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Governing Differential Equations for the Mechanics of Undamageable Materials

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In this work the mathematical foundations of the mechanics of elastic undamageable materials are presented. In particular the governing differential equations are derived for both the scalar and tensorial cases. In the isotropic case it is found that the resulting scalar differential equations are simple and easy to solve. However, in the anisotropic case the tensorial differential equations are complicated and unsolvable at this time. The current work presents the solution in the form of explicit nonlinear stress-strain relations for the simple one-dimensional case. However, the general solution of the three-dimensional case remains unattainable at the present time. Only the governing tensorial differential equations are derived for this latter case.

It is to be noted that the term "undamageable" is reflected in the context of the material stiffness and not the property of indestructibility due to various loading conditions. Thus, the undamageable material reflects that no microcracks or microvoids occur as well as no plastic yielding in the material. To illustrate this concept, a last section is added on applications.

Key words: damage, damage mechanics, undamageable material, differential equation, scalar, tensor, rubber, biological tissue, tissue, soft material, metallic glass, metal rubber.

1. Introduction

In order to introduce the proposed undamageable material, it is necessary first to review some basic issues of Damage Mechanics. The concept of effective stress for uniaxial tension was first introduced by Kachanov [10] and Rabotnov [24]. It has been argued [15, 16] that the assumption of isotropic damage is sufficient to give good predictions of the load carrying capacity, the number of cycles or the time to local failure in structural components. However, the development of anisotropic damage has been confirmed experimentally [17, 28] even if the virgin material is isotropic. For the case of isotropic damage mechanics, the damage variable is scalar and the evolution equations are easy to handle [3, 6, 13, 14, 17, 20, 22, 23, 34]. The thermodynamics of these materials has been investigated by Hansen and Shreyer [9].

Assume $\boldsymbol{\sigma}$ is the second-rank Cauchy stress tensor and $\overline{\boldsymbol{\sigma}}$ the corresponding effective stress tensor applied to a fictitious state of the material which is totally undamaged, i.e. all damage in this state has been removed. This state is assumed to be mechanically equivalent to the actual damaged state of the material [11, 12, 31, 32, 37, 38]. In this regard, the hypothesis of elastic energy equivalence is usually used. The authors have shown the microstructural link of damage mechanics by characterizing micro-cracks through the use of fabric tensors [35, 36]. Damage in micro- and mesomechanics of composite materials has also been extensively investigated [18, 19].

It has been postulated by VOYIADJIS and KATTAN [39, 40] that a hypothetical material may exist or may be manufactured that cannot be damaged at all under any type of loading. Such a material will maintain zero value for the damage variable throughout the deformation process. The damage variable is considered within the framework of Continuum Damage Mechanics and utilizing the concept of effective stress. Such materials that undergo no damage whatsoever were termed Undamageable Materials. In VOYIADJIS and KATTAN [39, 40] it was shown that a higher-order strain energy function is associated with these types of materials. This function has the strain raised to the n-th power where n is a positive integer greater than 1. It was proved mathematically that as the value of the exponent n approaches infinity, the damage variable remains identically zero throughout the deformation process. In Sec. 2, one considers several exponents in details including $n = 2$, $n = 3$, and a general n and eventually consider the ultimate case as n goes to infinity.

In this work the mathematical foundations of the mechanics of elastic undamageable materials are presented. In particular the governing differential equations are derived for both the scalar and tensorial cases. In the isotropic case it is found that the resulting scalar differential equations are simple and easy to solve. However, in the anisotropic case the tensorial differential equations are complicated and unsolvable at this time. The current work presents the solution in the form of explicit nonlinear stress-strain relations for the simple one-dimensional case. However, the general solution of the three-dimensional case remains unattainable at the present time. Only the governing tensorial differential equations are derived for this latter case.

It is to be noted that the term "undamageable" is reflected in the context of the material stiffness and not the property of indestructibility due to various loading conditions. Thus, the undamageable material reflects that no microcracks or microvoids occur as well as no plastic yielding in the material. To illustrate this concept, a last section is added on applications. In this section, four different existing and/or recently manufactured materials are outlined such as rubber-like materials, soft biological tissues, metallic glass and metal rubber. These materials come very close in their properties to the postulated "hypothet-

ical" undamageable material. Finally, a most recent development is mentioned. The recent manufacture of self-healing "Terminator" polymeric material is also outlined.

2. Scalar differential equations for the one-dimensional case

In this section the governing scalar differential equations will be derived for elastic undamageable materials. These equations apply to the simple one-dimensional case. In general, the strain energy function for an elastic material in one dimension is given by:

$$U = \int \sigma \, d\varepsilon, \tag{2.1}$$

where σ is the stress and ε is the strain. Both these variables are measured with respect to the deformed state of the material.

2.1. Formulation for $n = 2$

One now considers a type of elastic material that has a nonlinear strain energy function of the form:

$$U = \frac{1}{2}\sigma\varepsilon^2, \tag{2.2}$$

where the strain is raised to the second power. In the subsequent parts of this section, one will consider similar strain energy functions with an increasing power of the strain ($n = 2, 3, ...$). It should be noted that researchers in mechanics did consider nonlinear strain energy functions for certain types of materials. For example Fung [7, 8] used an exponential strain energy function to model the elastic behavior of biological tissue.

Let the stress σ be a function of the strain only, i.e.

$$\sigma = f(\varepsilon) \equiv f. \tag{2.3}$$

Substituting Eq. (2.3) into Eqs. (2.1) and (2.2), then substituting Eq. (2.2) on the left-hand-side of Eq. (2.1), one obtains:

$$\frac{1}{2}f\varepsilon^2 = \int f \, d\varepsilon. \tag{2.4}$$

Differentiating Eq. (2.4) with respect to ε and simplifying, one obtains:

$$\frac{df}{d\varepsilon}\varepsilon^2 + 2f\varepsilon = 2f. \tag{2.5}$$

Re-writing the above equation in terms of the derivative, one obtains:

$$\frac{df}{d\varepsilon} = \frac{2f(1-\varepsilon)}{\varepsilon^2}. \tag{2.6}$$

The expression given in Eq. (2.6) is the governing scalar differential equation for this type of material ($n = 2$) where n is the power of ε in the expression of the strain energy function. The solution to the differential Eq. (2.6) is given by:

$$f = \frac{C}{\varepsilon^2} e^{-2/\varepsilon}, \tag{2.7}$$

where C is a constant of integration. Substituting the expression of f above into Eq. (2.3), one obtains the general scalar nonlinear stress-strain relationship for this type of material:

$$\sigma = \frac{C}{\varepsilon^2} e^{-2/\varepsilon}. \tag{2.8}$$

The above expression appeared in previous publications [39, 40] with E instead of C where E is the elastic modulus. However, it will be shown later that such a constant elastic modulus does not exist in the type of elastic materials under consideration here. This is because the behavior of these materials is highly nonlinear. In order to determine the constant C, one applies the initial conditions $\sigma = \sigma_0$ when $\varepsilon = \varepsilon_0$ to Eq. (2.8). Thus one obtains:

$$\sigma = \sigma_0 \Big(\frac{\varepsilon_0}{\varepsilon}\Big)^2 e^{[2/\varepsilon_0 - 2/\varepsilon]}. \tag{2.9}$$

Next, one plots the stress-strain relation of Eq. (2.8) in order to study the characteristics of the elastic behavior of this material. First a plot is shown in Fig. 1 where the values of the strain vary up to a value of 0.2. It is seen from this figure that the stress is almost zero in this range (the vertical axis is multiplied by 10^{-3}). In Fig. 2, the stress-strain curve is shown up to a strain of 0.5. It is seen that the stress-strain curve is highly nonlinear. It is also seen that a strengthening of the elastic modulus occurs when the strain exceeds about 0.2. Thus, in these types of materials one cannot utilize a constant modulus of elasticity. Another measure of the material behavior will be used as will be seen later.

One needs to investigate what happens to the stress-strain curve when the strain exceeds 0.5. Thus, one makes another plot of Eq. (2.8) for larger values of strain – up to 3.0. The resulting plot is shown in Fig. 3. It is seen that the stress reaches a maximum value when $\varepsilon = 1$, then starts decreasing until it approaches zero at infinity. There is another characteristic that is clearly evident from the stress-strain curves of Figs. 1, 2 and 3. These types of materials experience large

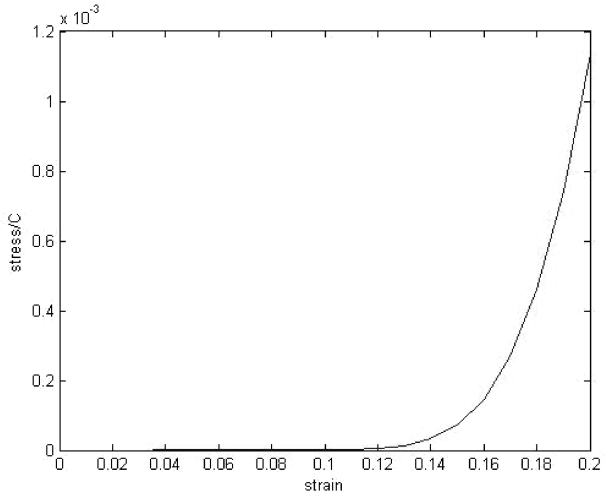

FIG. 1. Stress-strain curve for $n = 2$ (strain up to 0.2).

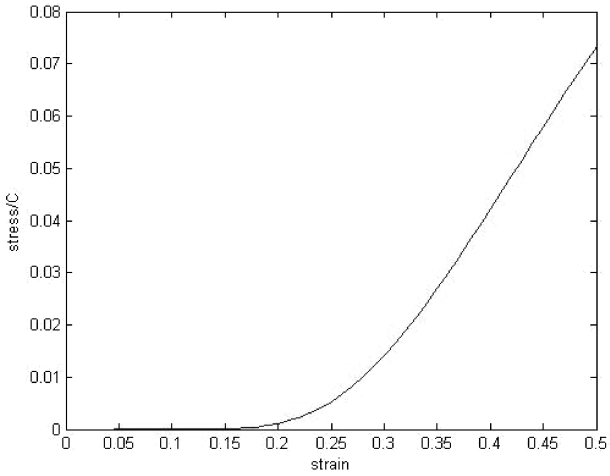

FIG. 2. Stress-strain curve for $n = 2$ (strain up to 0.5).

strains under relatively small values of stress. In particular, the stress remains almost zero until the strain reaches a value approximately 0.15 as seen from Fig. 2.

The maximum value of the stress in Fig. 3 can be computed by differentiating Eq. (2.8) and setting the derivative equal to zero. This maximum value turns out to be:

$$\sigma_{\max} = \frac{C}{e^2} \approx \frac{C}{7.39} \approx 0.135C. \tag{2.10}$$

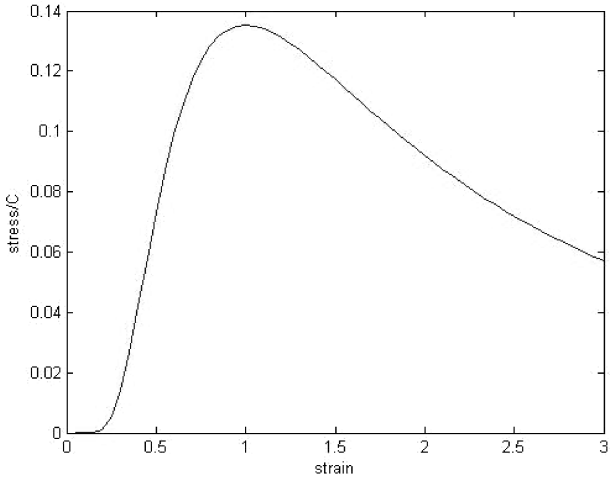

FIG. 3. Stress-strain curve for $n = 2$ (strain up to 3.0).

Since these types of materials experience large strains, let us solve Eq. (2.8) for the strain in order to find an explicit expression for the strain in terms of the stress. Thus, one obtains:

$$\varepsilon = -\frac{1}{W\left(\pm\sqrt{\sigma/C}\right)}, \tag{2.11}$$

where $W(x)$ is the Lambert W function [4]. This function is one of the special functions of mathematics.

As mentioned previously, one needs to find a suitable measure of the material parameters. FUNG [7, 8] used the slope of the stress-strain curve of biological tissue for this purpose. He plotted the slope vs. the stress and obtained a simple relation that he fitted first with a linear graph then with a quadratic graph. One will attempt to follow the same approach here for these types of materials.

Let us calculate first the slope of the stress-strain curve. This is accomplished by computing the derivative $\dfrac{d\sigma}{d\varepsilon}$ using Eq. (2.9). Thus one obtains:

$$\frac{d\sigma}{d\varepsilon} = 2\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\sigma. \tag{2.12}$$

Substituting for ε from Eq. (2.11) into Eq. (2.12) and simplifying, one obtains:

$$\frac{d\sigma}{d\varepsilon} = 2\left[W^2\left(\pm\sqrt{\sigma/C}\right) + W\left(\pm\sqrt{\sigma/C}\right)\right]\sigma. \tag{2.13}_1$$

The above expression presents an explicit relation between the slope $\dfrac{d\sigma}{d\varepsilon}$ and σ. Since the values of the stress are small, one tries to approximate the rela-

tion in Eq. (2.13) using the appropriate series expansions. It is more efficient computationally to re-write the above expression in the following form:

$$\frac{d\,(\sigma/C)}{d\varepsilon} = 2\left[W^2\left(\pm\sqrt{\sigma/C}\right) + W\left(\pm\sqrt{\sigma/C}\right)\right](\sigma/C)\,. \tag{2.13}_2$$

The Lambert W function has the following series expansion [4]

$$W(x) = x - x^2 + \frac{3}{2}x^3 - \ldots. \tag{2.14}$$

Using the first two terms only of Eq. (2.14) along with choosing the plus signs in Eq. (2.13), substituting the result into Eq. (2.13) and simplifying, one then obtains (using the plus signs in the expression of Eq. (13)):

$$\frac{d\,(\sigma/C)}{d\varepsilon} \approx 2\left(\frac{\sigma}{C}\right)\left[\left(\frac{\sigma}{C}\right)^2 + \sqrt{\frac{\sigma}{C}} - 2\frac{\sigma}{C}\sqrt{\frac{\sigma}{C}}\right]. \tag{2.15}$$

Equation (2.15) above can be further approximated by using the series expansion of the square root function as follows:

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \ldots. \tag{2.16}$$

Using the first two terms only of the expansion of Eq. (2.16), substituting the result into Eq. (2.15), and simplifying, one obtains:

$$\frac{d\,(\sigma/C)}{d\varepsilon} \approx \frac{\sigma}{C} - \left(\frac{\sigma}{C}\right)^2. \tag{2.17}$$

It is finally seen that the quadratic expression of Eq. (2.17) is an approximation of the relationship of Eq. (2.13) for small values of σ. It is clear that the approximation is quadratic in full agreement with Fung's results on the elasticity of biological tissue [7, 8]. The exact relation of Eq. (2.13) along with the quadratic approximation of Eq. (2.17) are plotted and compared in Fig. 4. It is seen that these curves closely resemble those obtained by Fung doing experiments on biological tissue [7, 8]. In fact, Fung used a quadratic curve fit for the experimental results.

Finally, by examining the stress-strain curve of Fig. 3, one finds that the area under the curve is finite, i.e. the total strain energy of these materials is a finite value unlike that of linear elastic solids which is infinite. Next, one calculates this finite total strain energy. The total strain energy is calculated using the following integral applied to Eq. (2.8):

$$U_{total} = \int_0^\infty \sigma\, d\varepsilon = \int_0^\infty \frac{C}{\varepsilon^2} e^{-2/\varepsilon} d\varepsilon = \frac{C}{2}. \tag{2.18}$$

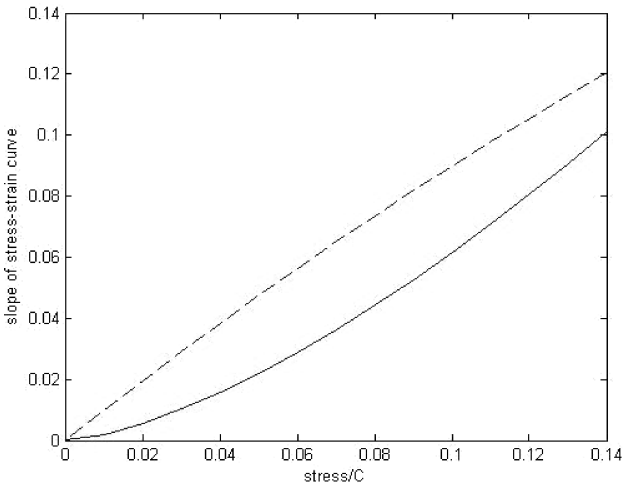

FIG. 4. Slope vs. stress for $n = 2$ (solid line is the exact relation, dashed line is the approximate relation).

2.2. Formulation for $n = 3$

Another type of elastic material is now considered that has a nonlinear strain energy function of the form:

$$U = \frac{1}{2}\sigma\varepsilon^3, \tag{2.19}$$

where the strain is raised to the third power. Let the stress σ be a function of the strain only as shown in Eq. (2.3). Substituting Eq. (2.3) into Eqs. (2.1) and (2.19), and substituting Eq. (2.19) on the left-hand-side of Eq. (2.1), one obtains:

$$\frac{1}{2}f\varepsilon^3 = \int f\, d\varepsilon. \tag{2.20}$$

Differentiating Eq. (2.20) with respect to ε and simplifying, results in the following:

$$\frac{df}{d\varepsilon}\varepsilon^3 + 3f\varepsilon^2 = 2f. \tag{2.21}$$

Re-writing the above equation in terms of the derivative, one obtains:

$$\frac{df}{d\varepsilon} = \frac{f\left(2 - 3\varepsilon^2\right)}{\varepsilon^3}. \tag{2.22}$$

The expression given in Eq. (2.22) is the governing scalar differential equation for this type of material ($n = 3$) where n is the power of ε in the expression of the strain energy function. The solution to the differential Eq. (2.22) is given by:

$$f = \frac{C}{\varepsilon^3} e^{-1/\varepsilon^2}, \tag{2.23}$$

where C is a constant of integration. Substituting the expression of f above into Eq. (2.3), the general scalar nonlinear stress-strain relationship is obtained for this type of material:

$$\sigma = \frac{C}{\varepsilon^3} e^{-1/\varepsilon^2}. \tag{2.24}$$

The above expression appeared in previous publications [39, 40] with E instead of C where E is the elastic modulus. However, it will be shown later that such a constant elastic modulus does not exist in the type of elastic materials under consideration here. This is because the behavior of these materials is highly nonlinear. In order to determine the constant C, the initial conditions are applied $\sigma = \sigma_0$ when $\varepsilon = \varepsilon_0$ to Eq. (2.24). Thus one obtains:

$$\sigma = \sigma_0 \left(\frac{\varepsilon_0}{\varepsilon}\right)^3 e^{\left[1/\varepsilon_0^2 - 1/\varepsilon^2\right]}. \tag{2.25}$$

Next, the stress-strain relation of Eq. (2.24) is plotted in order to study the characteristics of the elastic behavior of this material. First a plot is shown in Fig. 5 where the values of the strain vary up to a value of 0.2. It is seen from this

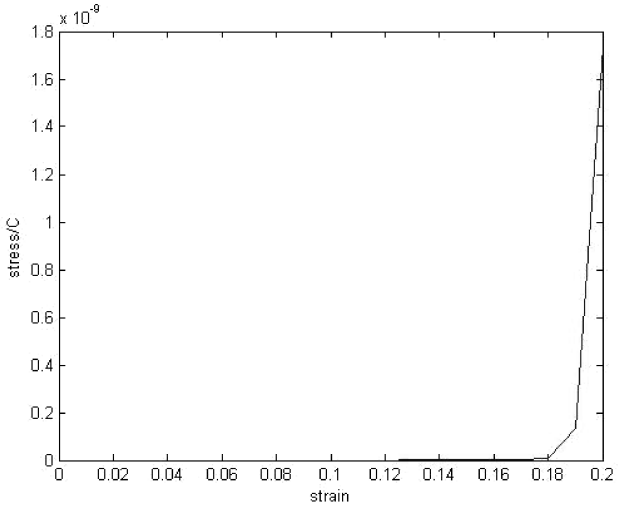

FIG. 5. Stress-strain curve for $n = 3$ (strain up to 0.2).

figure that the stress is almost zero in this range (the vertical axis is multiplied by 10^{-9}). In Fig. 6, the stress-strain curve is shown up to a strain of 0.5. It is seen that the stress-strain curve is highly nonlinear. It is also seen that a strengthening of the elastic modulus occurs when the strain exceeds about 0.3. Thus, in these types of materials one cannot utilize a constant modulus of elasticity. Another measure of the material behavior will be used as will be seen later.

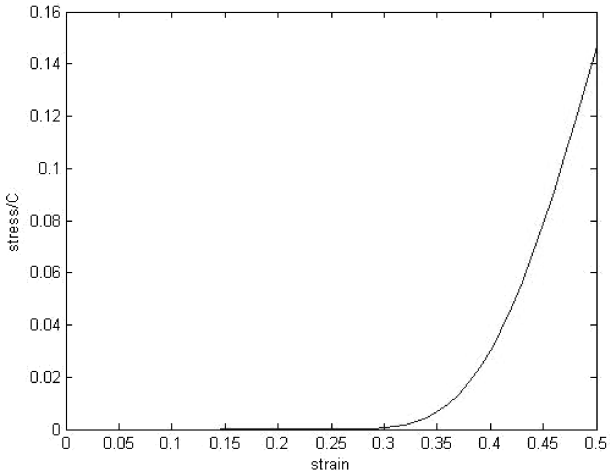

FIG. 6. Stress-strain curve for $n = 3$ (strain up to 0.5).

One needs to investigate what happens to the stress-strain curve when the strain exceeds 0.5. Thus, another plot is made of Eq. (2.24) for larger values of strain – up to 3.0. The resulting plot is shown in Fig. 7. It is seen that

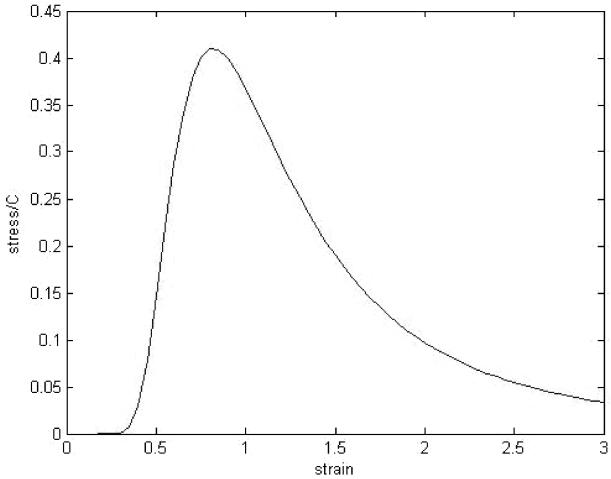

FIG. 7. Stress-strain curve for $n = 3$ (strain up to 3.0).

the stress reaches a maximum value when ε is between 0.5 and 1.0 (the exact value is calculated below), then starts decreasing until it approaches zero at infinity. There is another characteristic that is clearly evident from the stress-strain curves of Figs. 5, 6 and 7. These types of materials experience large strains under relatively small values of stress. In particular, the stress remains almost zero until the strain reaches a value approximately 0.30 as seen from Fig. 6.

The maximum value of the stress in Fig. 7 can be computed by differentiating Eq. (2.24) and setting the derivative equal to zero. This maximum value turns out to be:

$$\sigma_{\max} = \frac{C}{(2/3)^{3/2} e^{2/3}} \approx \frac{C}{2.44} \approx 0.410C \tag{2.26}$$

which occurs at $\varepsilon = \sqrt{2/3} = 0.816$.

Since these types of materials experience large strains, one now solves Eq. (2.24) for the strain in order to find an explicit expression for the strain in terms of the stress. Thus, one obtains:

$$\varepsilon = \frac{-i\sqrt{2/3}}{\sqrt{W\left(-\frac{2}{3}\left(\frac{\sigma}{C}\right)^{2/3}\right)}}, \tag{2.27}$$

where $W(x)$ is the Lambert W function [4]. Next, an explicit expression is derived between the stress and the slope of the stress-strain curve. One now calculates first the slope of the stress-strain curve. This is accomplished by computing the derivative $\dfrac{d\sigma}{d\varepsilon}$ using Eq. (2.24). Thus one obtains:

$$\frac{d\sigma}{d\varepsilon} = \left(\frac{2}{\varepsilon^3} - \frac{3}{\varepsilon}\right)\sigma. \tag{2.28}$$

Substituting for ε from Eq. (2.27) into Eq. (2.28) and simplifying, one obtains:

$$\frac{d\sigma}{d\varepsilon} = \left[\frac{2\,W^{3/2}\left(-\frac{2}{3}\left(\frac{\sigma}{C}\right)^{2/3}\right)}{i\left(\frac{2}{3}\right)^{3/2}} + \frac{3W^{1/2}\left(-\frac{2}{3}\left(\frac{\sigma}{C}\right)^{2/3}\right)}{i\left(\frac{2}{3}\right)^{1/2}}\right]\sigma. \tag{2.29}$$

The above expression presents an explicit relation between the slope $\dfrac{d\sigma}{d\varepsilon}$ and σ. Dividing both sides by C, the above expression can be re-written as follows:

$$\frac{d\,(\sigma/C)}{d\varepsilon} = \left[\frac{2\,W^{3/2}\left(-\frac{2}{3}\left(\frac{\sigma}{C}\right)^{2/3}\right)}{i\left(\frac{2}{3}\right)^{3/2}} + \frac{3W^{1/2}\left(-\frac{2}{3}\left(\frac{\sigma}{C}\right)^{2/3}\right)}{i\left(\frac{2}{3}\right)^{1/2}}\right](\sigma/C). \tag{2.30}$$

It is clear that the above relation is very complicated – its approximation using a polynomial will not even be attempted here. One is now content by sketching a graph of the above exact relation for small values of the stress ratio σ/C. This plot is shown in Fig. 8. Note that the imaginary number i appearing in the denominators will cancel out during the calculations and one obtains a positive real number as a result.

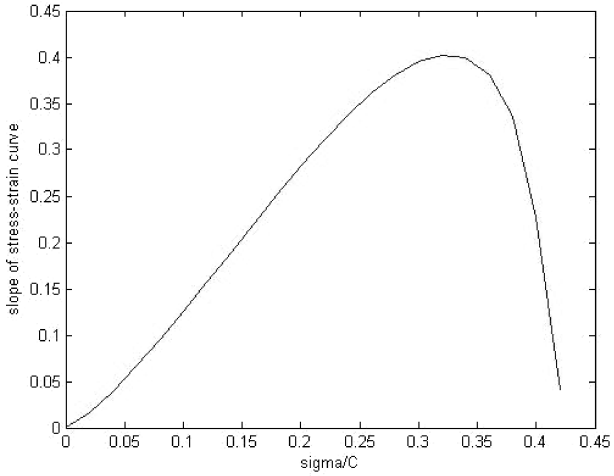

FIG. 8. Relation between stress ratio and slope of stress-strain curve for $n = 3$.

Finally, by examining the stress-strain curve of Fig. 7, one finds that the area under the curve is finite, i.e. the total strain energy of these materials is a finite value unlike that of linear elastic solids which is infinite. Next, the finite total strain energy is calculated. The total strain energy is obtained using the following integral applied to Eq. (2.24):

$$U_{total} = \int_0^\infty \sigma \, d\varepsilon = \int_0^\infty \frac{C}{\varepsilon^3} e^{-1/\varepsilon^2} d\varepsilon = \frac{C}{2}. \tag{2.31}$$

It is seen from the result obtained in Eq. (2.31) for the total strain energy that the same expression is obtained as that of Eq. (2.18). Thus, the total strain energy for the two cases $n = 2$ and $n = 3$ has the same expression although the constant of integration C may be different.

2.3. Formulation for n = a positive integer greater than one

A type of elastic material is considered that has a nonlinear strain energy function of the form:

$$U = \frac{1}{2}\sigma\varepsilon^n, \tag{2.32}$$

where the strain is raised to the n-th power, n is a positive integer greater than 1. In Secs. 2.1 and 2.2 the two special cases are considered when $n = 2$ and $n = 3$, respectively. In this section, the general case is considered for any positive integer n greater than 1. Let the stress σ be a function of the strain only as given by Eq. (2.3). Substituting Eq. (2.3) into Eqs. (2.1) and (2.32), and then substituting Eq. (2.32) on the left-hand-side of Eq. (2.1), one obtains:

$$\frac{1}{2} f \varepsilon^n = \int f d\varepsilon. \tag{2.33}$$

Differentiating Eq. (2.33) with respect to ε and simplifying, one obtains:

$$\frac{df}{d\varepsilon} \varepsilon^n + n f \varepsilon^{n-1} = 2f. \tag{2.34}$$

Re-writing the above equation in terms of the derivative, one obtains:

$$\frac{df}{d\varepsilon} = \frac{f\left(2 - n\varepsilon^{n-1}\right)}{\varepsilon^n}. \tag{2.35}$$

The expression given in Eq. (2.35) is the governing scalar differential equation for this type of material where n is the power of ε in the expression of the strain energy function. The solution to the differential Eq. (2.35) is given by:

$$f = \frac{C}{\varepsilon^n} e^{\frac{-2}{(n-1)\varepsilon^{(n-1)}}}, \tag{2.36}$$

where C is a constant of integration. Substituting the expression of f above into Eq. (2.3), the general scalar nonlinear stress-strain relationship is obtained for this type of material:

$$\sigma = \frac{C}{\varepsilon^n} e^{\frac{-2}{(n-1)\varepsilon^{(n-1)}}}. \tag{2.37}$$

The above expression appeared in previous publications [39, 40] with E instead of C where E is the elastic modulus. However, it will be shown later that such a constant elastic modulus does not exist in the type of elastic materials under consideration here. This is because the behavior of these materials is highly nonlinear. In order to determine the constant C, the initial conditions $\sigma = \sigma_0$ when $\varepsilon = \varepsilon_0$ are applied to Eq. (2.8). Thus one obtains:

$$\sigma = \sigma_0 \left(\frac{\varepsilon_0}{\varepsilon}\right)^n e^{\left[\frac{2}{(n-1)\varepsilon_0^{(n-1)}} - \frac{2}{(n-1)\varepsilon^{(n-1)}}\right]}. \tag{2.38}$$

Equation (2.38) is the nonlinear stress-strain relationship that governs the behavior of these types of materials. The expression in Eq. (2.38) is shown in several plots in Figs. 9, 10 and 11. In Fig. 9, the stress-strain relations are shown for the values $n = 2, 3, 4, 5$ in a strain range from 0.0 to 1.0. It is seen that the stress reaches a maximum value then starts decreasing to approach zero as the strain approaches infinity. In Fig. 10, the stress-strain relations are shown for the values $n = 5, 10, 100$. It is noted that for the high value of $n = 100$, the stress is zero everywhere except near the vicinity of the maximum value. Once the value

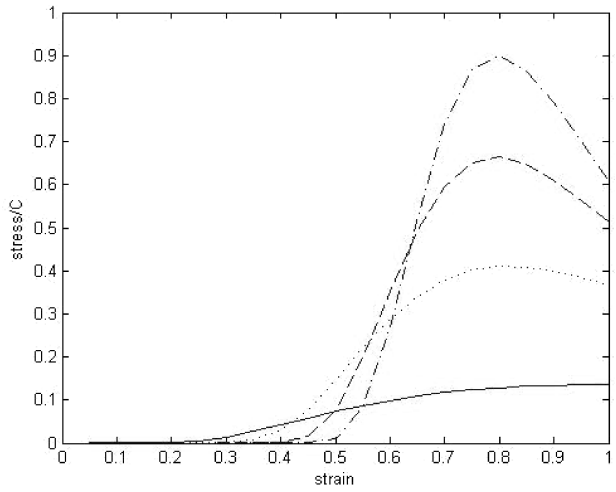

FIG. 9. Stress-strain curves for $n = 2, 3, 4, 5$.

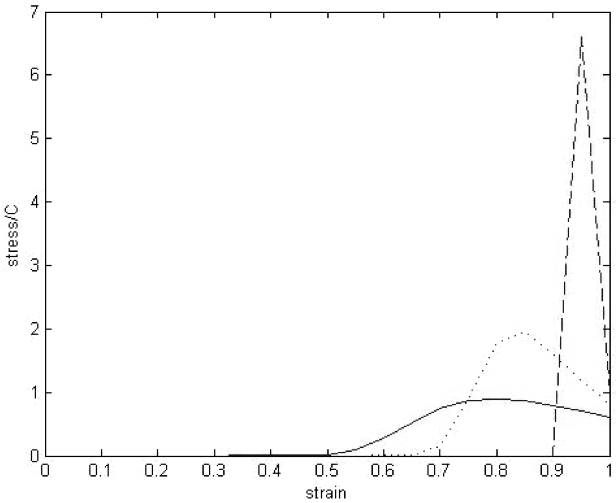

FIG. 10. Stress-strain curves for $n = 5, 10, 100$.

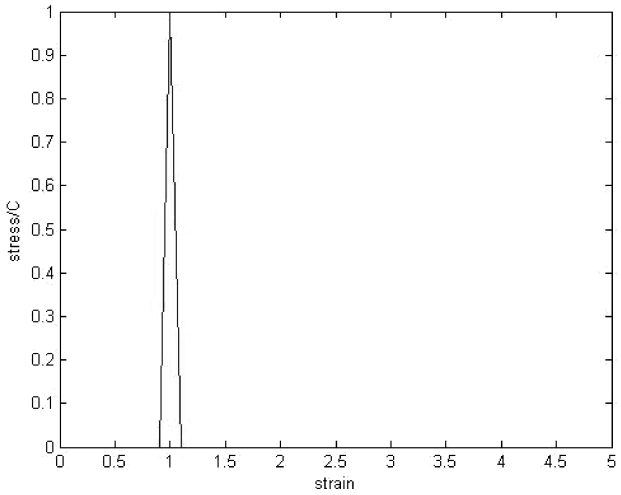

FIG. 11. Stress-strain curve for $n = 1000$.

of n increases more, it is seen that this behavior continues with zero stress almost everywhere. This is clear from Fig. 11 for $n = 1000$ where the stress attains a point-like maximum value and vanishes elsewhere. It is concluded that as $n \to \infty$, the stress approaches zero everywhere. This is one of the characteristics of the sought undamageable material as will be shown in Subsec. 2.4.

The maximum value of the stress is derived from Eq. (2.38) by calculating the derivative and setting it equal to zero. Thus one obtains the following expression for the maximum stress:

$$\sigma_{\max} = \frac{C}{\left(\frac{n}{2}\right)^{\frac{n-1}{n}} e^{\frac{n-1}{n}}}. \tag{2.39}$$

Actually Eq. (2.37) is the one used to derive the expression of the maximum stress of Eq. (2.39). The strain ε_m at which this maximum stress occurs is given by the expression:

$$\varepsilon_m = \frac{1}{\left(\frac{n}{2}\right)^{\frac{1}{n-1}}}. \tag{2.40}$$

Figure 12 shows a plot between $\sigma_{\max}/C$ versus n. It is seen that the maximum stress decreases with increasing values of n until the maximum stress approaches zero as the value of n approaches infinity. Figure 13 shows a plot between n and the strain ε_m at which the maximum stress occurs. It is seen that ε_m approaches the value of 1.0 as $n \to \infty$.

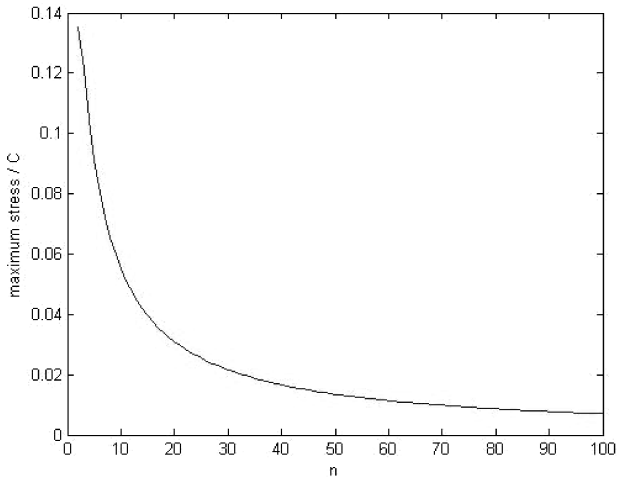

FIG. 12. Relation between n and the ratio of maximum stress/C.

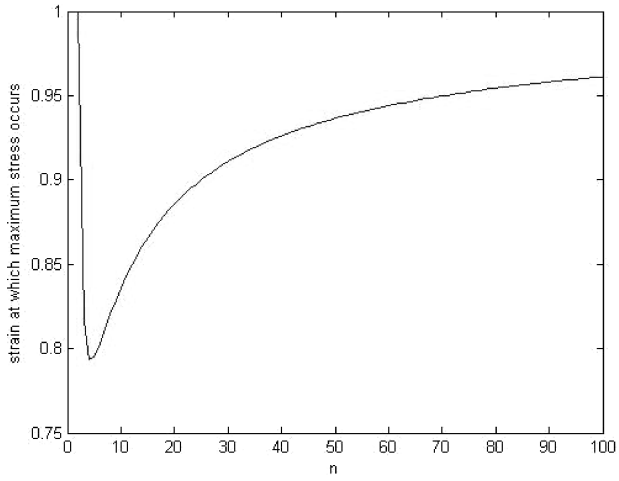

FIG. 13. Relation between n and the strain at which maximum stress occurs.

Since these materials exhibit large strains, one needs to solve Eq. (2.37) to get an explicit expression for the strain in terms of the stress. Thus, one obtains:

$$\varepsilon = \frac{2^{\frac{1}{n-1}}}{\left[-nW\left[-\dfrac{2^{\frac{n}{n-1}}\left(\dfrac{\sigma}{C}\right)}{n\left(2^{\frac{n}{n-1}}\left(\dfrac{\sigma}{C}\right)\right)^{\frac{1}{n}}}\right]\right]^{\frac{1}{n-1}}}, \tag{2.41}$$

where W is the Lambert W function [4].

Finally, by examining the stress-strain curves of Figs. 9 and 10, one finds that the area under each curve is finite, i.e. the total strain energy of these materials is a finite value unlike that of linear elastic solids which is infinite. Next, one calculates this finite total strain energy. The total strain energy is calculated using the following integral applied to Eq. (2.37):

$$U_{total} = \int_0^\infty \sigma \, d\varepsilon = \int_0^\infty \frac{C}{\varepsilon^n} e^{\frac{-2}{(n-1)\varepsilon^{(n-1)}}} \, d\varepsilon = \frac{C}{2}. \tag{2.42}$$

Thus, it is seen that the total strain energy has the same explicit value which is $C/2$ whatever the value of the integer n, where C is the constant of integration for each case. Thus the total strain energy of these types of materials is finite and is expressed by a simple calculation as shown above.

2.4. The case when n goes to infinity

In the previous Subsecs. 2.1, 2.2, and 2.3 it was shown that as the value of the exponent n in the strain energy function increases, the values of the stress decrease exponentially and approach zero as $n \to \infty$. The exact nonlinear stress-strain relationships for these types of materials with the specific strain energy functions indicated were derived by solving the governing scalar differential equations. It was also shown by Voyiadjis and Kattan [39, 40] that as $n \to \infty$, the value of the damage variable becomes identically zero irrespective of the deformation process. This means that the material as $n \to \infty$ becomes undamageable, i.e. the material cannot be damaged under any type of loading. Such a material does not exist at the present time but could be manufactured in the future based on the theoretical principles outlined here. In particular, the exact nonlinear stress-strain relationship is derived for these undamageable materials. The following characteristics are outlined of the sought undamageable material:

1. The material as $n \to \infty$ must be undamageable.
2. The value of the stress will remain equal to zero throughout the deformation process, for the specific case $n \to \infty$
3. The value of the damage variable will be equal to zero also throughout the deformation process.
4. The undamageable material has zero strain energy. This property is directly derived from property # 2 above.
5. The undamageable material has nonzero strain values. Thus, the undamageable material is a type of deformable material, not a rigid body.
6. The undamageable material is based on the proposed higher-order strain energy function of Eq. (2.32) taken in the limit when $n \to \infty$.

The stress-strain relationship for the undamageable may be obtained from the nonlinear elastic relation in Eq. (2.37) or (2.38) taken in the limit as $n \to \infty$. Items # 2, 4, and 5 above may be clearly deduced from the limit of Eq. (2.34). These characteristics are also clearly evident in Fig. 11 which was plotted based on Eq. (2.34).

2.5. Ratio of stresses for successive materials

It is interesting to explore the ratio of stresses for two successive materials of the above, i.e. compute the ratio of the stress σ when n equals to 2 and 3, then when n equals to 3 and 4, so on, until finally one reaches n and $n+1$.

One thus divides Eq. (2.24) for the stress when $n = 3$ by Eq. (2.8) for the stress when $n = 2$. Thus, one obtains:

$$\frac{\sigma^{(3)}}{\sigma^{(2)}} = C^{(3,2)} \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}\left(2-\frac{1}{\varepsilon}\right)}, \tag{2.43}$$

where $C^{(3,2)}$ is the ratio of the two constants of Eqs. (2.8) and (2.24). It is seen from Eq. (2.43) that the ratio of the stresses is a function of $1/\varepsilon$, i.e. the inverse of the strain. Since these materials undergo large strains, the ratio $1/\varepsilon$ is very small. Thus, Eq. (2.43) is approximated by a polynomial using the Taylor series expansion of the exponential function. The final approximation of Eq. (2.43) becomes:

$$\frac{\sigma^{(3)}}{\sigma^{(2)}} = C^{(3,2)} \frac{1}{\varepsilon} \left[1 + 2\left(\frac{1}{\varepsilon}\right) - \left(\frac{1}{\varepsilon}\right)^2 \right]. \tag{2.44}$$

Thus, it is seen that the ratio of the stresses is a quadratic polynomial function of the inverse of the strain.

Repeating the above procedure for the cases $n = 3$ and $n = 4$, one obtains:

$$\frac{\sigma^{(4)}}{\sigma^{(3)}} = C^{(4,3)} \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon^2}\left(1-\frac{2}{3}\frac{1}{\varepsilon}\right)}. \tag{2.45}$$

Again, one can make the same observation regarding Eq. (2.45) as was made before. The ratio of the stresses is a function of the inverse of the strain. Approximating Eq. (2.34) using the Taylor series expansion of the exponential function, one obtains:

$$\frac{\sigma^{(4)}}{\sigma^{(3)}} = C^{(4,3)} \frac{1}{\varepsilon} \left[1 + \left(\frac{1}{\varepsilon}\right)^2 - \frac{2}{3}\left(\frac{1}{\varepsilon}\right)^3 \right]. \tag{2.46}$$

Thus, it is seen from Eq. (2.46) that the ratio of the stresses is approximated by a cubic polynomial of the inverse of the strain.

Repeating the above procedure for the general case of n based on Eq. (2.37), one obtains the following general formula for the ratio of the stresses for any two successive materials discussed here:

$$\frac{\sigma^{(n+1)}}{\sigma^{(n)}} = C^{(n+1,n)} \frac{1}{\varepsilon} e^{\frac{2}{n-1}\left(\frac{1}{\varepsilon}\right)^{n-1}\left(1-\frac{n-1}{n}\frac{1}{\varepsilon}\right)}. \tag{2.47}$$

Using the Taylor series expansion of the exponential function and noting that the value of the inverse of the strain is small, one can approximate Eq. (2.47) by the following polynomial equation:

$$\frac{\sigma^{(n+1)}}{\sigma^{(n)}} = C^{(n+1,n)} \frac{1}{\varepsilon} \left[1 + \frac{2}{n+1}\left(\frac{1}{\varepsilon}\right)^{n-1} - \frac{2}{n}\left(\frac{1}{\varepsilon}\right)^{n}\right]. \tag{2.48}$$

Thus, the ratio of the stresses can be approximated by an n-th degree polynomial as shown in Eq. (2.48). The authors believe that the ratio of the stresses for two successive materials is significant. It can also be deduced from Eq. (2.48) that the ratio of the stresses goes to zero as the value of n goes to infinity.

3. Tensorial differential equations for the three-dimensional case

In this section an attempt is made to extend the scalar formulation of Sec. 2 to the general case of deformation and damage. In this case, the formulation is developed for anisotropic undamageable materials. For this purpose, tensors will be used instead of scalars. Also, in this formulation the indicial notation is utilized together with the summation convention.

In general the strain energy function is given by the following expression for a general state of deformation and damage:

$$U = \int \sigma_{ij}\, d\varepsilon_{ij}, \tag{3.1}$$

where ε_{ij} are the components of the strain tensor and σ_{ij} are the components of the stress tensor. The above expression is a generalization of the scalar expression of Eq. (2.1). One starts with the case when $n = 2$. In this case, the exponent n does not appear explicitly in the strain energy function like the scalar case but indicates the number of appearances of the strain components in the strain energy function. In this case, the strain energy function is postulated to take the following form:

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{jp}\varepsilon_{pi}. \tag{3.2}$$

It is clear that the above expression for the strain energy function is a generalization of the scalar expression of Eq. (2.2). Let the stress tensor be a function of the strain tensor only, i.e. consider the following equation which is a generalization of the scalar Eq. (2.3):

$$\sigma_{ij} = f_{ij}(\varepsilon) \equiv f_{ij}. \tag{3.3}$$

Substituting Eq. (3.3) into Eqs. (3.1) and (3.2), and then substituting Eq. (3.2) on the left-hand-side of Eq. (3.1), one obtains:

$$\frac{1}{2} f_{kl} \varepsilon_{jp} \varepsilon_{pi} = \int f_{kl}\, d\varepsilon_{ij}. \tag{3.4}$$

Taking the derivative of the above expression with respect to the components of the strain tensor ε_{mn} and simplifying, one obtains:

$$\frac{1}{2} \frac{df_{kl}}{d\varepsilon_{mn}} \varepsilon_{jp} \varepsilon_{pi} + \frac{1}{2} f_{kl} \varepsilon_{ni} \delta_{jm} + \frac{1}{2} f_{kl} \varepsilon_{jm} \delta_{in} = f_{kl} \delta_{im} \delta_{jn}, \tag{3.5}$$

where δ_{ij} are the components of the Kronecker delta tensor. Finally, one rewrites the above differential equation in the following form:

$$\frac{df_{kl}}{d\varepsilon_{mn}} \varepsilon_{jp} \varepsilon_{pi} = f_{kl}(2\delta_{im}\delta_{jn} - \delta_{jm}\varepsilon_{ni} - \delta_{in}\varepsilon_{jm}). \tag{3.6}$$

The above expression constitutes the governing tensorial differential equation for the case $n = 2$ for these types of materials. It is noted that the above differential equation reduces to the scalar differential Eq. (2.6) for the one-dimensional case. Currently no analytical method is found for solving the above differential equation in order to obtain a closed-form solution for the tensorial nonlinear stress-strain relationship for this case. In fact the above expression written in indicial notation represents a set of 81 simultaneous ordinary differential equations. Currently no method is available to solve them except numerically which is beyond the scope of this work.

One considers next the case when $n = 3$. In this case, the exponent n does not appear explicitly in the strain energy function like the scalar case but indicates the number of appearances of the strain components in the strain energy function. In this case, one postulates the strain energy function to take the following form:

$$U = \frac{1}{2} \sigma_{ij} \varepsilon_{jp} \varepsilon_{pq} \varepsilon_{qi}. \tag{3.7}$$

It is clear that the above expression for the strain energy function is a generalization of the scalar expression of Eq. (2.19). Let the stress tensor be a function of

the strain tensor only as given by Eq. (3.3) which is a generalization of the scalar Eq. (2.3). Substituting Eq. (3.3) into Eqs. (3.1) and (3.7), and then substituting Eq. (3.7) on the left-hand-side of Eq. (3.2), one obtains:

$$\frac{1}{2} f_{kl} \varepsilon_{jp} \varepsilon_{pq} \varepsilon_{qi} = \int f_{kl} \, d\varepsilon_{ij}. \tag{3.8}$$

Taking the derivative of the above expression with respect to the components of the strain tensor ε_{mn} and simplifying, one obtains:

$$\frac{1}{2} \frac{df_{kl}}{d\varepsilon_{mn}} \varepsilon_{jp} \varepsilon_{pq} \varepsilon_{qi} + \frac{1}{2} f_{kl} \varepsilon_{nq} \varepsilon_{qi} \delta_{jm} + \frac{1}{2} f_{kl} \varepsilon_{jm} \varepsilon_{ni} + \frac{1}{2} f_{kl} \varepsilon_{jp} \varepsilon_{pm} \delta_{in} = f_{kl} \delta_{im} \delta_{jn}. \tag{3.9}$$

Finally, the above differential equation is re-written in the following form:

$$\frac{df_{kl}}{d\varepsilon_{mn}} \varepsilon_{jp} \varepsilon_{pq} \varepsilon_{qi} = f_{kl} (2\delta_{im} \delta_{jn} - \varepsilon_{jm} \varepsilon_{ni} - \delta_{jm} \varepsilon_{nq} \varepsilon_{qi} - \delta_{in} \varepsilon_{jp} \varepsilon_{pm}). \tag{3.10}$$

The above expression constitutes the governing tensorial differential equation for the case $n = 3$ for these types of materials. It is noted that the above differential equation reduces to the scalar differential Eq. (2.22) for the one-dimensional case. No analytical method is currently available to solve the above differential equation in order to obtain a closed-form solution for the tensorial nonlinear stress-strain relationship for this case. In fact the above expression written in indicial notation represents a set of 81 simultaneous ordinary differential equations. No method is currently available to solve them except numerically which is beyond the scope of this work.

Finally, one considers next the general case for positive integer n greater than 1. In this case, the exponent n does not appear explicitly in the strain energy function like the scalar case but indicates the number of appearances of the strain components in the strain energy function. In this case, one postulates the strain energy function to take the following form:

$$U = \frac{1}{2} \sigma_{ip_1} \varepsilon_{p_1 p_2} \varepsilon_{p_2 p_3} \, \ldots \, \varepsilon_{p_n i}. \tag{3.11}$$

It is clear that the above expression for the strain energy function is a generalization of the scalar expression of Eqn. (2.32). It is noted that the components of the strain tensor appear exactly n times in the above strain energy function. Let the stress tensor be a function of the strain tensor only as given by Eq. (3.3) which is a generalization of the scalar Eq. (2.3). Substituting Eq. (3.3) into Eqs. (3.1) and (3.11), and then substituting Eq. (3.11) on the left-hand-side of Eq. (3.1), one obtains:

$$\frac{1}{2} f_{kl} \varepsilon_{p_1 p_2} \varepsilon_{p_2 p_3} \, \ldots \, \varepsilon_{p_n i} = \int f_{kl} \, d\varepsilon_{ip_1}. \tag{3.12}$$

Taking the derivative of the above expression with respect to the components of the strain tensor ε_{mn} and simplifying, one obtains:

$$\begin{aligned}(3.13)\quad &\frac{1}{2}\frac{df_{kl}}{d\varepsilon_{mn}}\varepsilon_{p_1p_2}\varepsilon_{p_2p_3}\ \dots\ \varepsilon_{p_ni}\\ &\qquad+\frac{1}{2}f_{kl}\delta_{p_1m}\delta_{p_2n}\varepsilon_{p_2p_3}\ \dots\ \varepsilon_{p_ni}+\frac{1}{2}f_{kl}\varepsilon_{p_1p_2}\delta_{p_2i}\delta_{p_3j}\varepsilon_{p_3p_4}\ \dots\ \varepsilon_{p_ni}\\ &\qquad+\ \dots\ +\frac{1}{2}f_{kl}\varepsilon_{p_1p_2}\varepsilon_{p_2p_3}\varepsilon_{p_3p_4}\ \dots\ \varepsilon_{p_{n-1}p_n}\delta_{p_nm}\delta_{in}=f_{kl}\delta_{im}\delta_{p_1n}.\end{aligned}$$

Finally, the above differential equation is re-written in the following form:

$$\begin{aligned}(3.14)\quad &\frac{df_{kl}}{d\varepsilon_{mn}}\varepsilon_{p_1p_2}\varepsilon_{p_2p_3}\ \dots\ \varepsilon_{p_ni}=f_{kl}(2\delta_{im}\delta_{p_1n}-\delta_{p_1m}\delta_{p_2n}\varepsilon_{p_2p_3}\ \dots\ \varepsilon_{p_ni}\\ &\quad-\varepsilon_{p_1p_2}\delta_{p_2i}\delta_{p_3j}\varepsilon_{p_3p_4}\ \dots\ \varepsilon_{p_ni}-\ \dots\ -\varepsilon_{p_1p_2}\varepsilon_{p_2p_3}\varepsilon_{p_3p_4}\ \dots\ \varepsilon_{p_{n-1}p_n}\delta_{p_nm}\delta_{in}).\end{aligned}$$

The above expression constitutes the governing tensorial differential equation for the general case for any positive integer n greater than 1 for these types of materials. It is noted that the above differential equation reduces to the scalar differential Eq. (2.35) for the one-dimensional case. No analytical method is available to solve the above differential equation in order to obtain a closed-form solution for the tensorial nonlinear stress-strain relationship for this case. In fact the above expression written in indicial notation represents a set of 81 simultaneous ordinary differential equations. Currently no method is available to solve them except numerically which is beyond the scope of this work.

Thus, the above set of three tensorial governing differential Eqs. (3.6), (3.10), and (3.14) remain unsolved at the present time. It is hoped that they will be solved in a forthcoming paper. They are presented here with full derivation in order to complete the theory of governing differential equations for the mechanics of undamageable materials.

4. Applications

In this section, four different existing materials are outlined that display a nonlinear elastic behavior close to that of the proposed undamageable material. These four types of materials include rubber and rubber-like materials, soft biological tissues, metallic glass, and metal rubber. It is speculated that the future undamageable material may have one or more of these materials as a prime constituent:

1. Rubber and rubber-like materials: The single most important property of rubber is its ability to undergo large elastic deformations and return to its original shape in a reversible way. This behavior also characterizes the

proposed undamageable material. The elastic stress-strain relationship for both rubber and undamageable materials is nonlinear but there are noticeable differences. For more details about rubber and rubber-like materials, the reader is referred to Arruda and Boyce [1].

2. Soft biological tissue: The nonlinear elastic response of soft biological tissue resembles that of rubber but there are significant differences. It is noted that the stress-strain relationship of soft biological tissue includes the exponential function which also appears in the stress-strain relationship of the proposed undamageable materials. For more details about the elasticity and mechanics of soft biological tissue, the reader is referred to Fung [7, 8].
3. Metallic glass (or amorphous metal or glassy metal): This is a solid metallic material, usually an alloy, with a disordered atomic-scale structure. These materials are non-crystalline and have a glass-like structure. Different types of these materials have been produced since 1960. These materials have higher tensile yield strengths and higher elastic strain limits. For more details about the behavior of metallic glass, the reader is referred to Castelvecchi [2], Das *et al.* [5], and Zhang *et al.* [42].
4. Metal rubber: This new and self-assembling nanocomposite is ultra flexible and durable to high and low pressures, temperatures, tensions, most chemical reactions, and retains all of its physical and chemical properties upon being returned to a ground state. Metal rubber has been produced since 2004 and has a number of industrial applications. For more details about metal rubber, the reader is referred to Science Daily [26].

In addition to the above four types of materials, there has been a very recent development in this regard. In 2013, researchers have been able to realize a self-healing polymer that can self-heal in two hours. This material has been called the "Terminator" polymer because it resembles the self-healing robot in the Terminator movies. For more details on this most recent development, the reader is referred to Martin *et al.* [21], Rekondo *et al.* [25], Templeton [29], and Science Daily [27]. In particular, Xu and Demkowicz [41] presented a self-healing model for nanocracks based on disclinations.

5. Conclusion

In this work both scalar and tensorial differential equations are formulated that govern the mechanics of undamageable materials. These types of hypothetical materials were recently proposed by the authors to be free of any damage effects whatsoever. It is hoped that the manufacturing industry will develop in the future such materials. The scalar differential equations are solved in great

detail and the resulting nonlinear stress-strain relationships are investigated. On the other hand, the complicated tensorial differential equations remain currently without solutions.

The authors admit that this work is entirely mathematical in its formulation and needs to be complemented by presenting the physical and metallurgical aspects. However, it is not clear to the authors what these issues are at the present time and how they can be approached. The authors anticipate that the physical and metallurgical aspects to be investigated in a future manuscript. The authors reiterate their viewpoint that this mathematical formulation lays a possible groundwork for any future development in this regard. The authors are still hopeful that some form of strengthened material may be realized in the near future. It may well be a "mathematical concept" at the present time that may lay the foundation to seek the proper material characterization as depicted in the various figures presented here based on these mathematical principles to become a reality. The use of the fabric tensor is the approach to link the mathematics with the material characterization. The authors have extensively published on this subject including several chapters in their last book [34–36].

It is to be noted that the term "undamageable" is reflected in the context of the material stiffness and not the property of indestructibility due to various loading conditions. Thus, the undamageable material reflects that no microcracks or microvoids occur as well as no plastic yielding in the material. To illustrate this concept, a last section is added on applications. Four different existing and/or recently manufactured materials are outlined in this work such as rubber-like materials, soft biological tissues, metallic glass and metal rubber. These materials come very close in their properties to the postulated "hypothetical" undamageable material. Finally, a most recent development is mentioned. The recent manufacture of self-healing "Terminator" polymeric material is also outlined.

References

1. Arruda E.M., Boyce M.C., *A Three-Dimensional Constitutive Model for the Large Stretch Behavior of Rubber Elastic Materials*, Journal of the Mechanics and Physics of Solids, Vol. **41**, 2, 389–412, 1993.

2. Castelvecchi D., *Glass-like Metal Performs Better Under Stress*, Physical Review Focus, **15**, 20, 1103/PhysRevFocus.15.20, 2005.

3. Celentano D.J., Tapia P.E., Chaboche J-L., *Experimental and Numerical Characterization of Damage Evolution in Steels*, Mecanica Computacional, Vol. XXIII, (edited by G. Buscaglia, E. Dari, O. Zamonsky), Bariloche, Argentina, 2004.

4. Corless R.M., Gonnet G.H, Hare D.E.G., Jeffrey D.J., Knuth E.E., *On the Lambert W Function*, Advances in Computational Mathematics, **5**, 329–359, 1996.

5. Das J., Tang M.B., Kim K.B., Theissmann R., Baier F., Wang W.H., Eckert J., *Work-Hardenable" Ductile Bulk Metallic Glass*, Physical Review Letters, **94**, 205501, DOI: 1103/PhysRevLett.94.205501, 2005.

6. Doghri I., *Mechanics of Deformable Solids: Linear and Nonlinear, Analytical and Computational Aspects*, Springer-Verlag, Germany, 2000.

7. Fung Y.C., *Elasticity of Soft Tissues in Simple Elongation*, American Journal of Physiology, **213**, 6, 1432–1544, 1967.

8. Fung Y.C., *Biomechanics: Mechanical Properties of living Tissues*, Second Edition, Springer, 1993.

9. Hansen N.R., Schreyer H.L., *A Thermodynamically Consistent Framework for Theories of Elastoplasticity Coupled with Damage*, International Journal of Solids and Strucutres, **31**, 3, 359–389, 1994.

10. Kachanov L., *On the Creep Fracture Time* [in Russian], Izv Akad, Nauk USSR Otd Tech., **8**, 26–31, 1958.

11. Kattan P.I., Voyiadjis G.Z., *A Coupled Theory of Damage Mechanics and Finite Strain Elasto-Plasticity – Part I: Damage and Elastic Deformations*, International Journal of Engineering Science, **28**, 5, 421–435, 1990.

12. Kattan P.I., Voyiadjis G.Z., *A Plasticity-Damage Theory for Large Deformation of Solids – Part II: Applications to Finite Simple Shear*, International Journal of Engineering Science, **31**, 1, 183–199, 1993.

13. Kattan P.I., Voyiadjis G.Z., *Decomposition of Damage Tensor in Continuum Damage Mechanics*, Journal of Engineering Mechanics, ASCE, **127**, 9, 940–944, 2001.

14. Kattan P.I., Voyiadjis G.Z., *Damage Mechanics with Finite Elements: Practical Applications with Computer Tools*, Springer-Verlag, Germany, 2001.

15. Ladeveze P., Poss M., Proslier L., *Damage and Fracture of Tridirectional Composites*, [in:] Progress in Science and Engineering of Composites. Proceedings of the Fourth International Conference on Composite Materials, Japan Society for Composite Materials, **1**, 649–658, 1982.

16. Ladeveze P., Lemaitre J., *Damage Effective Stress in Quasi-Unilateral Conditions*, The 16th International Cogress of Theoretical and Applied Mechanics, Lyngby, Denmark, 1984.

17. Lee H., Peng K., Wang J., *An Anisotropic Damage Criterion for Deformation Instability and its Application to Forming Limit Analysis of Metal Plates*, Engineering Fracture Mechanics, **21**, 1031–1054, 1985.

18. Lubineau G., *A Pyramidal Modeling Scheme for Laminates – Identification of Transverse Cracking*, International Journal of Damage Mechanics, **19**, 4, 499–518, 2010.

19. Lubineau G., Ladeveze P., *Construction of a Micromechanics-based Intralaminar Mesomodel, and Illustrations in ABAQUS/Standard*, Computational Materials Science, **43**, 1, 137–145, 2008.

20. Luccioni B., Oller S., *A Directional Damage Model*, Computer Methods in Applied Mechanics and Engineering, **192**, 1119–1145, 2003.

21. Martin R., Rekondo A., Echeberria J., Cabanero G., Grande H.J., Odriozola I., *Room Temperature Self-healing Power of Silicone Elastomers Having Silver Nanoparticles as Crosslinkers*, Chemical Communications, **48**, 66, 8255, DOI: 10.1039/c2cc32030d, 2012.

22. Nichols J.M., Abell A.B., *Implementing the Degrading Effective Stiffness of Masonry in a Finite Element Model*, North American Masonry Conference, Clemson, South Carolina, USA, 2003.

23. Nichols J.M., Totoev Y.Z., *Experimental Investigation of the Damage Mechanics of Masonry Under Dynamic In-plane Loads*, North American Masonry Conference, Austin, Texas, USA, 1999.

24. Rabotnov Y., *Creep Rupture*, [in:] Proceedings, Twelfth International Congress of Applied Mechanics, M. Hetenyi and W.G. Vincenti [Eds.], Stanford, 1968, Springer-Verlag, Berlin, pp. 342–349, 1969.

25. Rekondo A., Martin R., de Luzuriaga A.R., Cabanero G., Grande H.J., Odriozola I., *Catalyst-free Room Temperature Self-healing Elastomers Based on Aromatic Disulfide Metasthesis*, Materials Horizons, DOI: 10.1039/c3mh00061c, 2013.

26. Science Daily, *Chemists Create Self-assembling Conductive Rubber*, ScienceDaily, Published online at http://www.sciencedaily.com/videos/2007/0409-metal_rubber.htm, 2007.

27. Science Daily, *'Terminator' Polymer: Self-healing Polymer That Spontaneously and Indpendently Repairs Itself*, ScienceDaily, (appeared on September 13, 2013 online), http://www.sciencedaily.com/releases/2013/09/130913101819.htm, 2013.

28. Sidoroff F., *Description of Anisotropic Damage Application in Elasticity*, [in:] IUTAM Colloqium on Physical Nonlinearities in Structural Analysis, pp. 237–244, Springer-Verlag, Berlin, 1981.

29. Templeton G., *'Terminator' Polymer Can Spontaneously Self-heal in Just Two Hours*, ExtremeTech, Published online at http://www.extremetech.com/extreme/166656-terminator-polymer-can-spontaneously-self-heal-in-just-two-hours, 2013.

30. Voyiadjis G.Z., *Degradation of Elastic Modulus in Elastoplastic Coupling with Finite Strains*, International Journal of Plasticity, **4**, 335–353, 1988.

31. Voyiadjis G.Z., Kattan P.I., *A Coupled Theory of Damage Mechanics and Finite Strain Elasto-Plasticity – Part II: Damage and Finite Strain Plasticity*, International Journal of Engineering Science, **28**, 6, 505–524, 1990.

32. Voyiadjis G.Z., Kattan P.I., *A Plasticity-Damage Theory for Large Deformation of Solids – Part I: Theoretical Formulation*, International Journal of Engineering Science, **30**, 9, 1089–1108, 1992.

33. Voyiadjis G.Z., Kattan P.I., *Damage Mechanics*, Taylor and Francis (CRC Press), 2005.

34. Voyiadjis G.Z., Kattan P.I., *Advances in Damage Mechanics: Metals and Metal Matrix Composites with an Introduction to Fabric Tensors*, Second Edition, Elsevier, 2006.

35. Voyiadjis G.Z., Kattan P.I., *A New Fabric-Based Damage Tensor*, Journal of the Mechanical Behavior of Materials, **17**, 1, 31–56, 2006.

36. Voyiadjis G.Z., Kattan P.I., *Damage Mechanics with Fabric Tensors*, Mechanics of Advanced Materials and Structures, **13**, 4, 285–301, 2006.

37. Voyiadjis G.Z., Kattan P.I., *A Comparative Study of Damage Variables in Continuum Damage Mechanics*, International Journal of Damage Mechanics, **18**, 4, 315–340, 2009.

38. Voyiadjis G.Z., Kattan P.I., *A New Class of Damage Variables in Continuum Damage Mechanics*, Journal of Engineering Materials and Technology, ASME, in press, 2011.

39. Voyiadjis G.Z., Kattan P.I., *Introduction to the Mechanics and Design of Undamageable Materials*, International Journal of Damage Mechanics, **22**, 3, 323–335, 2012.

40. Voyiadjis G.Z., Kattan P.I., *On the Theory of Elastic Undamageable Materials*, Journal of Engineering Materials and Technology, ASME, Special Issue on Modeling Material Behavior at Multiple Scales, Xi Chen [Ed.], **135**, 2, 021002-1–021002-6, 2013.

41. Xu G.Q., Demkowicz M.J., *Healing of Nanocracks by Disclinations*, Physical Review Letters, **111**, 145501, DOI: 10.1103/PhysRevLett.111.145501, 2013.

42. Zhang B., Zhao D.Q., Pan M.X., Wang W.H., Greer A.L., *Amorphous Metallic Plastic*, Physical Review Letters, **94**, 205502, DOI: 10.1103/PhysRevLett.94.205502, 2005.

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