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CERTAIN PROBLEMS OF STATICS AND DYNAMICS OF POINT-SUPPORTED ELASTIC RECTANGULAR PLATES

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The paper presents an algorithm, based on the finite Fourier transforms technique, making it possible to derive formally accurate solutions of the following engineering problems of statics and dynamics of elastic, isotropic plates: (1) Harmonic vibrations and statical bending of rectangular plates, loaded symmetrically with respect to both axes of symmetry of the plate, supported at four corners and, along the sides, on elastic ribs; (2) Statical bending of rectangular plates, loaded as before and supported at four or two points at equal distances from the opposite edges, or supported at a single point, at the center of symmetry of the plate; (3) Statical bending of a plate having the form of an isosceles rightangled triangle, the hypotenuse being either free or supported on an elastic rib, simply supported along the legs, and loaded symmetrically with respect to the symmetry axis. In the case of rectangular plates supported at four corners and on elastic ribs, the solution presented is confined to symmetric loadings for the sake of simplicity only, since it is easily observed that the formulae derived in the first part of the paper may be generalized to arbitrary static or dynamic loadings.

All the considerations and formulae presented in the paper may easily be generalized to the case of orthotropy, as also to the practically important case of rectangular plates supported on columns having finite cross-sections, and located at certain distances from the edges of the plate, provided the loads are symmetric.

1. Introduction

Problems of rectangular plates supported at isolated points fall within the class of classical boundary-value problems of the plate theory which is characterized by a relatively small number of solutions, most of them only approximate and concerning the cases of point-supports located at the corners of the plate. The following solutions should be mentioned here: (1) A formally accurate solution [1] of statical bending of a rectangular plate supported at the corners, subject to arbitrary loads distributed along the edges; (2) Approximate solutions [2–5] concerning rectangular plates supported at the corners; (3) An approximate solution [6] (derived by the method of finite differences) of the problem of statical bending of a rectangular plate loaded uniformly and supported at four symmetric points at a certain distance from the edges.

In this paper, we shall present a certain algorithm (based on the finite Fourier transforms technique) enabling us to derive formally accurate solutions of certain practical problems of statics and dynamics of elastic, isotropic, rectangular plates, supported at several points, as also certain problems concerning the statics of triangular plates. The range of applicability of the algorithm comprises: (1) Harmonic vibrations and statical bending of rectangular plates loaded symmetrically with respect to the two axes of symmetry of the plate, supported at four corners and resting on elastic ribs along the edges; (2) Statics of rectangular plates subject to the same loading and supported at four or two points at equal distances from the opposite edges, or supported at a single point, at the center of the plate; (3) Statical bending of a plate in the form of an isosceles rightangled triangle, the hypotenuse being either free or supported on an elastic rib, and the legs being simply supported; the load is symmetric with respect to the axis perpendicular to the hypotenuse.

It should be stressed that the solutions presented in this paper have been obtained by means of effective application of the finite Fourier transforms technique; its first application to the boundary-value problems of rectangular plates may be found in the well-known paper [7].

2. Application of the Fourier method to the problem

Let us consider an elastic, homogeneous, isotropic rectangular plate, arbitrarily supported along the edges and subject to arbitrary loads producing forced harmonic vibrations.

The differential equation describing the deflection amplitudes of such a plate has the form:

$$\frac{\partial^2 M_{11}}{\partial x^2}+2\frac{\partial^2 M_{12}}{\partial x\,\partial y}+\frac{\partial^2 M_{22}}{\partial y^2}+\rho\omega^2 w+p=0. \tag{2.1}$$

Here, $w=w(x,y)$ is the deflection amplitude of the middle surface of the plate, ρ — mass per unit area of the middle surface, ω — angular frequency of forced harmonic vibrations, $p=p(x,y)$ — amplitude of the loads producing forced vibrations.

The amplitudes of bending and twisting moments are expressed by the well-known formulae:

$$\begin{gathered}M_{11}=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\qquad M_{22}=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),\\ M_{12}=-D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y},\end{gathered} \tag{2.2}$$

with the bending rigidity

$$D=\frac{Eh^3}{12(1-\nu^2)}. \tag{2.3}$$

Here, E is the Young modulus, h — thickness of the plate, ν — Poisson's ratio.

In order to obtain the most general possible (with respect to the boundary conditions) solution, let us apply the finite Fourier transform. Let us assume that the functions to be transformed satisfy the Dirichlet conditions in the region $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$.

The finite sine and cosine Fourier transforms are expressed by the following formulae:

$$
\begin{aligned}
T^1_{mn}\{f(x,y)\} &= \int_0^a \int_0^b f(x,y)\sin\alpha_m x \sin\beta_n y\,dx\,dy, \\
T^2_{mn}\{f(x,y)\} &= \int_0^a \int_0^b f(x,y)\cos\alpha_m x \cos\beta_n y\,dx\,dy,
\end{aligned} \tag{2.4}
$$

with

$$\alpha_m = \frac{m\pi}{a}, \qquad \beta_n = \frac{n\pi}{b}.$$

Applying the finite sine transform to the functions $w(x, y)$ and $p(x, y)$, we obtain from the Eqs. (2.4):

$$T^1_{mn}\{w\} = w_{mn}, \qquad T^1_{mn}\{p\} = p_{mn}. \tag{2.5}$$

The inverse transforms of w and p are written in terms of a double Fourier series:

$$T^{-1}_{mn}\{w_{mn}\} = w(x,y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\alpha_m x \sin\beta_n y, \tag{2.6}$$

$$T^{-1}_{mn}\{p_{mn}\} = p(x,y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\sin\alpha_m x \sin\beta_n y. \tag{2.7}$$

Applying the finite sine transform to the Eq. (2.1), we obtain

$$T^1_{mn}\left\{\frac{\partial^2 M_{11}}{\partial x^2} + 2\frac{\partial^2 M_{12}}{\partial x\,\partial y} + \frac{\partial^2 M_{22}}{\partial y^2}\right\} + \rho\omega^2 w_{mn} + p_{mn} = 0. \tag{2.8}$$

Performing the integration by parts, we obtain:

$$
\begin{aligned}
T^1_{mn}\left\{\frac{\partial^2 M_{11}}{\partial x^2}\right\} &= -\alpha_m^2 T^1_{mn}\{M_{11}\} - \alpha_m M^{(1)}_{mn}, \\
T^1_{mn}\left\{\frac{\partial^2 M_{12}}{\partial x\,\partial y}\right\} &= \alpha_m \beta_n T^2_{mn}\{M_{12}\}, \\
T^1_{mn}\left\{\frac{\partial^2 M_{22}}{\partial y^2}\right\} &= -\beta_n^2 T^1_{mn}\{M_{22}\} - \beta_n M^{(2)}_{mn}.
\end{aligned} \tag{2.9}
$$

Here,

$$
\begin{aligned}
M^{(1)}_{mn} &= \int_0^b [(-1)^m M_{11}(a,y) - M_{11}(0,y)]\sin\beta_n y\,dy, \\
M^{(2)}_{mn} &= \int_0^a [(-1)^n M_{22}(x,b) - M_{22}(x,0)]\sin\alpha_m x\,dx.
\end{aligned} \tag{2.10}
$$

Similarly, by taking into account the Eqs. (2.2), we may write the transforms of moment amplitudes M_{11}, M_{12}, M_{22} in the form:

$$\begin{aligned} T^1_{mn}\{M_{11}\} &= D\,[(\alpha_m^2+\nu\beta_n^2)\,w_{mn}+\alpha_m W^{(1)}_{mn}+\nu\beta_n W^{(2)}_{mn}], \\ T^2_{mn}\{M_{12}\} &= D(\nu-1)\,[\alpha_m\beta_n w_{mn}+\alpha_m W^{(2)}_{mn}+\beta_n W^{(1)}_{mn}+W^{(0)}_{mn}], \\ T^1_{mn}\{M_{22}\} &= D\,[(\nu\alpha_m^2+\beta_n^2)\,w_{mn}+\nu\alpha_m W^{(1)}_{mn}+\beta_n W^{(2)}_{mn}], \end{aligned} \tag{2.11}$$

with the notations

$$\begin{aligned} W^{(1)}_{mn} &= \int_0^b [(-1)^m w(a,y)-w(0,y)]\sin\beta_n y\,dy, \\ W^{(2)}_{mn} &= \int_0^a [(-1)^n w(x,b)-w(x,0)]\sin\alpha_m x\,dx, \\ W^{(0)}_{mn} &= (-1)^{m+n} w(a,b)+w(0,0)-(-1)^m w(a,0)-(-1)^n w(0,b). \end{aligned} \tag{2.12}$$

Taking into account that transformation, we obtain

$$T^1_{mn}\left\{\frac{\partial^2 M_{11}}{\partial x^2}+2\frac{\partial^2 M_{12}}{\partial x\,\partial y}+\frac{\partial^2 M_{22}}{\partial y^2}\right\}=-D\Delta_{mn}w_{mn}-B_{mn}, \tag{2.13}$$

where

$$\Delta_{mn}=(\alpha_m^2+\beta_n^2)^2, \tag{2.14}$$

$$\begin{aligned} B_{mn} &= \alpha_m M^{(1)}_{mn}+\beta_n M^{(2)}_{mn}+D\alpha_m[\alpha_m^2+\beta_n^2(2-\nu)]W^{(1)}_{mn}+ \\ &\quad +D\beta_n[\beta_n^2+\alpha_m^2(2-\nu)]W^{(2)}_{mn}+2\alpha_m\beta_n D(1-\nu)W^{(0)}_{mn}. \end{aligned} \tag{2.15}$$

The expression for B_{mn} is easily seen to depend solely on the bending moments and deflections along the edges of the plate.

Using the expression (2.13), we may transform the Eq. (2.8) to the form

$$w_{mn}D(\Delta_{mn}-c^2)=p_{mn}-B_{mn}, \tag{2.16}$$

with the notation

$$c^2=\frac{\rho\omega^2}{D}. \tag{2.17}$$

The functions describing the amplitudes of bending moments and deflections may be represented by single Fourier series:

$$\begin{aligned} M_{11}(a,y) &= \frac{2}{b}\sum_{j=1}^{\infty} e^{(1)}_j \sin\beta_j y, & M_{11}(0,y) &= \frac{2}{b}\sum_{j=1}^{\infty} e^{(3)}_j \sin\beta_j y, \\ M_{22}(x,b) &= \frac{2}{a}\sum_{i=1}^{\infty} e^{(2)}_i \sin\alpha_i x, & M_{22}(x,0) &= \frac{2}{a}\sum_{i=1}^{\infty} e^{(4)}_i \sin\alpha_i x, \end{aligned} \tag{2.18}$$

(2.19)
$$w(a, y)=\frac{2}{b}\sum_{j=1}^{\infty} w_j^{(1)}\sin\beta_j y, \qquad w(0, y)=\frac{2}{b}\sum_{j=1}^{\infty} w_j^{(3)}\sin\beta_j y,$$
$$w(x, b)=\frac{2}{a}\sum_{i=1}^{\infty} w_i^{(2)}\sin\alpha_i x, \qquad w(x, 0)=\frac{2}{a}\sum_{i=1}^{\infty} w_i^{(4)}\sin\alpha_i x.$$

Substituting the series (2.18), (2.19) into the Eqs. (2.10), (2.12), we obtain

(2.20) $$M_{mn}^{(1)}=(-1)^m e_n^{(1)}-e_n^{(3)}, \qquad M_{mn}^{(2)}=(-1)^n e_m^{(2)}-e_m^{(4)},$$

(2.21) $$W_{mn}^{(1)}=(-1)^m w_n^{(1)}-w_n^{(3)}, \qquad W_{mn}^{(2)}=(-1)^n w_m^{(2)}-w_m^{(4)}.$$

3. Vibration and bending of rectangular, point-supported plates

3.1. Forced and free vibration of a plate supported at the corners and on elastic ribs

In order to simplify the problem considered, let us assume the load introducing forced vibrations to be symmetric with respect to the two axes of symmetry of the plate (Fig. 1).

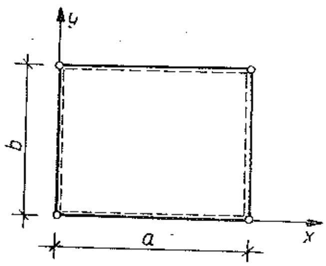


Fig. 1

Along the edges of the plate, the following conditions hold true:

(3.1) $$\left[\frac{d^4 w}{dy^4}-c_1^2 w\right]_{\substack{x=0\\x=a}}=\frac{Q_1}{EJ_1}\bigg|_{\substack{x=0\\x=a}}, \qquad \left[\frac{d^4 w}{dx^4}-c_2^2 w\right]_{\substack{y=0\\y=b}}=\frac{Q_2}{EJ_2}\bigg|_{\substack{y=0\\y=b}},$$

with the Kirchhoff forces

(3.2) $$Q_1=\frac{\partial M_{11}}{\partial x}+2\frac{\partial M_{12}}{\partial y}, \qquad Q_2=\frac{\partial M_{22}}{\partial y}+2\frac{\partial M_{12}}{\partial x},$$

and with the following notations:

(3.3) $$c_1^2=\frac{\mu_1\omega^2}{EJ_1}, \qquad c_2^2=\frac{\mu_2\omega^2}{EJ_2},$$

μ_1, J_1, μ_2, J_2 — masses per unit length of the stiffeners and the cross-sectional moments of inertia of the ribs.

Amplitudes of forces Q_1 and Q_2 are expressed by the following Fourier series:

$$(3.4)\qquad \begin{aligned} Q_1 &= \frac{4}{ab}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} Q_{mn}^{(1)}\lambda_m \cos\alpha_m x \sin\beta_n y,\\ Q_2 &= \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} Q_{mn}^{(2)}\lambda_n \sin\alpha_m x \cos\beta_n y, \end{aligned}$$

with the notations

$$\lambda_m = 1-0{,}5\delta_{m0}, \qquad \lambda_n = 1-0{,}5\delta_{n0},$$

δ_{m0} and δ_{n0} denoting the Kronecker symbols.

Making use of the series (3.4) and (2.19), the boundary conditions (3.1) are reduced to the form

$$(3.5)\qquad \frac{2}{a}\sum_{m=1,3,\dots}^{\infty} Q_{mn}^{(1)} = EJ_1 w_n^{(1)} \Lambda_n, \qquad \frac{2}{b}\sum_{n=1,3,\dots}^{\infty} Q_{mn}^{(2)} = EJ_2 w_m^{(2)} \Lambda_m.$$

Here

$$(3.6)\qquad \Lambda_m = \alpha_m^4 - c_2^2, \qquad \Lambda_n = \beta_n^4 - c_1^2.$$

According to the formula (3.2) we obtain

$$(3.7)\qquad \begin{aligned} Q_{mn}^{(1)} &= \int_0^a\int_0^b \left(\frac{\partial M_{11}}{\partial x} + 2\frac{\partial M_{12}}{\partial y}\right)\cos\alpha_m x \sin\beta_n y\, dx\, dy,\\ Q_{mn}^{(2)} &= \int_0^a\int_0^b \left(\frac{\partial M_{22}}{\partial y} + 2\frac{\partial M_{12}}{\partial x}\right)\sin\alpha_m x \cos\beta_n y\, dx\, dy. \end{aligned}$$

On integrating by parts and taking into account the relations

$$\begin{gathered} w(x,b) = w(x,0), \qquad w(a,y) = w(0,y),\\ w(0,0) = w(a,b) = w(a,0) = w(0,b) = 0,\\ M_{11}(0,y) = M_{11}(a,y) = M_{22}(x,0) = M_{22}(x,b) = 0 \end{gathered}$$

we find:

$$(3.8)\qquad \begin{aligned} Q_{mn}^{(1)} &= D\{[(2-\nu)\beta_n^2\alpha_m + \alpha_m^3]\, w_{mn} - 2[2(1-\nu)\beta_n^2 + \alpha_m^2]\, w_n^{(1)} - 2(2-\nu)\alpha_m\beta_n w_m^{(2)}\},\\ Q_{mn}^{(2)} &= D\{[(2-\nu)\alpha_m^2\beta_n + \beta_n^3]\, w_{mn} - 2[2(1-\nu)\alpha_m^2 + \beta_n^2]\, w_m^{(2)} - 2(2-\nu)\alpha_m\beta_n w_n^{(1)}\}. \end{aligned}$$

The expression (2.15) is now simplified and assumes the form:

$$(3.9)\qquad B_{mn} = D\alpha_m(\alpha_m^2 + 2\beta_n^2 - \nu\beta_n^2) W_{mn}^{(1)} + D\beta_n(\beta_n^2 + 2\alpha_m^2 - \nu\alpha_m^2) W_{mn}^{(2)},$$

in which, resulting from the Eqs. (2.21), the following relations hold true:

$$(3.10)\qquad W_{mn}^{(1)} = -2w_n^{(1)}, \qquad W_{mn}^{(2)} = -2w_m^{(2)}.$$

Taking into account the expression (3.9) in the Eq. (2.16), we transform the formulae (3.8) to the form:

$$Q_{mn}^{(1)}=\frac{(2-\nu)\,\alpha_m\beta_n^2+\alpha_m^3}{\Omega_{mn}}p_{mn}-2D(1-\nu)\frac{[2\beta_n^2+(1+\nu)\,\alpha_m^2]\,\beta_n^4}{\Omega_{mn}}w_n^{(1)}+$$

(3.11)
$$+2D(1-\nu)^2\frac{\alpha_m^3\beta_n^3}{\Omega_{mn}}w_m^{(2)}+2Dc^2\frac{[2(1-\nu)\,\beta_n^2+\alpha_m^2]\,w_n^{(1)}+(2-\nu)\,\alpha_m\beta_n w_m^{(2)}}{\Omega_{mn}},$$

$$Q_{mn}^{(2)}=\frac{(2-\nu)\,\alpha_m^2\beta_n+\beta_n^3}{\Omega_{mn}}p_{mn}-2D(1-\nu)\frac{[2\alpha_m^2+(1+\nu)\,\beta_n^2]\,\alpha_m^4}{\Omega_{mn}}w_m^{(2)}+$$

$$+2D(1-\nu)^2\frac{\alpha_m^3\beta_n^3}{\Omega_{mn}}w_n^{(1)}+2Dc^2\frac{[2(1-\nu)\,\alpha_m^2+\beta_n^2]\,w_m^{(2)}+(2-\nu)\,\alpha_m\beta_n w_n^{(1)}}{\Omega_{mn}},$$

where

(3.12)
$$\Omega_{mn}=\Delta_{mn}-c^2.$$

Making use of the Eqs. (3.11) in the Eqs. (3.5), we obtain

$$[(2-\nu)\,\beta_n^2F_n^{(1)}+F_n^{(3)}]-2D(1-\nu)\,\beta_n^4\,[2\beta_n^2S_n^{(0)}+(1+\nu)\,S_n^{(2)}]\,w_n^{(1)}+$$

(3.13)
$$+2D(1-\nu)^2\beta_n^3\sum_{m=1,3,\ldots}^{\infty}\frac{\alpha_m^3}{\Omega_{mn}}w_m^{(2)}+2Dc^2\Bigg\{[2(1-\nu)\,\beta_n^2S_n^{(0)}+S_n^{(2)}]\,w_n^{(1)}+$$

$$+(2-\nu)\,\beta_n\sum_{m=1,3,\ldots}^{\infty}\frac{\alpha_m}{\Omega_{mn}}w_m^{(2)}\Bigg\}=\frac{EJ_1a}{2}\Delta_n w_n^{(1)},$$

$$[(2-\nu)\,\alpha_m^2F_m^{(1)}+F_m^{(3)}]-2D(1-\nu)\,\alpha_m^4\,[2\alpha_m^2S_m^{(0)}+(1+\nu)\,S_m^{(2)}]\,w_m^{(2)}+$$

$$+2D(1-\nu)^2\alpha_m^3\sum_{n=1,3,\ldots}^{\infty}\frac{\beta_n^3}{\Omega_{mn}}w_n^{(1)}+2Dc^2\Bigg\{[2(1-\nu)\,\alpha_m^2S_m^{(0)}+S_m^{(2)}]\,w_m^{(2)}+$$

$$+(2-\nu)\,\alpha_m\sum_{n=1,3,\ldots}^{\infty}\frac{\beta_n}{\Omega_{mn}}w_n^{(1)}\Bigg\}=\frac{EJ_2b}{2}\Delta_m w_m^{(2)}.$$

Here

$$S_n^{(0)}=\sum_{m=1,3,\ldots}^{\infty}\frac{1}{\Omega_{mn}}=\frac{a}{8c}\left(\frac{\operatorname{th}\Psi_n\frac{a}{2}}{\Psi_n}-\frac{\operatorname{th}\Phi_n\frac{a}{2}}{\Phi_n}\right),$$

$$S_n^{(2)}=\sum_{m=1,3,\ldots}^{\infty}\frac{\alpha_m^2}{\Omega_{mn}}=\frac{a}{8c}\left(\Phi_n\operatorname{th}\Phi_n\frac{a}{2}-\Psi_n\operatorname{th}\Psi_n\frac{a}{2}\right),$$

(3.14)
$$F_n^{(1)}=\sum_{m=1,3,\ldots}^{\infty}\frac{p_{mn}\alpha_m}{\Omega_{mn}}=\int_0^a\int_0^b p(x,y)f_n^{(1)}(x)\sin\beta_ny\,dx\,dy,$$

$$f_n^{(1)}(x)=\sum_{m=1,3,\ldots}^{\infty}\frac{\alpha_m\sin\alpha_mx}{\Omega_{mn}}=\frac{a}{8c}\left[\frac{\operatorname{ch}\Psi_n\left(\frac{a}{2}-x\right)}{\operatorname{ch}\Psi_n\frac{a}{2}}-\frac{\operatorname{ch}\Phi_n\left(\frac{a}{2}-x\right)}{\operatorname{ch}\Phi_n\frac{a}{2}}\right],$$

$$F_n^{(3)} = \sum_{m=1,3,\dots}^{\infty} \frac{p_{mn}\alpha_m^3}{\Omega_{mn}} = \int_0^a \int_0^b p(x,y) f_n^{(3)}(x) \sin\beta_n y \, dx\, dy,$$

$$f_n^{(3)}(x) = \sum_{m=1,3,\dots}^{\infty} \frac{\alpha_m^3 \sin\alpha_m x}{\Omega_{mn}} = \frac{a}{8c}\left[\Phi_n^2 \frac{\operatorname{ch}\Phi_n\left(\frac{a}{2}-x\right)}{\operatorname{ch}\Phi_n \frac{a}{2}} - \Psi_n^2 \frac{\operatorname{ch}\Psi_n\left(\frac{a}{2}-x\right)}{\operatorname{ch}\Psi_n\frac{a}{2}}\right],$$

$$S_m^{(0)} = \sum_{n=1,3,\dots}^{\infty} \frac{1}{\Omega_{mn}} = \frac{b}{8c}\left(\frac{\operatorname{th}\Psi_m \frac{b}{2}}{\Psi_m} - \frac{\operatorname{th}\Phi_m\frac{b}{2}}{\Phi_m}\right),$$

$$S_m^{(2)} = \sum_{n=1,3,\dots}^{\infty} \frac{\beta_n^2}{\Omega_{mn}} = \frac{b}{8c}\left(\Phi_m \operatorname{th}\Phi_m\frac{b}{2} - \Psi_m \operatorname{th}\Psi_m \frac{b}{2}\right),$$

(3.14) [c.d.]

$$F_m^{(1)} = \sum_{n=1,3,\dots}^{\infty} \frac{p_{mn}\beta_n}{\Omega_{mn}} = \int_0^a\int_0^b p(x,y) f_m^{(1)}(y)\sin\alpha_m x\, dx\, dy,$$

$$f_m^{(1)}(y) = \sum_{n=1,3,\dots}^{\infty} \frac{\beta_n \sin\beta_n y}{\Omega_{mn}} = \frac{b}{8c}\left[\frac{\operatorname{ch}\Psi_m\left(\frac{b}{2}-y\right)}{\operatorname{ch}\Psi_m\frac{b}{2}} - \frac{\operatorname{ch}\Phi_m\left(\frac{b}{2}-y\right)}{\operatorname{ch}\Phi_m\frac{a}{2}}\right],$$

$$F_m^{(3)} = \sum_{n=1,3,\dots}^{\infty} \frac{p_{mn}\beta_n^3}{\Omega_{mn}} = \int_0^a\int_0^b p(x,y) f_m^{(3)}(y)\sin\alpha_m x\, dx\, dy,$$

$$f_m^{(3)}(y) = \sum_{n=1,3,\dots}^{\infty} \frac{\beta_n^3 \sin\beta_n y}{\Omega_{mn}} = \frac{b}{8c}\left[\Phi_m^2 \frac{\operatorname{ch}\Phi_m\left(\frac{b}{2}-x\right)}{\operatorname{ch}\Phi_m\frac{b}{2}} - \Psi_m^2\frac{\operatorname{ch}\Psi_m\left(\frac{b}{2}-x\right)}{\operatorname{ch}\Psi_m\frac{b}{2}}\right].$$

In the Eqs. (3.14), the following notations are introduced:

$$\Phi_n, \Psi_n = \sqrt{\beta_n^2 \pm c}, \qquad \Phi_m, \Psi_m = \sqrt{\alpha_m^2 \pm c}.$$

The system of Eqs. (3.13) may now be written in the form:

(3.15)

$$C_n^{(1)} w_n^{(1)} - \sum_{m=1,3,\dots}^{\infty} \frac{(1-\nu)^2\beta_n^2\alpha_m^2 + c^2(2-\nu)}{\Omega_{mn}}\alpha_m w_m^{(2)} = Z_n^{(1)}, \qquad n = 1, 3, 5, \dots,$$

$$-\sum_{n=1,3,\dots}^{\infty} \frac{(1-\nu)^2\beta_n^2\alpha_m^2 + c^2(2-\nu)}{\Omega_{mn}}\beta_n w_n^{(1)} + C_m^{(1)} w_m^{(2)} = Z_m^{(1)}, \qquad m = 1, 3, 5, \dots,$$

in which, with $\alpha_m^2 \geqslant c$, $\beta_n^2 \geqslant c$, the relations

(3.16)

$$C_n^{(1)} = \frac{a}{8\beta_n c}\left[\Psi_n(\Phi_n^2 - \nu\beta_n^2)^2 \operatorname{th}\Psi_n\frac{a}{2} - \Phi_n(\Psi_n^2-\nu\beta_n^2)^2 \operatorname{th}\Phi_n\frac{a}{2} + 8\kappa_1 c\Lambda_n\right],$$

$$C_m^{(1)} = \frac{b}{8\alpha_m c}\left[\Psi_m(\Phi_m^2 - \nu\alpha_m^2)^2 \operatorname{th}\Psi_m\frac{b}{2} - \Phi_m(\Psi_m^2 - \nu\alpha_m^2)^2 \operatorname{th}\Phi_m\frac{b}{2} + 8\kappa_2 c\Lambda_m\right],$$

$$Z_n^{(1)}=\frac{\alpha}{16D\beta_n c}\int_0^a\int_0^b p(x,y)\left[(\Phi_n^2-\nu\beta_n^2)\frac{\operatorname{ch}\Psi_n\left(\frac{a}{2}-x\right)}{\operatorname{ch}\Psi_n\frac{a}{2}}-\right.$$

(3.17)
$$\left.-(\Psi_n^2-\nu\beta_n^2)\frac{\operatorname{ch}\Phi_n\left(\frac{a}{2}-x\right)}{\operatorname{ch}\Phi_n\frac{a}{2}}\right]\sin\beta_n y\,dx\,dy,$$

$$Z_m^{(1)}=\frac{b}{16D\alpha_m c}\int_0^a\int_0^b p(x,y)\left[(\Phi_m^2-\nu\alpha_m^2)\frac{\operatorname{ch}\Psi_m\left(\frac{b}{2}-y\right)}{\operatorname{ch}\Psi_m\frac{b}{2}}-\right.$$

$$\left.-(\Psi_m^2-\nu\alpha_m^2)\frac{\operatorname{ch}\Phi_m\left(\frac{b}{2}-y\right)}{\operatorname{ch}\Phi_m\frac{b}{2}}\right]\sin\alpha_m x\,dx\,dy,$$

hold, with the notations:

(3.18)
$$\kappa_1=\frac{EJ_1}{4D},\qquad \kappa_2=\frac{EJ_2}{4D}.$$

With $\beta_n^2<c$ and $\alpha_m^2<c$ we use the relations

$$\sqrt{x}=i\sqrt{|x|},\quad \operatorname{th} iy=i\operatorname{tg}y,\quad \operatorname{sh} iy=i\sin y,\quad \operatorname{ch} iy=\cos y,$$

$$\operatorname{sh} i\Psi_n x\operatorname{th} i\Psi_n\frac{a}{2}=-\sin\Psi_n x\operatorname{tg}\Psi_n\frac{a}{2},$$

$$i\Psi_n\operatorname{th} i\Psi_n\frac{a}{2}=-\Psi_n\operatorname{tg}\Psi_n\frac{a}{2},\qquad \text{for }\ \beta_n^2<c,$$

$$\operatorname{ch} i\Psi_n x=\cos\Psi_n x,$$

$$\operatorname{sh} i\Psi_m y\operatorname{th} i\Psi_m\frac{b}{2}=-\sin\Psi_m y\operatorname{tg}\Psi_m\frac{b}{2},$$

$$i\Psi_m\operatorname{th} i\Psi_m\frac{b}{2}=-\Psi_m\operatorname{tg}\Psi_m\frac{b}{2},\qquad \text{for }\ \alpha_m^2<c;$$

$$\operatorname{ch} i\Psi_m y=\cos\Psi_m y,$$

Here,

$$\Psi_n=\sqrt{|\beta_n^2-c|},\qquad \Psi_m=\sqrt{|\alpha_m^2-c|}.$$

From the set of Eqs. (3.15) (for a certain finite number of equations) the coefficients $w_m^{(2)}$ and $w_n^{(1)}$ may be calculated which in turn enable us to determine [by means of the series (2.19)] the deflection amplitudes at the edges of the plate.

The coefficients $w_m^{(2)}$ and $w_n^{(1)}$ being known, the expression (3.9) may be determined; coefficients w_{mn} are then found by means of the formula (2.16) to yield the deflection and stress amplitudes occurring at various points of the middle surface of the plate.

Equating to zero the principal determinant of the Eqs. (3.15), we obtain the characteristic equation making it possible to determine the angular frequencies of free vibrations of the plate.

Example 1

Let us calculate the frequency of free vibrations of a square plate supported at the corners. In such a case

$$a=b, \quad \alpha_m=\frac{m\pi}{a}, \quad \beta_n=\frac{n\pi}{a}, \quad \kappa_1=\kappa_2=0, \quad w_n^{(1)}=w_n, \quad w_m^{(2)}=w_m,$$

and the set of Eqs. (3.15) together with (3.16) is reduced to a single, homogeneous system which, after transformation, may be written in the form:

$$(3.19)\quad \frac{\pi}{8\lambda n}\Big[\sqrt{n^2-\lambda}\,(n^2+\lambda-\nu n^2)^2 \operatorname{th}\frac{\pi}{2}\sqrt{n^2-\lambda}- -\sqrt{n^2+\lambda}\,(n^2-\lambda-\nu n^2)^2 \operatorname{th}\frac{\pi}{2}\sqrt{n^2+\lambda}\Big] w_n- -\sum_{m=1,3,\dots}^{\infty}\frac{(1-\nu)^2 m^2 n^2+\lambda^2(2-\nu)}{(m^2+n^2)^2-\lambda^2}\, m w_m=0, \qquad n=1,3,5,\dots,$$

with the notations

$$\lambda^2=c^2\frac{a^4}{\pi^4}=\omega^2\frac{\rho a^4}{D\pi^4}.$$

Taking into account three equations in the set (3.18) for $m=1, 3, 5$, the lowest angular frequency of free vibrations corresponding to the symmetric form of the deflection surface is found to be equal to

$$(3.20)\qquad \omega=0{,}72\frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho}}.$$

This result differs only by 0.01% from the free vibration frequency calculated on the basis of two equations in (3.19) — i.e., for $m=1,3$.

3.2. Bending of a rectangular plate supported at the corners and on the ribs

Let us consider the static bending of a rectangular plate symmetrically loaded and supported at the corners and on elastic ribs. Assuming in the Eq. (2.16) $\omega=0$, and making use of the expressions (3.9), (3.10), we obtain:

$$(3.21)\quad w_{mn}=\frac{p_{mn}}{D\Delta_{mn}}+\frac{2}{\Delta_{mn}}\{\alpha_m[\alpha_m^2+\beta_n^2(2-\nu)]\,w_n^{(1)}+\beta_n[\beta_n^2+\alpha_m^2(2-\nu)]\,w_m^{(2)}\}.$$

The equation of the middle, deformed surface of the plate is then found by means of the Eqs. (2.6), (3.21), and (2.2), as also the expressions for bending moments:

$$w(x,y)=w^*(x,y)-\frac{1}{b}\sum_{n=1,3,\dots}^{\infty}\frac{w_n^{(1)}\sin\beta_n y}{\operatorname{ch}\beta_n a+1}[(1-\nu)H_n(x)+I_n(x)]+ \tag{3.22}$$

$$+\frac{1}{a}\sum_{m=1,3,\dots}^{\infty}\frac{w_m^{(2)}\sin\alpha_m x}{\operatorname{ch}\alpha_m b+1}[(1-\nu)H_m(y)+I_m(y)].$$

$$M_{11}(x,y)=M_{11}^*(x,y)-\frac{1}{b}D(1-\nu)^2\sum_{n=1,3,\dots}^{\infty}\frac{w_n^{(1)}\beta_n^2\sin\beta_n y}{\operatorname{ch}\beta_n a+1}H_n(x)+$$

$$+\frac{1}{a}D\sum_{m=1,3,\dots}^{\infty}\frac{w_m^{(2)}\alpha_m^2\sin\alpha_m x}{\operatorname{ch}\alpha_m b+1}[(1-\nu)^2H_m(y)+(1-\nu^2)I_m(y)], \tag{3.23}$$

$$M_{22}(x,y)=M_{22}^*(x,y)-\frac{1}{a}D(1-\nu)^2\sum_{m=1,3,\dots}^{\infty}\frac{w_m^{(2)}\alpha_m^2\sin\alpha_m x}{\operatorname{ch}\alpha_m b+1}H_m(y)+$$

$$+\frac{1}{b}D\sum_{n=1,3,\dots}^{\infty}\frac{w_n^{(1)}\beta_n^2\sin\beta_n y}{\operatorname{ch}\beta_n a+1}[(1-\nu)^2H_n(x)+(1-\nu^2)I_n(x)],$$

where

$$\begin{aligned}H_n(x)&=\beta_n[x\operatorname{sh}\beta_n(a-x)+(a-x)\operatorname{sh}\beta_n x], & I_n(x)&=2[\operatorname{ch}\beta_n(a-x)+\operatorname{ch}\beta_n x],\\ H_m(y)&=\alpha_m[y\operatorname{sh}\alpha_m(b-y)+(b-y)\operatorname{sh}\alpha_m y], & I_m(y)&=2[\operatorname{ch}\alpha_m(b-y)+\operatorname{ch}\alpha_m y],\end{aligned} \tag{3.24}$$

and $w^*(x,y)$, $M_{11}^*(x,y)$, $M_{22}^*(x,y)$ denote the corresponding functions describing the deflections and bending moments occurring in the middle of deformed plate simply supported along the edges. The functions are expressed by the following, well-known formulae:

$$w^*(x,y)=\frac{4}{abD}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{\sin\alpha_m x\sin\beta_n y}{\Delta_{mn}}\times \tag{3.25}$$

$$\times\int_0^a\int_0^b p(x,y)\sin\alpha_m x\sin\beta_n y\,dx\,dy,$$

$$M_{11}^*(x,y)=\frac{4}{ab}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{\alpha_m^2+\nu\beta_n^2}{\Delta_{mn}}\times$$

$$\times\sin\alpha_m x\sin\beta_n y\int_0^a\int_0^b p(x,y)\sin\alpha_m x\sin\beta_n y\,dx\,dy, \tag{3.26}$$

$$M_{22}^*(x,y)=\frac{4}{ab}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{\nu\alpha_m^2+\beta_n^2}{\Delta_{mn}}\sin\alpha_m x\sin\beta_n y\times$$

$$\times\int_0^a\int_0^b p(x,y)\sin\alpha_m x\sin\beta_n y\,dx\,dy.$$

The simple Fourier series appearing in the Eqs. (3.22), (2.23) are found by means of the corresponding double Fourier series in which summation over one of the indices is performed; this is of primary importance for improving the convergence of results. Summation over one of the indices may also be performed in the Eqs. (3.25), (3.26) (provided the series are uniformly convergent to allow for the change of order of summation and integration). Calculation on a computer proved, however, that this fact is unimportant.

In order to determine the unknown coefficients appearing in the Eqs. (3.22), (3.23), $w_m^{(2)}$ and $w_n^{(1)}$, we shall use the set of Eqs. (3.15) which, for $\omega \to 0$, may be reduced by means of the d'Hospital rule to the form:

$$
\begin{aligned}
&C_n^{(2)} w_n^{(1)} - (1-\nu)^2 \beta_n^2 \sum_{m=1,3,\ldots}^{\infty} \frac{\alpha_m^3}{\Delta_{mn}} w_m^{(2)} = Z_n^{(2)}, \qquad n=1,3,5,\ldots, \\
&-(1-\nu)^2 \alpha_m^2 \sum_{n=1,3,\ldots}^{\infty} \frac{\beta_n^3}{\Delta_{mn}} w_n^{(1)} + C_m^{(2)} w_m^{(2)} = Z_m^{(2)}, \qquad m=1,3,5,\ldots .
\end{aligned}
\tag{3.27}
$$

Here,

$$
\begin{aligned}
&C_n^{(2)} = \lim_{\omega\to 0} C_n^{(1)} = \frac{(1-\nu)\beta_n^2 a}{4} \left\{ \frac{1}{2(\operatorname{ch}\beta_n a + 1)} [(3+\nu)\operatorname{sh}\beta_n a - (1-\nu)\beta_n a] + \right. \\
&\qquad\qquad \left. + r_1 (1+\nu)\beta_n a \right\}, \\
&C_m^{(2)} = \lim_{\omega\to 0} C_m^{(1)} = \frac{(1-\nu)\alpha_m^2 b}{4} \left\{ \frac{1}{2(\operatorname{ch}\alpha_m b + 1)} [(3+\nu)\operatorname{sh}\alpha_m b - (1-\nu)\alpha_m b] + \right. \\
&\qquad\qquad \left. + r_2 (1+\nu)\alpha_m b \right\},
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
&Z_n^{(2)} = \lim_{\omega\to 0} Z_n^{(1)} = \frac{a}{16D\beta_n(\operatorname{ch}\beta_n a + 1)} \int_0^a \int_0^b p(x, y)\,[(1-\nu)H_n(x) + \\
&\qquad\qquad + I_n(x)] \sin\beta_n y \, dx\, dy, \\
&Z_m^{(2)} = \lim_{\omega\to 0} Z_m^{(1)} = \frac{b}{16D\alpha_m(\operatorname{ch}\alpha_m b + 1)} \int_0^a \int_0^b p(x, y)\,[(1-\nu)H_m(y) + \\
&\qquad\qquad + I_m(y)] \sin\alpha_m x \, dx\, dy .
\end{aligned}
\tag{3.29}
$$

$$
r_1 = \frac{4\kappa_1}{(1-\nu^2)a} = \frac{s_1 t_1^3}{ah^3}, \qquad r_2 = \frac{4\kappa_2}{(1-\nu^2)b} = \frac{s_2 t_2^3}{bh^3} .
\tag{3.30}
$$

In the Eqs. (3.30), s_1, s_2, t_1, t_2 denote the respective widths and hights of the ribs, and h — the thickness of the plate.

If the thickness ribs along the edges of the plate are disregarded — i.e., if we put $r_1 = r_2 = 0$ in the Eqs. (3.28), we obtain the corresponding static and geometric magnitudes of a plate supported at the corners.

Let us consider a very specialized case of a plate supported at the corners and on elastic ribs, loaded uniformly over the entire surface. In that case, the expression (3.25) occurring in the formula (3.22) assumes the form:

$$w^*(x, y) = \frac{16p}{abD} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin\alpha_m x \sin\beta_n y}{\alpha_m \beta_n \Delta_{mn}}. \tag{3.31}$$

If the function $w^*(x, y)$ is known, we may directly determine the corresponding expressions for bending moments $M_{11}^*(x, y)$ and $M_{22}^*(x, y)$ entering into the formulae (3.23).

On the basis of the Eq. (3.29), we obtain:

$$\begin{aligned} Z_n^{(2)} &= \frac{pa}{4D\beta_n^3(\operatorname{ch}\beta_n a+1)}[(3-\nu)\operatorname{sh}\beta_n a-(1-\nu)\beta_n a], \\ Z_m^{(2)} &= \frac{pb}{4D\alpha_m^3(\operatorname{ch}\alpha_m b+1)}[(3-\nu)\operatorname{sh}\alpha_m b-(1-\nu)\alpha_m b]. \end{aligned} \tag{3.32}$$

Example 2

Let us calculate the deflections and moments of a square, uniformly loaded plate simply supported on elastic ribs. Here,

$$a=b, \quad \alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{a}, \quad r_1 = r_2 = r, \quad w_n^{(1)} = w_n, \quad w_m^{(2)} = w_m.$$

Using the expressions (3.32), we reduce the sets of equations (3.27) to a single set:

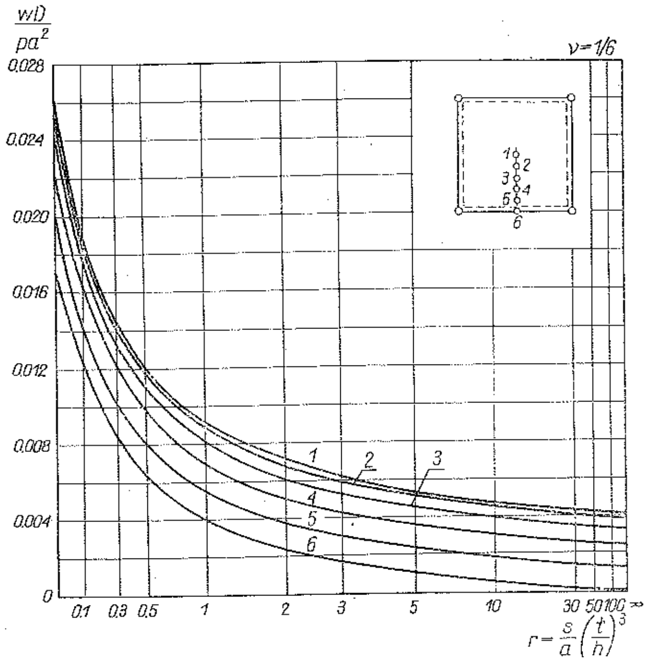
$$\begin{aligned} &\frac{\pi}{4(1-\nu)}\left[\frac{(3+\nu)\operatorname{sh} n\pi-(1-\nu)n\pi}{2(\operatorname{ch} n\pi+1)}+r(1+\nu)n\pi\right]w_n- \\ &-\sum_{m=1,3,\ldots}^{\infty}\frac{m^3}{(m^2+n^2)^2}w_m=\frac{pa^5[(3-\nu)\operatorname{sh} n\pi-(1-\nu)n\pi]}{4\pi^4(1-\nu)^2Dn^5(\operatorname{ch} n\pi+1)}, \qquad n=1,3,5,\ldots, \end{aligned} \tag{3.33}$$

The coefficients w_m, w_n are determined from that system, and the Eqs. (3.22), (3.23), (3.31) are used to evaluate the deflections and bending moments.

Note that, owing to the good convergence of the series, taking into account merely the three coefficients w_m yields the deflection with an accuracy sufficient for practical purposes. In calculating the bending moments, however, seven terms of the series should be retained, and the coefficients w_m should be found from the solution of a set of seven equations of (3.33).

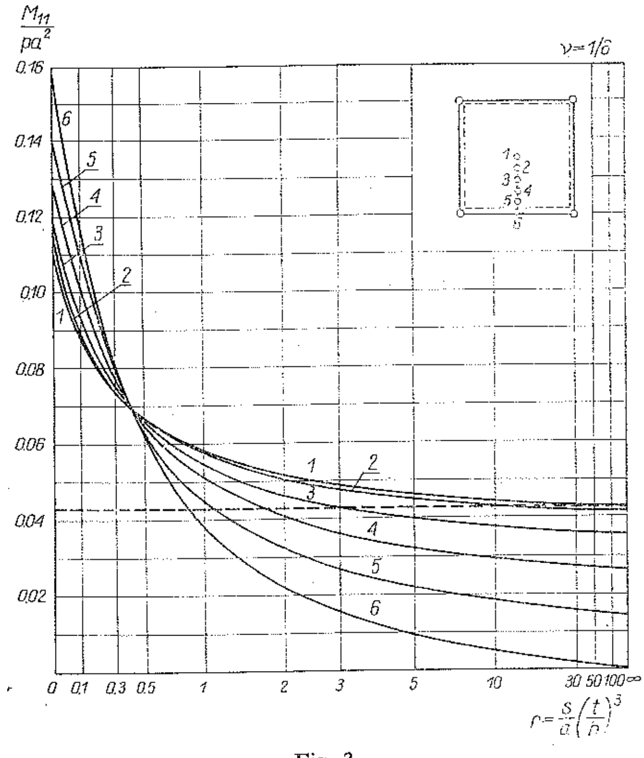
The graphs in Figs. 2 and 3 demonstrate the values of deflections and bending moments at several points of the plate, and for various geometric parameters of the plate and the stiffener. It is seen that with $r \to \infty$ we obtain the deflections and bending moments of a simply supported plate.

In Fig. 3 dashed line shows the value of the bending moment in the middle of the plate freely supported at the circumference $M_{11}\left(\frac{a}{2}, \frac{a}{2}\right) = 0.0429\ pa^2$.



wD/pa^2
ν=1/6
0.028
0.024
0.020
0.016
0.012
0.008
0.004
0
1
2
3
4
5
6
0.1 0.3 0.5 1 2 3 5 10 30 50 100 ∞
r = s/a (t/h)^3

Fig. 2



M11/pa^2
ν=1/6
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
1
2
3
4
5
6
0 0.1 0.3 0.5 1 2 3 5 10 30 50 100 ∞
r = s/a (t/h)^3

Fig. 3

Example 3

Let us determine the deflections and bending moments of a uniformly loaded rectangular plate, supported at the corners.

To that end we shall use the formulae (3.31), (3.32). In the calculations, we assume:

$$\frac{a}{b}=2, \quad r_1=r_2=0, \quad \nu=\frac{1}{6}.$$

In Fig. 4 are shown the deflections and bending moments occurring along the edges and in the middle cross-sections of the plate.

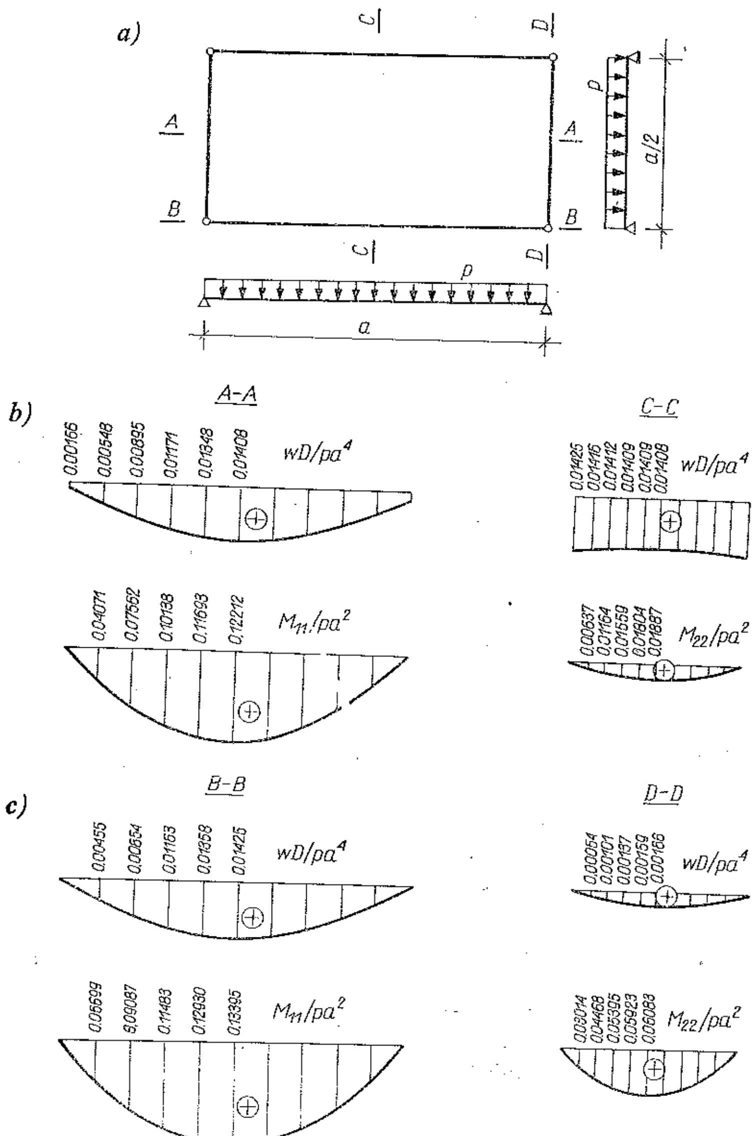


Fig. 4

Table 1 contains certain results concerning the plate under the assumption that $\nu=0$ as compared with the results derived in [4] and Tables published in [3].

Table 1

	Finite Fourier transformation	A. S. Kałmanok	St. Lee, P. Ballesteros	
$w\left(\frac{a}{2}, \frac{b}{2}\right)$	0.01433	0.0143	0.01447	
$w\left(0, \frac{b}{2}\right)$	0.00180	0.0018	0.00210	$\frac{pa^4}{D}$
$w\left(\frac{a}{2}, 0\right)$	0.01394	0.0140	0.01431	
$M_{11}\left(\frac{a}{2}, \frac{b}{2}\right)$	0.12205	0.1218	0.11979	
$M_{22}\left(\frac{a}{2}, \frac{b}{2}\right)$	0.01533	0.0152	0.01042	pa^2
$M_{11}\left(\frac{a}{2}, 0\right)$	0.13024	0.1300	0.13542	
$M_{22}\left(0, \frac{b}{2}\right)$	0.06539	0.0655	0.07292	

Substantial differences of results may be observed in that Table. This is probably due to the approximate charater of boundary conditions in the solution [4].

3.3. Bending of a symmetrically loaded, rectangular plate supported at points lying at a certain distance from the edges

Let us first consider the problem of bending of a rectangular plate with free edges, supported at four corners and symmetrically loaded by four concentrated forces P. The forces are assumed to act at the distances u, v from the edges (Fig. 5).

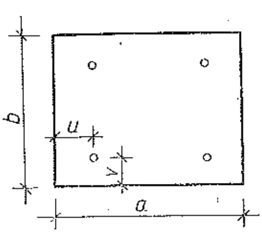


Fig. 5

The load is expressed by the following function:

$$p(x, y)=P\delta[x-u, y-v]+ \\ +P\delta[x-u, y-(b-v)]+P\delta[x-(a-u), y-v]+ \\ +P\delta[x-(a-u), y-(b-v)]. \tag{3.34}$$

$\delta[\ ]$ denoting the Dirac delta-functions.

With such loading, the Eq. (3.25) yields the function:

$$w^*(x, y)=\frac{16P}{abD}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin\alpha_m u \sin\beta_n v}{\Delta_{mn}}\sin\alpha_m x\sin\beta_n y, \tag{3.35}$$

which enables a simple determination of the corresponding formulae for the moments $M^*_{11}(x, y)$ and $M^*_{22}(x, y)$. The form of the remaining terms of the Eqs. (3.22) and (3.23) is unchanged.

Now we have $r_1 = r_2 = 0$, and hence, on the basis of the Eqs. (3.28), we obtain the expressions:

$$C_n^{(3)} = \frac{(1-\nu)\beta_n^2 a}{8(\operatorname{ch}\beta_n a + 1)}[(3+\nu)\operatorname{sh}\beta_n a - (1-\nu)\beta_n a],$$

(3.36)

$$C_m^{(3)} = \frac{(1-\nu)\alpha_m^2 b}{8(\operatorname{ch}\alpha_m b + 1)}[(3+\nu)\operatorname{sh}\alpha_m b - (1-\nu)\alpha_m b].$$

Introducing the load function (3.34) into the Eqs. (3.29), we obtain:

$$Z_n^{(2)} = \frac{Pa\sin\beta_n v}{4D\beta_n(\operatorname{ch}\beta_n a + 1)}[(1-\nu)H_n(u) + I_n(u)],$$

(3.37)

$$Z_m^{(2)} = \frac{Pb\sin\alpha_m u}{4D\alpha_m(\operatorname{ch}\alpha_m b + 1)}[(1-\nu)H_m(v) + I_m(v)].$$

If the forces P expressed by the Eqs. (3.34) are assumed to counterbalance the load acting on the plate

(3.38) $$4P = -\int_0^a\int_0^b p(x, y)\,dx\,dy,$$

then the corner reactions must vanish.

Equations (3.25), (3.35) and (3.38) yield

(3.39) $$w^*(x, y) = \frac{4}{abD}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin\alpha_m x\sin\beta_n y}{\Delta_{mn}}\int_0^a\int_0^b p(x, y)\times$$
$$\times(\sin\alpha_m x\sin\beta_n y - \sin\alpha_m u\sin\beta_n v)\,dx\,dy.$$

The set of Eqs. (3.27) for the balanced load is reduced to

$$C_n^{(3)} w_n^{(1)} - (1-\nu)^2\beta_n^2\sum_{m=1,3,\ldots}^{\infty}\frac{\alpha_m^3}{\Delta_{mn}} w_m^{(2)} = Z_n^{(3)}, \qquad n = 1, 3, 5, \ldots,$$

(3.40)

$$-(1-\nu)^2\alpha_m^2\sum_{n=1,3,\ldots}^{\infty}{}' \frac{\beta_n^3}{\Delta_{mn}} w_n^{(1)} + C_m^{(3)} w_m^{(2)} = Z_m^{(3)}, \qquad m = 1, 3, 5, \ldots,$$

with the notations

$$Z_n^{(3)} = \frac{a}{16D\beta_n(\operatorname{ch}\beta_n a+1)} \int_0^a \int_0^b p(x,y)\{[(1-\nu)H_n(x)+I_n(x)]\sin\beta_n y - -[(1-\nu)H_n(u)+I_n(u)]\sin\beta_n v\}\,dx\,dy,$$

(3.41)

$$Z_m^{(3)} = \frac{b}{16D\alpha_m(\operatorname{ch}\alpha_m b+1)} \int_0^a \int_0^b p(x,y)\{[(1-\nu)H_m(y)+I_m(y)]\sin\alpha_m x - -[(1-\nu)H_m(v)+I_m(v)]\sin\alpha_m u\}\,dx\,dy,$$

$C_n^{(3)}$, $C_m^{(3)}$ are expressed by the formulae (3.36).

Displacements of the middle surface of the plate referred to the plane passing through the points of support located at a distance from the edges are determined from the relation:

$$\bar{w}(x,y) = w(x,y) - w(u,v). \tag{3.42}$$

Here $w(x,y)$ is the function of deflection (3.22) in which $w^*(x,y)$ is expressed by (3.39); $w(u,v)$ denotes the displacement calculated from the Eq. (3.22) at the points of application of the forces P.

In the case of a uniformly loaded plate, the expressions (3.39) and (3.41) are reduced to the form:

$$w^*(x,y) = \frac{4p}{abD} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin\alpha_m x \sin\beta_n y}{\alpha_m \beta_n \Delta_{mn}} (4 - ab\alpha_m\beta_n \sin\alpha_m u \sin\beta_n v), \tag{3.43}$$

$$Z_n^{(3)} = \frac{pa}{16D\beta_n^3(\operatorname{ch}\beta_n a+1)} \{4[(3-\nu)\operatorname{sh}\beta_n a - (1-\nu)\beta_n a] - -ab\beta_n^2[(1-\nu)H_n(u)+I_n(u)]\sin\beta_n v\},$$

(3.44)

$$Z_m^{(3)} = \frac{pb}{16D\alpha_m^3(\operatorname{ch}\alpha_m b+1)} \{4[(3-\nu)\operatorname{sh}\alpha_m b - (1-\nu)\alpha_m b] - -ab\alpha_m^2[(1-\nu)H_m(v)+I_m(v)]\sin\alpha_m u\}.$$

Example 4

Let us determine the deflections and bending moments in a square plate, loaded uniformly and supported at four points lying inside the plate.

In the calculations it is assumed that

$$a=b, \quad u=v=0.2a, \quad \nu=1/6.$$

Owing to the symmetry of the load, two systems of Eqs. (3.40) are reduced to a single set enabling the calculation of the coefficients w_m necessary for the determination of deflections and bending moments. In order to determine the deflection with an accuracy sufficient for practical purposes, eight terms of the series (3.22) are necessary. The series (3.23) (bending moments) converge rather slowly, the convergence markedly decreasing in the vicinity of the points of support. A satisfactory accuracy of results requires more than sixty terms of the series. Graphs of deflections and bending moments in certain cross-sections of the plate are shown in Fig. 6.

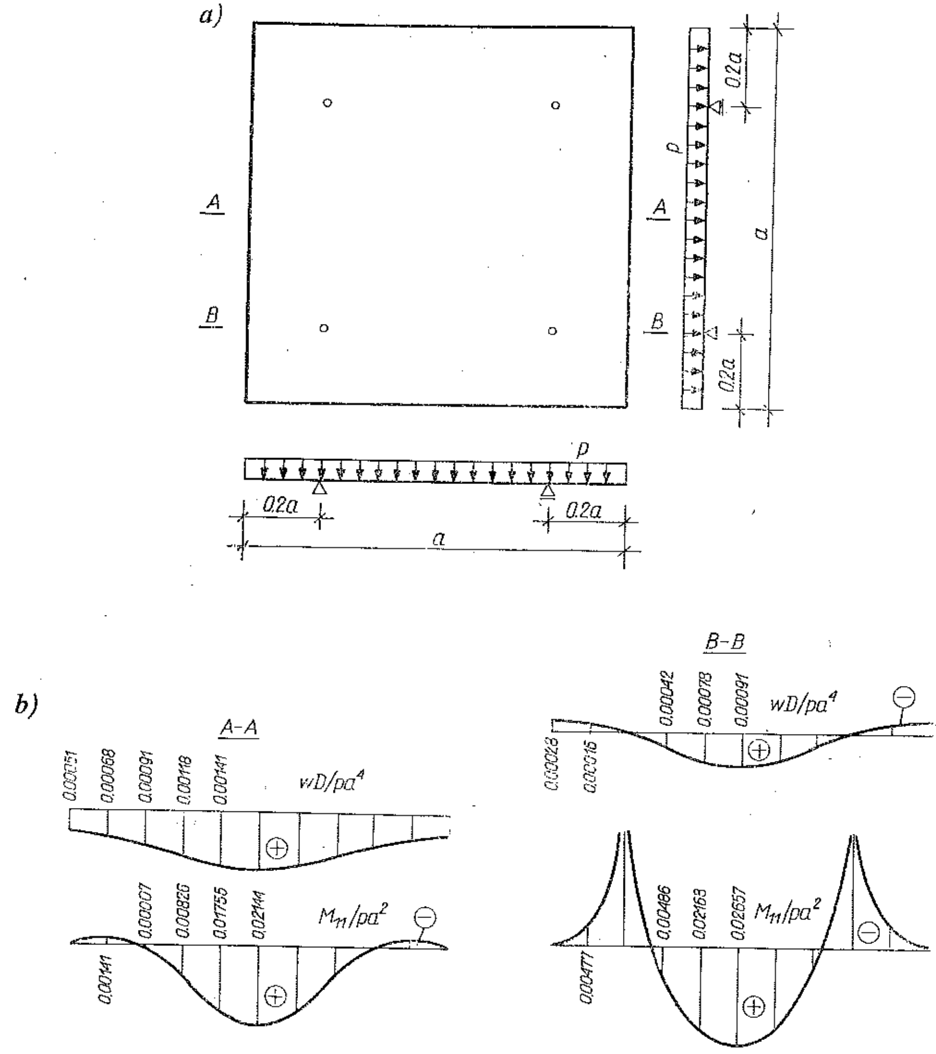


Fig. 6

Example 5

Let us calculate the deflections and bending moments in a rectangular, uniformly loaded plate, supported at two points.

For the calculations, let us apply the formulae (3.43) and (3.44) with the following parameters:

$$a/b=1.5\,, \quad u=0.2a\,, \quad v=0.5b\,, \quad \nu=1/6\,.$$

In Fig. 7 are shown the graphs of deflections and bending moments in certain cross-sections of the plate.

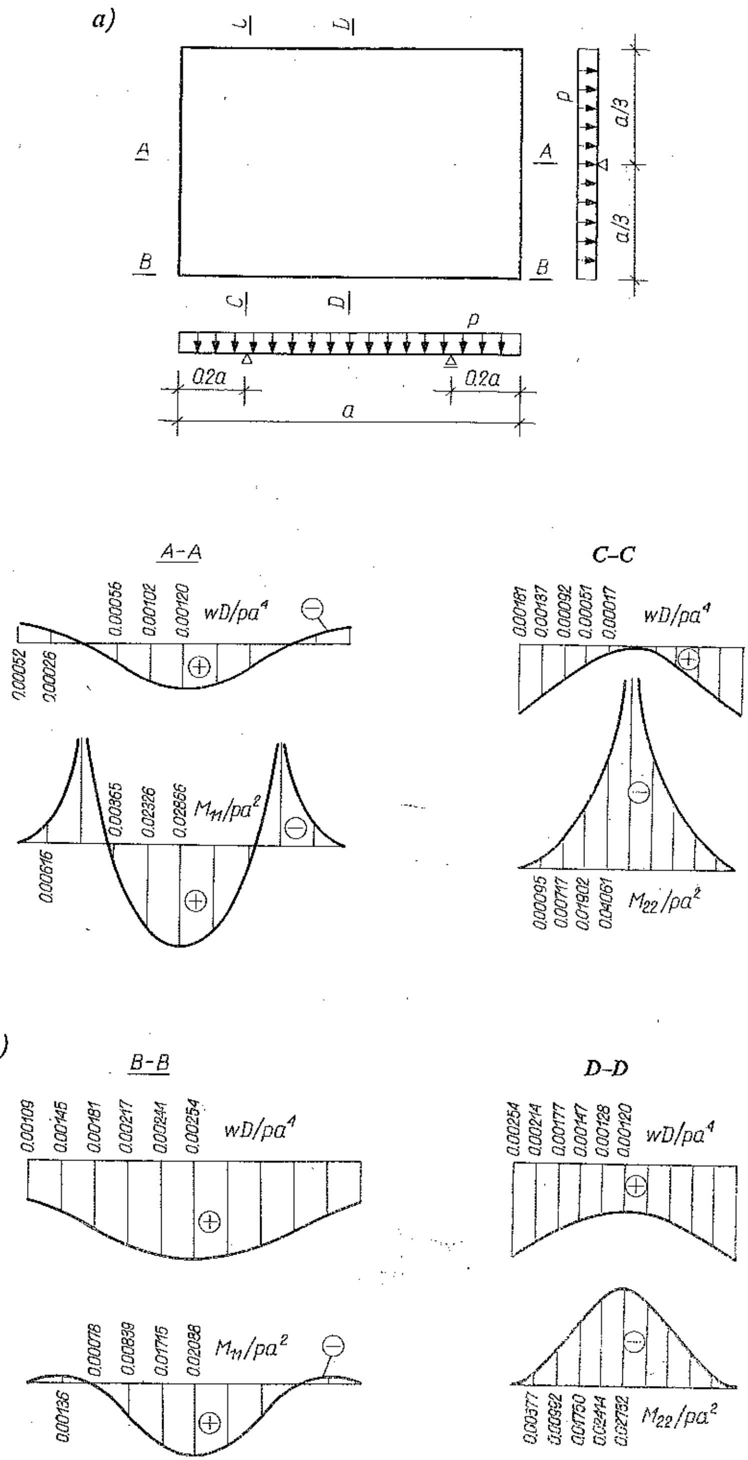

a)
A
A
B
B
C
D
C
D
p
a/3
a/3
p
0.2a
0.2a
a
b)
A–A
0.00056
0.00102
0.00120
wD/pa^4
0.00052
0.00026
0.00365
0.02326
0.02856
M11/pa^2
0.00616
C–C
0.00161
0.00137
0.00092
0.00051
0.00017
wD/pa^4
0.00095
0.00717
0.01902
0.04061
M22/pa^2
c)
B–B
0.00109
0.00145
0.00181
0.00217
0.00241
0.00254
wD/pa^4
0.00076
0.00839
0.01715
0.02098
M11/pa^2
0.00136
D–D
0.00254
0.00214
0.00177
0.00147
0.00128
0.00120
wD/pa^4
0.00377
0.00962
0.01750
0.02414
0.02732
M22/pa^2

Fig. 7

4. Bending of a triangular plate

Let us consider a plate in the form of a rightangled triangle, simply supported on the edges adjacent to the right angle, and supported on an elastic beam along the hypotenuse. Such case may easily be constructed on the basis of solutions derived previously, by means of a suitable symmetric and antisymmetric loading of a square plate (Fig. 8).

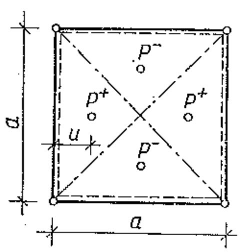


Fig. 8

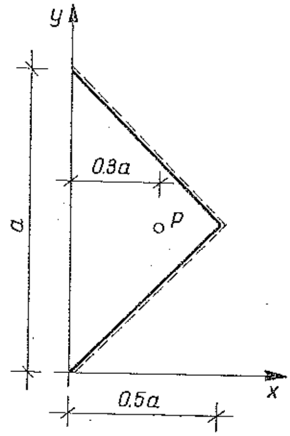


Fig. 9

The load is expressed by the formula:

$$p(x,y)=P\delta\left(x-u,\,y-\frac{a}{2}\right)+P\delta\left[x-(a-u),\,y-\frac{a}{2}\right]-\\ -P\delta\left(x-\frac{a}{2},\,y-u\right)-P\delta\left[x-\frac{a}{2},\,y-(a-u)\right]. \tag{4.1}$$

Taking into account the expression (4.1) in the formula (3.25), we obtain:

$$w^*(x,y)=\frac{8P}{a^2D}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin\alpha_m x\sin\beta_n y}{\Delta_{mn}}\times\\ \times\left(\sin\frac{n\pi}{2}\sin\alpha_m u-\sin\frac{m\pi}{2}\sin\beta_n u\right). \tag{4.2}$$

Here,

$$\alpha_m=\frac{m\pi}{a},\qquad \beta_n=\frac{n\pi}{a}.$$

The right-hand sides (3.29) of the Eqs. (3.27) have the form

$$Z_n^{(2)}=\frac{Pa}{8D\beta_n(\operatorname{ch}\beta_n a+1)}\left\{[(1-\nu)H_n(u)+I_n(u)]\sin\frac{n\pi}{2}-\right.\\ \left.-\left[(1-\nu)\beta_n a\operatorname{sh}\frac{n\pi}{2}+4\operatorname{ch}\frac{n\pi}{2}\right]\sin\beta_n u\right\}, \tag{4.3}$$

(4.3) [cont.]
$$Z_m^{(2)} = \frac{Pa}{8D\alpha_m(\operatorname{ch}\alpha_m a+1)}\left\{\left[(1-\nu)\alpha_m a \operatorname{sh}\frac{m\pi}{2}+4\operatorname{ch}\frac{m\pi}{2}\right]\sin\alpha_m u - \right.$$
$$\left. -[(1-\nu)H_m(u)+I_m(u)]\sin\frac{m\pi}{2}\right\}.$$

Functions $H_m(u)$, $H_n(u)$, $I_m(u)$, $I_n(u)$ being expressed by the Eqs. (3.24). The coefficients (3.28) of the Eqs. (3.27) are unchanged.

Example 6

Determine the deflections of the plate considered, simply supported along the legs of the triangle, the edge opposite to the right angle being free ($r=0$), acted on by a concentrated force at the axis of symmetry, at the distance $u=0, 3a$ from the free edge (Fig. 9), with $\nu=1/6$.

The numerical results are given in Table 2.

Table 2

y/a \ x/a	0	0.1	0.2	0.3	0.4	0.5
0	0.009106	0.006956	0.005067	0.003254	0.000999	0.000000
0.1	0.008537	0.006314	0.004243	0.002132	0.000000	
0.2	0.007000	0.004645	0.002328	0.000000		
0.3	0.004865	0.002425	0.000000			
0.4	0.002466	0.000000				$\frac{Pa^2}{D}$
0.5	0.000000					

5. Concluding remarks

The examples discussed in the paper indicate the practical applicability of an algorithm which may be used (by means of electronic computers) in solving numerous engineering problems concerning rectangular plates supported at isolated points, either at the corners or within the plate.

It should be added that in the case of rectangular plates supported at the corners and on elastic stiffeners, the assumed symmetry of the load substantially simplifies the calculations but does not represent a necessary condition. It is easily seen that the solutions presented may be generalized to yield a corresponding system of equations describing the general case of arbitrary static or dynamic loading.

All the solutions derived in this paper may also be generalized to the case of orthotropy, and to the practically important cases of plates supported at columns of finite cross-sections, at points lying within the region of the plate, provided the symmetry of the load is preserved.

A different approach will be required in the case of a plate supported at arbitrary distances from the edges, although even such problems, slightly more complicated than those considered in this paper, may be tackled by the method of finite Fourier transforms.

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Streszczenie

NIEKTÓRE ZAGADNIENIA STATYKI I DYNAMIKI SPRĘŻYSTYCH PŁYT PROSTOKĄTNYCH PODPARTYCH PUNKTOWO

W rozprawie opracowano na podstawie metody skończonej transformacji Fouriera pewien algorytm, umożliwiający otrzymywanie formalnie ścisłych rozwiązań następujących praktycznych zagadnień ze statyki i dynamiki sprężystych płyt izotropowych: 1. drgań harmonicznych i zginania statycznego płyt prostokątnych, obciążonych symetrycznie względem dwóch osi symetrii płyty oraz podpartych w czterech narożach i na sprężystych żebrach wzdłuż brzegów płyty, 2. zginania statycznego płyt prostokątnych, obciążonych jak wyżej i podpartych w czterech, albo w dwóch punktach jednakowo oddalonych od przeciwległych brzegów płyty lub podpartych tylko w jednym punkcie, tj. w środku geometrycznym płyty, 3. zginania statycznego płyty w kształcie równoramiennego trójkąta prostokątnego o krawędzi przeciwprostokątnej swobodnej lub podpartej na sprężystym żebrze i wzdłuż krawędzi przyprostokątnych swobodnie podpartej oraz obciążonej symetrycznie względem wysokości, prostopadłej do krawędzi przeciwprostokątnej. W przypadku płyt prostokątnych podpartych w czterech narożach i na sprężystych żebrach, ograniczenie przedstawionego w pracy rozwiązania problemu do symetrycznego stanu obciążenia zostało wprowadzone wyłącznie w celu uproszczenia i skrócenia rozważań. Łatwo bowiem zauważyć, że na podstawie wzorów wyprowadzonych w pierwszej części pracy można w wymienionym wyżej przypadku podparcia płyt otrzymać bez trudności odpowiedni układ równań, zezwalający na uzyskanie rozwiązań ogólniejszych, tj. dla dowolnego obciążenia statycznego lub dynamicznego.

Wszystkie zamieszczone w pracy rozważania i wzory można bardzo łatwo uogólnić dla przypadku ortotropii oraz dla ważnego praktycznie przypadku płyt prostokątnych, podpartych na słupach o skończonych wymiarach poprzecznych, oddalonych od brzegów płyty — oczywiście przy zachowaniu, jak wyżej, symetrii stanu obciążenia.

Резюме

НЕКОТОРЫЕ ПРОБЛЕМЫ СТАТИКИ И ДИНАМИКИ УПРУГИХ ПРЯМОУГОЛЬНЫХ ПЛИТ ТОЧЕЧНО ПОДПЕРТЫХ

В работе разработан, на основе метода конечного преобразования Фурье, некоторый алгоритм дающий возможность получения формально точных решений следующих практических проблем статики и динамики упругих изотропных плит: 1. гармонические колебания и статический изгиб прямоугольных плит, нагруженных симметрически по отношению к двум осям симметрии плиты и подпертых в четырех углах и на упругих ребрах вдоль границ плиты, 2. статический изгиб прямоугольных плит, нагруженных так как выше и подпертых в четырех или в двух точках одинаково удаленных от противолежащих границ плиты или подпертых только в одной точке т.е. в геометтрическом центре плиты, 3. статический изгиб плиты в форме равнобедренного прямогольного треугольника с гранью гипотенузы свободной или подпертой на упругом ребре, а вдоль граней катетов свободно подпертой, а также нагруженной симметрически по отношению к высоте перпендикулярной к грани гипотенузы. В случае прямоугольных плит подпертых в четырех углах и на упругих ребрах ограничение представленного в работе решения проблемы к симметричному состоянию нагрузки введено исключительно с целью упрощения и сокращения рассуждений. Ибо легко можно заметить, что на основе формул выведенных в первой части работы можно, в перечисленных выше случаях опирания плит, получить без затруднений соответствующую систему уравнений, которая позволяет получить более общее решения т.е. для произвольной статической или динамической нагрузки.

Все помещенные в работе рассуждения и формулы можно очень легко обобщить на случай ортотропии, а также на важный практически случай прямоугольных плит подпертых на столбах, с конечными поперечными размерами, удаленных от грацни плиты — конечно при сохранении, как выше, симметрии состояния нагрузки.

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