3),$$

(2.15)

$$\gamma_{\alpha3}=A_\alpha^\sigma(u^1,u^2,u_0^3)\,\bar{\gamma}_{\sigma3}(u^1,u^2,u_0^3),\qquad \gamma_{33}=\bar{\gamma}_{33}(u^1,u^2,u_0^3).$$

the uniqueness of which can be proved on the basis of suitable theorems on differential equations. The inverse formulae of (2.15) we are interested in take the form

(2.16) $$\bar{\gamma}_{\alpha\beta}=(A^{-1})_\alpha^\sigma(A^{-1})_\beta^\delta\gamma_{\sigma\delta},\qquad \bar{\gamma}_{\alpha3}=(A^{-1})_\alpha^\sigma\gamma_{\sigma3},\qquad \bar{\gamma}_{33}=\gamma_{33}.$$

The shifters A_β^α defined with the aid of Eq. $(1.1)_3$ are resolvents of the parallel transfer equations. This fact was proved by N. A. KILCHEVSKY [4] in the case when the lines u^1, u^2 coincide with the lines of curvature.

We shall illustrate the physical meaning of the tensor $\bar{\gamma}_{ij}$ and Eqs. (2.16). To this end we shall introduce a new coordinate system $u^{k'}$ in the point $S(u^1,u^2,u^3)$ so that the base vectors $\mathbf{R}_k'$ are parallel to the base vectors $\mathbf{r}_k$ of the point $P(u^1,u^2,0)$ on the middle surface of the shell.

$$u^{\alpha'}=\delta_\alpha^{\alpha'}A_\sigma^\alpha u^\sigma,\qquad u^{3'}=u^3,\qquad \mathbf{R}_{\alpha'}=\delta_{\alpha'}^\alpha\mathbf{r}_\alpha,\qquad \mathbf{R}_{3'}=\mathbf{r}_3,$$

where $\delta_\alpha^{\alpha'}$—two-point Kronecker symbol.

The tensor $\gamma_{\alpha'\beta'}$ and the physical components—$\gamma_{\alpha'\beta'}^*$ related to the base $\mathbf{R}_{k'}$ are equal to $\bar{\gamma}_{\alpha\beta}$ and $\bar{\gamma}_{\alpha\beta}^*$, respectively. The physical components $\gamma_{\alpha'\beta'}^*$ express stretch and change of the angle between the material fibres of the shell parallel to the lines u^1, u^2 on the middle surface. The physical components $\gamma_{\alpha\beta}^*$ of the tensor $\gamma_{\alpha\beta}$ related to the base $\mathbf{R}_k$ determine extensions and change of the angle between the material fibres of the shell lying along the lines u^1, u^2 on the parallel surface $u^3=\text{const}$. In view of the mutual twist of the bases $\mathbf{R}_k$ and $\mathbf{r}_k$ (thus $\mathbf{R}_k$ and $\mathbf{R}_{k'}$ as well), the components $\gamma_{\alpha\beta}^*$ and $\gamma_{\alpha'\beta'}^*$ describe the strain in the same point $S(u^1,u^2,u^3)$ with respect to the different directions determined by the bases. If the directions assigned by the base vectors $\mathbf{R}_k$ and $\mathbf{R}_{k'}$ coincide, what holds true in the particular case of curvature coordinates, it is not difficult to prove that

(2.17) $$\gamma_{\alpha\beta}^*=\gamma_{\alpha'\beta'}^*\qquad(\gamma_{\alpha'\beta'}^*=\bar{\gamma}_{\alpha\beta}^*).$$

The equality (2.17) is not satisfied, however, when the middle surface of the shell is referred to the orthogonal noncurvature coordinate system.

2.4. Generalized Taylor series method (by N. A. Kilchevsky)

One of the general methods to reduce a three-dimensional boundary value problem to the two-dimensional one is based on the expansion of unknown functions which determine the motion of the body into the power series in u^3. This idea was made us of by A. Cauchy and S. Poisson in their first studies on the theory of plates. N. A. KILCHEVSKY applied this method to the theory of shells [3].

The discussed method permits to determine the state of strain in an arbitrary point of the shell $S(u^1, u^2, u^3)$ by means of a certain sequence of functions related to the middle surface. To this end we expand the tensor $\gamma(u^1, u^2, u^3)$ into the power series in u^3.

$$\gamma(u^1, u^2, u^3) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k \gamma}{\partial (u^3)^k} \right)_{u^3=0} (u^3)^k. \tag{2.18}$$

Using the differentiation rule of tensors and then referring the objects of the latter equation to the base on the middle surface, we have

$$\left[\bar{\gamma}_{ij} - \sum_{k=0}^{\infty} \frac{1}{k!} (\gamma_{ij} \uparrow \underbrace{_{33\dots3}}_{k \text{ times}})_{u^3=0} \cdot (u^3)^k \right] \mathbf{r}^i \otimes \mathbf{r}^j = 0. \tag{2.19}$$

Making use of the linear independence of the tensor products $\mathbf{r}_i \otimes \mathbf{r}_j$ we obtain the expansion in a generalized Taylor series:

$$\bar{\gamma}_{ij} = \sum_{k=0}^{\infty} \frac{1}{k!} (\gamma_{ij} \uparrow {}_{33\dots3})_{u^3=0} (u^3)^k. \tag{2.20}$$

The quantities $\bar{\gamma}_{ij}$ related to the base in the point $P(u^1, u^2, 0)$ (see Eq. (2.10)) determine the state of strain in the point $S(u^1, u^2, u^3)$. The formula (2.20) is the desired solution of the parallel tansfer along an arbitrary curve joining these two points; it is thus more general than the "operator" formulae (2.16) describing the solution of the parallel shift problem along the straight line u^3.

Making the tensor γ_{ij} dependent upon the displacements V_i by means of Eq. (2.6) and then taking into account the commutativeness of the covariant differentiation operators, it is possible to observe on the grounds of the formula (2.20) that the components $\bar{\gamma}_{ij}$ are determined by the sequence of functions of two variables

$$f_i^{(k)}(u^1, u^2) = (V_i \uparrow \underbrace{_{33\dots3}}_{k \text{ times}})_{u^3=0}, \qquad i=1, 2, 3; \qquad k=1, 2, \dots$$

referred to the point $P(u^1, u^2, 0)$. The functions f_i^k can be interpreted as degrees of freedom of the point P.

Let us compare the generalized Taylor series with the "operator" formulae (2.16) when the undeformed configuration of the shell is referred to a normal coordinate system. The Taylor series determines coefficients at the succesive powers of u^3 in terms of the functions $f_i^k(u^1, u^2)$. However, the "operator" equations together with the formulae (2.7) do not permit to determine these coefficients in a direct way. Unknown are the functions $v_i(u^1, u^2, u^3)$. Assuming the form of the functions f_i^k consistently with the kinematic Kirchhoff–Love constraints and using Eq. (2.3), one can prove that the series (2.20) is an expansion into the power series in u^3 of, the right hand side of Eq. (2.16).

3. Simplifications of the equations describing deformation in the first approximation theory of shells

3.1. State of strain of a shell due to N. A. Kilchevsky. Reference to the results of V. Z. Vlasov

The concept of a N. A. Kilchevsky of describing the state of strain across the shell is based on the generalized Taylor series idea (2.20). Retaining the three first terms in this series we have

$$\bar{\gamma}_{\alpha\beta}=\mathring{\gamma}_{\alpha\beta}+u^3\,\bar{\rho}_{\alpha\beta}+(u^3)^2\,\bar{\mu}_{\alpha\beta}, \tag{3.1}$$

where

$$\bar{\rho}_{\alpha\beta}=(\gamma_{\alpha\beta}\uparrow 3)_{u^3=0}, \qquad \bar{\mu}_{\alpha\beta}=\frac{1}{2}\,(\gamma_{\alpha\beta}\uparrow 33)_{u^3=0}. \tag{3.2}$$

To obtain the formula given above, the Kirchhoff–Love assumptions $(2.3)_2$ have been used.

Differentiating covariantly the tensor $\gamma_{\alpha\beta}$ according to the Eq. $(3.2)_2$ and taking into account Eq. (2.3), the formulae for the Christoffel symbols of the second kind (1.4) and the known relation (see W. T. Koiter [16] p. 16 Eq. (4.6))

$$\bar{\rho}_{\alpha\beta}=\rho_{\alpha\beta}+2b^{\gamma}_{(\alpha}\,\mathring{\gamma}_{\alpha\beta)}, \tag{3.3}$$

one obtains an important(¹) relationship between the tensors $\bar{\mu}_{\alpha\beta}$, $\bar{\rho}_{\alpha\beta}$ and $\mathring{\gamma}_{\alpha\beta}$ as follows:

$$\bar{\mu}_{\alpha\beta}=b^{\sigma}_{(\alpha}\,\bar{\rho}_{\delta\beta)}+[c^{\sigma}_{(\alpha}\,\delta^{\delta}_{\beta)}-b^{\sigma}_{\alpha}\,b^{\delta}_{\beta}]\,\mathring{\gamma}_{\sigma\delta}. \tag{3.4}$$

We shall introduce the known (see e.g. [12] or [6], equations (4.4.6), (4.4.13), (4.4.16)) unsymmetric strain tensors which will be necessary in further considerations:

$$\eta_{\alpha\beta}=w_{\alpha}|_{\beta}, \qquad \kappa_{\alpha\beta}=\beta_{\alpha}\|_{\beta}. \tag{3.5}$$

The tensor $\kappa_{\alpha\beta}$ assumes the form

$$\kappa_{\alpha\beta}=-w\|_{\alpha\beta}-b^{\gamma}_{\alpha}\,\eta_{\gamma\beta}-b^{\gamma}_{\beta}\|_{\alpha}w_{\gamma}-c_{\alpha\beta}\,w, \qquad w\equiv w_3. \tag{3.6}$$

The latter equation follows from substituting β_{α} in Eq. $(3.5)_2$ by means of the formula (2.9).

The infinitesimal rotation vector $\mathbf{\Omega}$ is determined by the expression (see e.g. [5], Eq. (6.40.6))

$$\mathbf{\Omega}=e^{\alpha\beta}\,\beta_{\alpha}\,\mathbf{r}_{\beta}-\Phi\mathbf{r}_3, \tag{3.7}$$

where

$$\Phi=\frac{1}{2}\,e^{\alpha\beta}\,\eta_{\alpha\beta} \tag{3.8}$$

(¹) The relationship (3.4) allows to estimate directly the quantities $\mu_{\alpha\beta}$ without using compatibility equations (see [9], p. 19) is a measure of the rotation of a neighbourhood of the point $P(u^1, u^2, 0)$ on the middle surface around the normal.

The tensor $\bar{\rho}_{\alpha\beta}$ $(3.2)_1$ can be expressed as follows (see [9] p.3.2)

$$\bar{\rho}_{\alpha\beta}=\kappa_{(\alpha\beta)}-b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi+b^{\gamma}_{(\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta)} \tag{3.9}$$

or, utilizing Eq. (3.6) in the form

$$\bar{\rho}_{\alpha\beta}=-w\|_{\alpha\beta}-b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma}-2b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi-c_{\alpha\beta}\, w. \tag{3.10}$$

The first term in this equation has a plate character and is independent of the tangent displacements w_1 and w_2 and the exterior geometry of the middle surface. The second term describes an influence of tangent displacements of the middle surface on the change of curvature of this surface. In the case when the curvature tensor $b_{\alpha\beta}$ is constant (covariantly), what is satisfied, say for a spherical shell, this term vanishes identically. The third term in Eq. (3.10) determines an effect of the rigid rotation of the neighbourhood of the point on the middle sufrace around the normal on a change of a curvature. The last term expresses a change of curvature in the case of the uniform displacement w of the middle surface (compare the explanations by W. FLÜGGE [8] p.226, about the deformation of a cylindrical shell). The independence of the tensor $\bar{\rho}_{\alpha\beta}$ of deformations of the middle surface is worth mentioning. In the case of a spherical shell, the tensor $\bar{\rho}_{\alpha\beta}$ turns to be independent of the displacements w_1, w_2 (the second and third terms in Eq. (3.10) vanish).

Equations (3.1), (3.10) and (3.4) can be treated as a generalization on an arbitrary parametrization of the middle surface formulae given by V. Z. VLASOV ([2] pp. 221–223) in the lines of curvature, obtained on the basis of Eq. (2.5). The latter fact follows directly from the equality (2.17). Therefore, V. Z. Vlasov's measures of strain ([2] pp 221, 222, Eq. (7.11)) $-\kappa_{\alpha}$, $\tau/2$ and φ_{α}, $\psi/2$ are equal to the physical components of the tensors $\bar{\rho}_{\alpha\beta}$ and $\bar{\mu}_{\alpha\beta}$ respectively.

3.2. Assumptions of first-order similitude of geometries of the middle and parallel surfaces

In this section and the next one the consequences of certain assumptions will be discussed, which simplify the relationships between the tensors determining geometries of the middle surface and parallel surfaces (u^3=const), respectively. The approximations applied will be independent of a state of strain and displacement as they will be concerned with the geometry of the underformed configuration.

The following approximations will be treated as a first-order similarity of geometries of the middle and parallel surfaces respectively:

$$A^{\sigma}_{\alpha}\approx\delta^{\sigma}_{\alpha}. \tag{3.11}$$

The latter equation involves such simplifications:

$$\begin{gathered} R_{\alpha}\approx r_{\alpha}, \quad G_{\alpha\beta}\approx g_{\alpha\beta}, \quad G^{\alpha\beta}\approx g^{\alpha\beta}, \quad B_{\alpha\beta}\approx b_{\alpha\beta}, \\ \hat{\Gamma}^{\gamma}_{\alpha\beta}\approx \Gamma^{\gamma}_{\alpha\beta}-u^3\, b^{\gamma}_{\alpha}\|_{\beta}, \quad \hat{\Gamma}^{3}_{\alpha\beta}\approx b_{\alpha\beta}, \quad \hat{\Gamma}^{\alpha}_{3\beta}\approx b^{\alpha}_{\beta}. \end{gathered} \tag{3.12}$$

Equations (1.1)–(1.5) were utilized to obtain Eq. (3.12). In the case when the middle surface is referred to curvature coordinates, the latter simplifications are equivalent to neglecting terms of the type u^3/R (R—the smallest radius of curvature) small as compared with unity in the process of formulating the equations describing deformations of the shell. Recalling Eqs. (2.16) and (3.11), we have

$$\gamma_{ij} \approx \bar{\gamma}_{ij}. \tag{3.13}$$

Thus it is not purposeful to distinguish the tensors γ_{ij} and $\bar{\gamma}_{ij}$. One can prove that the formula (2.3) is equivalent to the following relationships between $\gamma_{\alpha\beta}$ and the tensors (3.4) (see e.g. [12] or [6] (4.12), (4.15))

$$\gamma_{\alpha\beta} = A^{\sigma}_{(\alpha}\, \eta_{\sigma\beta)} + u^3 A^{\sigma}_{(\alpha}\, \kappa_{\sigma\beta)}. \tag{3.14}$$

Substituting Eq. (3.5) in Eq. (3.14) and considering Eq. (3.11) we obtain

$$\gamma_{\alpha\beta} \approx \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \rho^{\mathrm{I}}_{\alpha\beta}, \tag{3.15}$$

$$\rho^{\mathrm{I}}_{\alpha\beta} = -w\|_{\alpha\beta} - b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma}. \tag{3.16}$$

Hence, considering Eqs. (3.13) and (3.15) we observe that the distribution of deformations across the thickness of the shell is represented by a linear expansion in u^3. The symmetry of the tensor $\rho^{\mathrm{I}}_{\alpha\beta}$ follows from the Mainardi-Codazzi equation (1.6) and from the commutativeness of covariant differentiation (symmetry of $\bar{\rho}_{\alpha\beta}$ followed from its definition (3.2).

Addition of terms of the type $\pm b^{\gamma}_{(\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta)}$ to the tensor $\rho^{\mathrm{I}}_{\alpha\beta}$ is negligible in view of the approximations (3.11)

$$\gamma^{1}_{\alpha\beta} = \overset{\circ}{\gamma}_{\alpha\beta} + u^3 (\rho^{\mathrm{I}}_{\alpha\beta} \pm b^{\gamma}_{\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta}) = (\delta^{\gamma}_{\alpha} \pm u^3 b^{\gamma}_{\alpha})\, \overset{\circ}{\gamma}_{\gamma\beta} + u^3 \rho^{\cdot}_{\alpha\beta} \approx \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \rho^{\mathrm{I}}_{\alpha\beta} = \gamma_{\alpha\beta}. \tag{3.17}$$

Analogously, a negligible effect is produced by adding to the tensor $\rho^{\mathrm{I}}_{\alpha\beta}$ terms of the type $b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi$

$$\gamma^{1}_{\alpha\beta} = \overset{\circ}{\gamma}_{\alpha\beta} + u^3 (\rho^{\mathrm{I}}_{\alpha\beta} + b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi) \approx A^{\gamma}_{\alpha}\, \eta_{\gamma\beta} + u^3 (\rho^{\mathrm{I}}_{\alpha\beta} + b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi + b^{\gamma}_{(\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta)}) = \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \rho^{\mathrm{I}}_{\alpha\beta} \approx \gamma_{\alpha\beta}, \tag{3.18}$$

where Eqs. (3.11) and (3.17) have been used. In the same way one can prove that the terms $c_{\alpha\beta}\, w$ can be omitted.

3.2.1. Comparison of the formulae (3.15), (3.16) *with the results of A. E. Love*

The approximation (3.11) was first employed in the work of A. E. Love [1] and then repeated by many investigators (see e.g. V. V. Novozhilov [6] p. 31 (4.30, 4.31). The formulae of A. E. Love have the form

$$\begin{aligned} \gamma_{\alpha\alpha} &= \overset{\circ}{\gamma}_{\alpha\alpha} + u^3 \kappa_{\alpha\alpha} \quad (\alpha \text{ not summed}), \\ \gamma_{12} &= \overset{\circ}{\gamma}_{12} + u^3 \rho_{12}, \end{aligned} \tag{3.19}$$

where $\kappa_{\alpha\alpha}$ and ρ_{12} have been defined above by Eqs. (3.6) and $(2.4)_2$ respectively. Making use of Eqs. (3.3), (3.6), (3.10) and (3.16) it is easy to prove that

$$(3.20) \quad \begin{aligned} &\rho^{\mathrm{I}}_{\alpha\alpha}-\kappa_{\alpha\alpha}=b^{\gamma}_{\alpha}\,\gamma_{\gamma\alpha}+b^{\gamma}_{\alpha}\,e_{\gamma\alpha}\,\Phi+c_{\alpha\alpha}\,w \qquad (\alpha \text{ not summed}),\\ &\rho^{\mathrm{I}}_{12}-\rho_{12}=2b^{\gamma}_{(1}\,\overset{\circ}{\gamma}_{\gamma 2)}+2b^{\gamma}_{(1}\,e_{\gamma 2)}\,\Phi+c_{12}\,w\,. \end{aligned}$$

With the aid of the previous results (3.17), (3.18), one observes that the terms of the right hand sides of (3.20) can be omitted. Assuming the formulae (3.19) instead of Eq. (3.15), the tensorial invariance of (3.15), (3.16) is lost. We mention here, however, that the tensors $\rho^{\mathrm{I}}_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ do not vanish identically under rigid motion of the shell. This property characterizes instead, the tensors $\overset{\circ}{\gamma}_{\alpha\beta}$, $\rho_{\alpha\beta}$ and $\bar{\rho}_{\alpha\beta}$.

3.3. *The assumption of second-order similitude of geometries of the middle surface and parallel surfaces ($u^3 = const$)*

Let us suppose that the relations (3.11), (3.12) hold true. Additionally we assume that

$$(3.21) \qquad \hat{\Gamma}^{\gamma}_{\alpha\beta}\approx\Gamma^{\gamma}_{\alpha\beta}-u^3\,b^{\gamma}_{\alpha}\|_{\beta}\approx\Gamma^{\gamma}_{\alpha\beta}\,.$$

The formula (3.15) can be written as follows:

$$(3.22) \qquad \gamma_{\alpha\beta}\approx w_{(\alpha,\,\beta)}-(\Gamma^{\gamma}_{\alpha\beta}+u^3\,b^{\gamma}_{\alpha}\|_{\beta})\,w_{\gamma}-b_{\alpha\beta}\,w+u^3\,(-w\|_{\alpha\beta})\,.$$

With the aid of the approximations (3.21) we obtain

$$(3.23) \qquad \gamma_{\alpha\beta}\approx\overset{\circ}{\gamma}_{\alpha\beta}+u^3\,\rho^{\mathrm{II}}_{\alpha\beta}\,,\qquad \rho^{\mathrm{II}}_{\alpha\beta}=-w\|_{\alpha\beta}\,.$$

The approximations (3.11) and (3.21) are analogous to those of the theory of shallow shells (see [7], Sect. 11.3). The tensor $\rho^{\mathrm{II}}_{\alpha\beta}$ has been expressed by an analogous formula as a tensor of changes of curvature of a shallow shell (see [7], Point 11.4, Formula 11.4.6).

A. E. Green and W. Zerna [7] assume for the tensor of flexible deformation for a shell of an arbitrary shape, the considerable simplified version $(3.23)_2$. The derivation described above, leading to Eqs. (3.23), provides a geometric interpretation of the approximations utilized by the authors of the monograph [7].

3.4. *Linearity of strain distribution across the thickness of the shell*

As it was pointed out in the Sect. 3.2, the linearity of strain distribution (3.15) follows directly from the assumptions (3.11). We now proceed to prove that the linear distribution of deformations described by means of the equation

$$(3.24) \qquad \bar{\gamma}_{\alpha\beta}=\overset{\circ}{\gamma}_{\alpha\beta}+u^3\,\bar{\rho}_{\alpha\beta}$$

is consistent with considering the assumption $h/R \ll 1$ during the derivation of constitutive equations (since the assumption $h/R \ll 1$ refers to the approximation of constitutive equations).

We know from the literature (see e.g. [12] Eq. (6.14), (6.15)) that the constitutive equations of the theory of elastic shells, isotropic and homogeneous across the thickness can be reduced, with the aid of Eqs. (2.16) and (3.14), to the following form:

$$N^{\alpha\beta}=E^{\nu\beta\sigma\delta}\int_{-\frac{h}{2}}^{\frac{h}{2}}(\delta^{\alpha}_{\nu}+u^3\, d^{\alpha}_{\nu})\,\bar{\gamma}_{\sigma\delta}\,du^3,$$

(3.25)

$$M^{\alpha\beta}=E^{\nu\beta\sigma\delta}\int_{-\frac{h}{2}}^{\frac{h}{2}}(\delta^{\alpha}_{\nu}+u^3\, d^{\alpha}_{\nu})\,\bar{\gamma}_{\sigma\delta}\,u^3\,du^3,$$

where by $E^{\nu\beta\sigma\delta}$ the tensor of elastic moduli has been denoted; d^{α}_{β} has been defined by Eq. $(1.3)_2$. Substituting the formulae (3.1) in Eq. (3.25), integrating with respect to u^3 from $-1/2\ h$ to $1/2\ h$ and making use of the assumption $h/R \ll 1$, we obtain

$$N^{\alpha\beta}=hE^{\alpha\beta\sigma\delta}\,\overset{\circ}{\gamma}_{\sigma\delta},\quad M^{\alpha\beta}=\frac{h^3}{12}\,E^{\nu\beta\sigma\delta}\,\rho^{\alpha}_{\nu\sigma\delta},\tag{3.26}$$

where

$$\rho^{\alpha}_{\nu\sigma\delta}=\delta^{\alpha}_{\gamma}\,\bar{\rho}_{\sigma\delta}+d^{\alpha}_{\gamma}\,\overset{\circ}{\gamma}_{\sigma\delta}\tag{3.27}$$

is an „exact" expression of the tensor of flexible deformation in the first-approximation theory(²). Substituting the linear equations (3.24) in Eqs. (3.25), we obtain relationships between stress resultants and strain measures which are identical with Eqs. (3.26). Thus we have proved the correctness of the assumption (3.24) in the first approximation theory of shells.

3.5. Equations describing deformations in the „improved" first approximation theories

We shall discuss briefly equations describing the state of strain of two base versions of the first approximation theories, in which the following conditions hold true: tensors of stress and couple resultants as well as strain and change of curvature tensors are symmetric, constitutive equations are not coupled, the sixth equation of equilibrium is satisfied identically and the rigid motion of the shell does not cause a state of stress in the shell.

3.5.1. The theory of P. M. Naghdi [13], J, L. Sanders and B. Budiansky [11]

In the papers [11, 13] the measure $\rho_{\alpha\beta}$ $(2.4)_2$ was assumed as a tensor of flexible deformation namely

$$\rho_{\alpha\beta}=-w\|_{\alpha\beta}-b^{\gamma}_{\alpha}\|_{\beta}\,w_{\gamma}-2b^{\gamma}_{(\alpha}\,e_{\gamma\beta)}\,\Phi-2b^{\gamma}_{(\alpha}\,\overset{\circ}{\gamma}_{\gamma\beta)}-c_{\alpha\beta}\,w\,.\tag{3.28}$$

(²) P. M. Naghdi states that the assumption $h/R \ll 1$ involves the simplification $\rho^{\alpha}_{\nu\sigma\delta}=\delta^{\alpha}_{\nu}\,\varkappa_{(\sigma\delta)}$. In our opinion, the omission of terms of the type Φ/R and ε/R requires an additional justification.

The above equation follows from Eqs. (3.3), (3.10). In view of Eqs. (3.28) and (3.3) one observes that the tensor $\rho_{\alpha\beta}$ differs in terms of the type ε/R from $\bar{\rho}_{\alpha\beta}$.

3.5.2. Theory due to W. T. Koitar [9], J. L. Sanders and B. Budiansky [10, 11]

In the paper [9] it was pointed out, on the basis of an analysis od strain energy per unit volume of a shell, that the terms of type ε/R are unimportant in the definition of the tensor of changes of curvature. Making use of this „free choice", the following expression for the tensor of flexible deformation was assumed ([9, 10, 11], see [14] as well)

$$\hat{\rho}_{\alpha\beta}=\kappa_{(\alpha\beta)}-b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi\,, \tag{3.29}$$

which differs from $\bar{\rho}_{\alpha\beta}$ Eq. (3.9) in the term $b^{\gamma}_{(\alpha}\, \mathring{\gamma}_{\gamma\beta)}$.

With the aid of Eq. (3.10) we have

$$\hat{\rho}_{\alpha\beta}=-w\|_{\alpha\beta}-b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma}-2b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi-b^{\gamma}_{(\alpha}\, \mathring{\gamma}_{\gamma\beta)}-c_{\alpha\beta}\, w\,. \tag{3.30}$$

As it is possible to observe from Eqs. (3.29) and $(3.5)_2$, the tensor $\hat{\rho}_{\alpha\beta}$ depends explicitly only on the components of the vector $\mathbf{\Omega}$ (3.7).

The tensors $\rho_{\alpha\beta}$ (3.28) and $\hat{\rho}_{\alpha\beta}$ (3.30) differ in the terms of the type ε/R which can be neglected according to W. T. Koiter's arguments [9] in the fiist approximation theory (see remarks of P. M. Naghdi [13], Sect. 6, p. 521).

4. Conclusions

The methods of describing the state of strain in a shell discussed in point 2 of the present paper can be appreciated considering their usefulness in a certain theory or estimating their possibility of generalization. The derivation presented as a generalization of the one given by A. E. Love is clear and brief. The formulae (2.3) have a simple form, the tensors $\mathring{\gamma}_{\alpha\beta}$ and $\rho_{\alpha\beta}$ have an explicit physical interpretation following directly from the definition (2.4). The tensor $\mu_{\alpha\beta}$ can be treated as the set of components of strain of Gauss sphere related to the given point of the middle surface of the shell.

The certain disadvantage of the latter derivation, and simultaneously, the reason of its simple form is the assumption to refer the actual configuration to a normal coordinate system, see Eq. (2.2). This fact restricts the application of this derivation to the theories with Kirchhoff–Love constraints.

The method of V. Z. Vlasov and P. M. Naghdi leading to Eqs. (2.3) does not include any additional restrictions on a deformation up till the „exact" formulae (2.7) are obtained. The latter formulae can be utilized for the formulation of the theory of shells with transverse shear deformations.

The essential role in the theory of shells is fulfilled by the tensor $\bar{\gamma}_{\alpha\beta}$ which describes extensions and a change of the angle between material fibres of a parallel surface (u^3=const), parallel to the base vectors $\mathbf{r}_1$, $\mathbf{r}_2$ related to the middle surface. The components $\bar{\gamma}_{\alpha\beta}$ can be obtained by solving the set of differential equations of the parallel transfer or by means of the generalized Taylor series. If the configuration of the shell is referred to a normal coordinate system, the shifters A^{α}_{β} or their inverses are resolvents of the set of differential equations of the parallel transfer. Then it is needless to employ the generalized Taylor series.

Introducing into our considerations the tensor $\bar{\gamma}_{\alpha\beta}$ allows for an examination of strain distribution along the line u^3, independent of changes in the normal coordinate system across the thickness of the shell. The tensor $\bar{\gamma}_{\alpha\beta}$ was employed in the formulation of the assumption (3.24) of linear distribution of strain across the thickness of the shell.

The tensor $\bar{\rho}_{\alpha\beta}$, occurring in Eq. (3.24), is difficult to associate with the name of a single worker. As it has already been mentioned previously, the physical components of this tensor in the case of curvature coordinates are equal to the measures κ_α, $\tau/2$ of V. Z. Vlasov. The tensorially invariant definition $(3.2)_1$ was given indepently by N. A. Kilchevsky [3] and W. T. Koiter [9]. The independence of this tensor of deformations of the middle surface of the shell is worth mentioning.

In point 3.2 of the present paper the effect of the assumption (3.11) of the first-order similitude between geometries of parallel surfaces (u^3=const) and the middle surface was examined. In the next point 3.3 an effect of the powerful assumptions of second-order similarity (3.11), (3.21) on strain distribution was discussed. The assumptions (3.11) involve the independence of the tensor $\rho^{\mathrm{I}}_{\alpha\beta}$ (3.16) of the rigid rotation Φ of an element of the middle surface around the normal and of an effect of a uniform displacement $w_3=w$ of points of the middle surface (compare Eqs. (3.16) and (3.10)).

Introducing simultaneously the assumptions (3.11) and (3.21), which have a form analogous to the simplifications employed in the theory of shallow shells, provides the independence of the tensor $\rho^{\mathrm{II}}_{\alpha\beta}$, Eq. $(3.23)_2$ of the tangent displacements w_α and of the second fundamental form of the middle surface. The derivation leading to Eqs. (3.23) can be treated as an attempt to state the known derivation of A. E. Green and W. Zerna ([7], p. 106) more precisely.

The assumptions of the first and second-order similitude Eqs. (3.11) and (3.11), (3.21) are equivalent in the particular case of a spherical shell.

It is not indifferent, whether the assumption of first-approximation $h/R \ll 1$ are applied to the derivation of strain measures or to the formulation of constitutive equations (i.e. after integrating Eq. (3.25)).

The first order approximation of the constitutive equations (3.25) permits to assume the linear distribution of strain across the thickness of the shell (3.24).

The approximations of the equations describing deformation of a shell discussed in points 3.2 and 3.3 involve vanishing differences between the tensors $\gamma_{\alpha\beta}$ and $\bar{\gamma}_{\alpha\beta}$ lead to the linearization of Eq. (3.1) in respect to u^3 (compare Eqs. (3.19) and (3.23)) and modify the tensor $\bar{\rho}_{\alpha\beta}$, Eqs. (3.16) and $(3.23)_2$.

In the recapitulation we shall present discussed versions of description of strain in a shell recalling to this end some equations from the present paper.

Deformations in an arbitrary point of the shell, referred to the base in this point, are determined by the formula (see Eq. (2.3)).

$$\gamma_{\alpha\beta} = \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \rho_{\alpha\beta} + (u^3)^2 \mu_{\alpha\beta}$$

derived by A. E. LOVE [1] in the particular case of curvature coordinates and in the form given above by P. M. NAGHDI [12].

However, deformations in an arbitrary point of the shell referred to the base on the middle surface can be expressed by means of the tensor (3.1)

$$\bar{\gamma}_{\alpha\beta} = \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \bar{\rho}_{\alpha\beta} + (u^3)^2 \bar{\mu}_{\alpha\beta} + \dots$$

introduced to the shell theory by N. A. KILCHEVSKY [3].

The first-order similarity assumption (3.11) between geometries of parallel and middle surfaces leads to the following linear equation in respect to u^3:

$$\gamma_{\alpha\beta} \approx \bar{\gamma}_{\alpha\beta} \approx \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \overset{\mathrm{I}}{\rho}_{\alpha\beta},$$

which is equivalent, considering Eq. (3.11), to A. E. Love's approximation.

The second-order similarity assumptions (3.11), (3.21) involve the formula

$$\gamma_{\alpha\beta} \approx \bar{\gamma}_{\alpha\beta} \approx \overset{\circ}{\gamma}_{\alpha\beta} + u^3 \overset{\mathrm{II}}{\rho}_{\alpha\beta}$$

given in [7] by A. E. GREEN and W. ZERNA.

Tensors of changes of curvature depend in the following way on the components of the displacement vector of the middle surface:

$$\rho_{\alpha\beta} = -w\|_{\alpha\beta} - b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma} - 2b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi - c_{\alpha\beta}\, w - 2b^{\gamma}_{(\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta)},$$

$$\bar{\rho}_{\alpha\beta} = -w\|_{\alpha\beta} - b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma} - 2b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi - c_{\alpha\beta}\, w,$$

$$\overset{\mathrm{I}}{\rho}_{\alpha\beta} = -w\|_{\alpha\beta} - b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma}, \qquad \overset{\mathrm{II}}{\rho}_{\alpha\beta} = -w\|_{\alpha\beta}.$$

The tensors $\mu_{\alpha\beta}$ and $\overset{\circ}{\mu}_{\alpha\beta}$ can be expressed by means of $\overset{\circ}{\gamma}_{\alpha\beta}$, $\rho_{\alpha\beta}$, $\bar{\rho}_{\alpha\beta}$

$$\mu_{\alpha\beta} = -b^{\sigma}_{(\alpha}\, \rho_{\sigma\beta)} - b^{\sigma}_{\alpha}\, b^{\delta}_{\beta}\, \overset{\circ}{\gamma}_{\sigma\delta},$$

$$\bar{\mu}_{\alpha\beta} = b^{\sigma}_{(\alpha}\, \bar{\rho}_{\sigma\beta)} + [c^{\sigma}_{(\alpha}\, \delta^{\delta}_{\beta)} - b^{\sigma}_{\alpha}\, b^{\delta}_{\beta}]\, \overset{\circ}{\gamma}_{\sigma\delta}.$$

We mention, however, that some tensors of flexible deformation encountered in the literature, derived in a variational way, differ from the tensors $\rho_{\alpha\beta}$, $\bar{\rho}_{\alpha\beta}$, $\overset{\mathrm{I}}{\rho}_{\alpha\beta}$ in terms of the type Φ/R or ε/R.

For example, E. REISSNER [19] assumed for the flexible deformation tensor the symmetric part of $\kappa_{\alpha\beta}$ Eq. $(3.5)_2$.

$$\kappa_{(\alpha\beta)} = -w\|_{\alpha\beta} - b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma} - b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi - c_{\alpha\beta}\, w - b^{\gamma}_{(\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta)},$$

whereas W. T. KOITER [9], J. L. SANDERS and B. BUDIANSKY [11] obtained the tensor $\hat{\rho}_{\alpha\beta}$, Eq. (3.30) in the form

$$\hat{\rho}_{\alpha\beta} = -w\|_{\alpha\beta} - b^{\gamma}_{\alpha}\|_{\beta}\, w_{\gamma} - 2b^{\gamma}_{(\alpha}\, e_{\gamma\beta)}\, \Phi - c_{\alpha\beta}\, w - b^{\gamma}_{(\alpha}\, \overset{\circ}{\gamma}_{\gamma\beta)}.$$

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Streszczenie

METODY OPISU STANU ODKSZTAŁCENIA POWŁOK W LINIOWEJ TEORII TYPU KIRCHHOFFA-LOVE'A

W pierwszej części pracy przedstawiono i porównano kilka metod opisu stanu odkształcenia w powłoce cienkiej z więzami typu Kirchhoffa-Love'a. Omówiono między innymi sposób N. A. Kilczewskiego opisu odkształcenia w dowolnym punkcie powłoki za pomocą składowych tensora deformacji w bazie na jej środkowej powierzchni. Z matematycznego punktu widzenia jest to przesunięcie równoległe. Zwrócono uwagę na sens fizyczny tensora N. A. Kilczewskiego. Przedstawiono i przedyskutowano dwie postacie rozwiązania problemu przeniesienia równoległego, tj. wzory «operatorowe» oraz uogólniony szereg Taylora. W drugiej części pracy omówiono geometryczne i fizyczne konsekwencje założenia $h/R \ll 1$ pierwszego przybliżenia. Przeprowadzono analizę kilku wersji związków geometrycznych, w szczególności w odniesieniu do tensora deformacji zgięciowej.

Резюме

О РАЗЛИЧНЫХ МЕТОДАХ ОПИСАНИЯ ДЕФОРМАЦИИ В ЛИНЕЙНОЙ ТЕОРИИ ОБОЛОЧЕК ТИПА ЛЯВА-КИРХГОФФА

В первой части работы приводятся и сравниваются несколькие методы описания деформации в тонкой оболочке со связами типа Лява-Кирхгоффа. Внимание уделяется методу Н. А. Кильчевского, описывающему деформацию в произвольной точке посредством составляющих тензора деформации в базисе определенном на серединной поверхности. С математической точки зрения имеем здес дело с параллельным переносом Обращается внимание на физической смысл тензора Н. А. Кильчевского. Приводятся и рассматриваются два вида параллельного переноса т.е. „операторные" формулы и обобщенный ряд Тейлора. Во второй части работы рассматриваются геометрические физические следствия первого приближения при $h/R \ll 1$. Обсуждаются несколькие варианты геометрических зависимостей, в частности в отношении тензора изгибной деформации.

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