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ON THE EXACT ANALYSIS OF THE STEADY CREEP OF A THIN TUBE UNDER COMBINED LOADING

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An exact analysis is presented for the creep deformation of a long, thin, constant thickness circular cross-section cylindrical shell under various loadings. Results are presented in the form of a design chart which can be used for several load combinations, and as flexibility factors. Previous approximate analyses are discussed.

Notation

M applied bending moment loading,
P applied internal pressure loading,
S $=(Pr/2h)(\dot{\varepsilon}_0/\sigma_0)$
$\mathcal{S}$ $=Pr^3/M$,
T applied torsion loading,
W $=T/\pi M$,
W_1 $=(T/4h\pi r^2)(\dot{\varepsilon}_0/\sigma_0)$,
β $=(\mathcal{S}^2+W^2)^{1/2}$,
$\dot{\gamma}$ end rotation rate of tube under combined loading,
$\dot{\gamma}_0$ end rotation rate of tube under bending alone;

suffices

Φ circumferential direction,
l longitudinal direction,
Φ_l shear direction,

All other relevant symbols are either defined in the text and fig. 1 or in reference [2].

1. Introduction

The exact analysis of the steady creep behaviour of a thin walled tube was considered by Edstam and Hult [1] using an isothermal steady creep law of the form

$$\frac{\dot{\bar{\varepsilon}}}{\dot{\varepsilon}_0}=\left(\frac{\bar{\sigma}}{\sigma_0}\right)^n. \tag{1.1}$$

The object of their work was to obtain equivalent stress levels under combined pressure and bending loading for comparison with the linear elastic case, the results being presented as a useful design chart. However, their solution was restricted to the case $n=2$. It is the purpose of this investigation to derive a corrected design chart based on a more general analysis. Further it is shown how various loadings, with bending taken as the primary loading, can be incorporated into the same chart. Finally, results are presented in terms of flexibility factors with reference to the case of bending alone.

2. Stress ratio analysis

Consider a thin tube of circular cross-section under internal pressure P and subjected to a bending moment M and a torque T, as defined in Fig. 1. The radial stress component is negligibly small if $2h/r \ll 1$. The circumferential stress σ_Φ and the shear stress $\sigma_{\Phi l}$ are isostatic and may be determined directly from equilibrium alone

$$\sigma_\Phi = \frac{Pr}{2h}, \tag{2.1}$$

$$\sigma_{\Phi l} = T/(4\pi h r^2), \tag{2.2}$$

both stresses being taken as constant through the thickness.

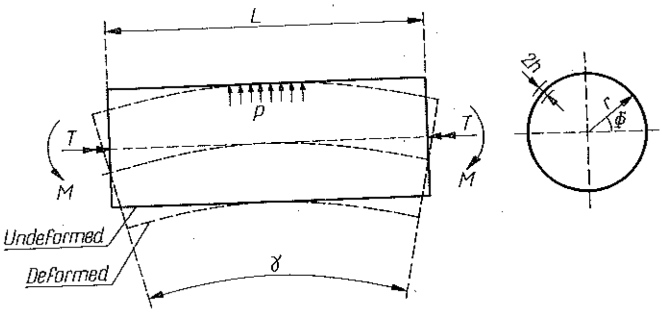

Fig. 1

It is found convenient to write the longitudinal stress

$$\sigma_l = \frac{Pr}{4h} + \rho(\Phi), \tag{2.3}$$

in a manner similar to [1].

This stress field must satisfy equilibrium with the applied moment

$$M = 2hr^2 \int_0^{2\pi} \sigma_l \sin\Phi \, d\Phi,$$

which yields

$$\int_0^{2\pi} R(\Phi) \sin\Phi \, d\Phi = 1, \tag{2.4}$$

on substitution of σ_l from (2.3), where $R(\Phi) = B_1 \rho(\Phi)$ and $B_1 = 2hr^2/M$.

If a pure bending moments alone were acting, the longitudinal strain rate would be

(2.5) $$\dot{\varepsilon}_l = r\dot{\gamma}\sin(\Phi)/L,$$

where $\dot{\gamma}$ is the end rotation rate, related to the curvature rate $\dot{\kappa}$

(2.6) $$\dot{\kappa} = \frac{\dot{\gamma}}{L}.$$

If on the other hand, if pressure and torque alone were acting, the longitudinal strain rate would be zero everywhere. Thus, under combined loading, the longitudinal strain rate would be as given by (2.5).

Further, the constitutive relation (1.1) yields an expression for $\dot{\varepsilon}_l$ in terms of the stresses (2.1), (2.2) and (2.3) [see Appendix II, Eq. (b)], and comparison with (2.5) produces the equation

(2.7) $$\left(\frac{3}{4}\beta^2 + R^2\right)^{n-1} R^2 = \left(\frac{r\dot{\kappa}(\sigma_0 B_1)^n}{\dot{\varepsilon}_0}\right)\sin^2\Phi,$$

where $\beta = (\mathcal{S}^2 + W^2)^{1/2}$.

Edstam and Hult [1] have proposed the use of the non-dimensional design parameter

$$\Theta = \frac{\bar{\sigma}^{(n)}}{\bar{\sigma}^{(1)}},$$

known as the "stress ratio", where $\bar{\sigma}^{(n)}$ is the maximum effective stress occurring in the tube, creeping according to the constitutive relation (1.1). Assuming that this occurs when $\Phi = \pi/2$ it follows that

(2.8) $$\Theta^2 = \frac{3/4 + R^2(\pi/2)/\beta^2}{3/4 + 1/(\pi\beta)^2},$$

and hence that Θ depends on β and n alone.

The suggested design chart plots lines of constant Θ on a diagram with $1/n$ along a horizontal axis, and the quantities β, $1/\beta$ along a vertical axis. Consideration of the moment equilibrium equation (2.4) with the equation (2.7) for $R(\Phi)$ and the relation (2.8) reduces the construction of the chart to the following problem:

Let Θ be given, then for each n solve the algebraic equation

$$F(\bar{\beta}) = \int_0^{2\pi} \bar{R}(\Phi)\sin\Phi\, d\Phi - \bar{\beta} = 0,$$

for $\bar{\beta} = 1/\beta$, where $\bar{R}(\Phi)$ satisfies:

$$\left(\frac{3}{4} + \bar{R}^2(\Phi)\right)^{n-1} \bar{R}^2(\Phi) = \left(\frac{3}{4} + \bar{R}^2(\pi/2)\right)^{n-1} \bar{R}^2(\pi/2)\sin^2\Phi,$$

with

$$\bar{R}^2(\pi/2)=\Theta^2(3/4+\bar{\beta}^2/\pi^2)-3/4,$$

and $\bar{R}(\Phi)=\bar{\beta}R(\Phi)$.

The solution of this problem is outlined in Appendix I. The results of such a calculation are given as a design chart in Fig. 2.

3. Comparison with a previous result

For the case of combined pressure and bending, with no applied torque, $W=0$ and $\beta=\mathcal{S}$. The design chart (Fig. 2) differs significantly from that given in [1] for the same problem. This difference is due to two factors: firstly, the previous analysis [1] was only legitimate for the special case $n=2$, secondly, the authors confined their attention to the calculation of values of Θ for given $\mathcal{S}$ and n (instead of $\mathcal{S}$ being found for given Θ and n) and hence their design chart involved considerable interpolation.

4. Use of the design chart for various loadings

It is important to note that the same design chart can be used for the three loading cases

(a) combined pressure and bending,
(b) combined torsion and bending,
(c) combined pressure, torsion and bending,

if β is simply replaced by $\mathcal{S}=Pr^3/M$, or $W=T/\pi M$ or $(\mathcal{S}^2+W^2)^{1/2}$, respectively.

5. Flexibility factor analysis

A further useful design parameter, used extensively in [2 and 3], is the "flexibility factor"

$$K=\frac{\text{end rotation rate of pipe in combined loading}}{\text{end rotation rate of pipe in bending alone}}=\frac{\dot{\gamma}}{\dot{\gamma}_0}.$$

It is easily shown that, with reference to (2.6) and (2.7),

$$K=D_0^n\cdot r\dot{\kappa}(\sigma_0 B_1)^n/\dot{\varepsilon}_0,$$

where $D_0=4\int_0^{\pi/2}(\sin\Phi)^{(1+1/n)}\,d\Phi$.

It is of interest to determine the value of K for a given β and n; since the moment equilibrium equation (2.4) has to be satisfied the problem reduces to the following:

Given β and n, find K such that the algebraic equation

$$G(K)=\int_0^{2\pi}R(\Phi)\sin\Phi\,d\Phi-1=0,$$

is satisfied, with $R(\Phi)$ the solution of the equation

$$\left(\frac{3}{4}\beta^2+R^2(\Phi)\right)^{n-1} R^2(\Phi)=\left(\frac{K\sin\Phi}{D_0^n}\right)^2.$$

The method of solution of this problem is similar to that of the previous stress ratio problem. Flexibility factor results are presented in Fig. 3.

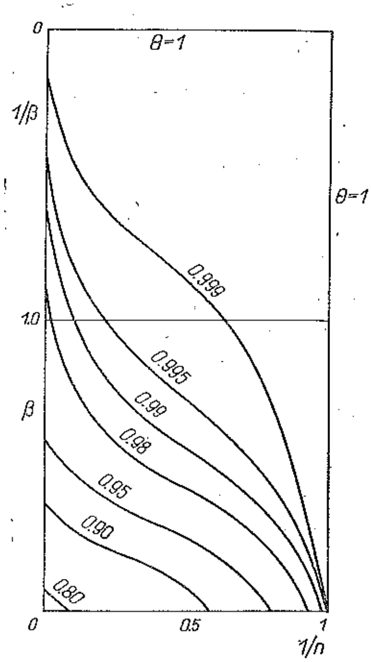

Fig. 2

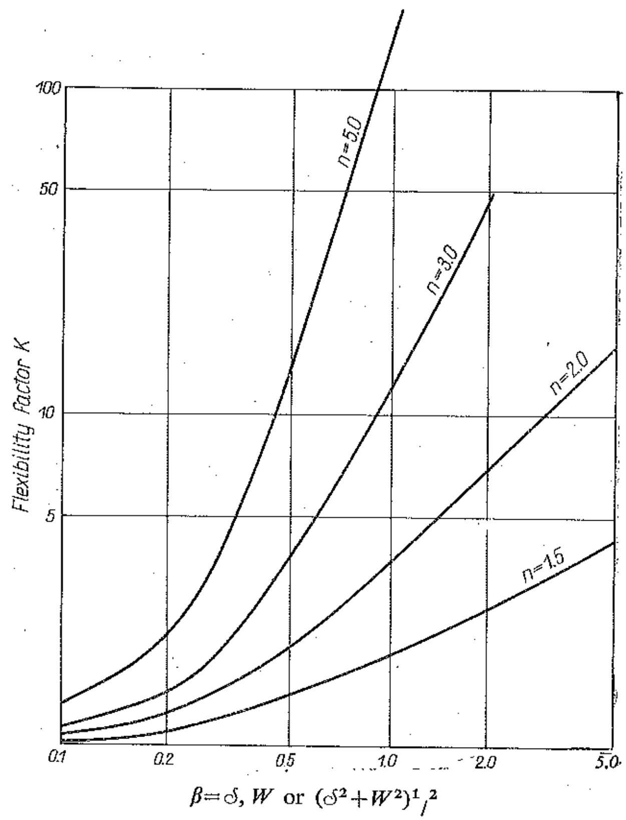

Fig. 3

6. Comparison with previous results

For the case of combined pressure and bending a comparison can be made with the results of [2]. At high δ values the three analyses — "type 1", "type 2" and "exact" — tend to become the same. The assumptions of a "type 2" analysis are not unlike those of the "exact" analysis and consequently the results are fairly close for all δ values. The "type 1" analysis shows a marked disagreement with the "exact" analysis; this shall be discussed further in the next section. For the case of combined loadings a comparison with the results of [3] is possible. The "exact" analysis confirms the simple β replacement scheme outlined in [3] for a "type 2" analysis such that the flexibility graphs can be used for various loadings.

7. Deformation results

It is worth commenting further here on the results of a "type 1" energy analysis as used in [2 and 3]. When the stress field is of the form (2.1), (2.2) and (2.3), the strain-stress relations [Appendix II, Eqs. (a) and (c)] can be written as

(7.1)
$$\dot{\varepsilon}_\phi=\left[\frac{3}{4}(S^2+4W_1^2)+\left(\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}\rho(\Phi)\right)^2\right]^{(n-1)/2}\frac{3}{4}S-\frac{\dot{\varepsilon}_l}{2},$$
$$\dot{\varepsilon}_{\phi l}=\left[\frac{3}{4}(S^2+4W_1^2)+\left(\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}\rho(\Phi)\right)^2\right]^{(n-1)/2}\frac{3}{2}W_1,$$

where

$$S=\frac{Pr}{2h}\,\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0},\qquad W_1=\frac{T}{4h\pi r^2}\,\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}.$$

However, the assumed strain rate field of a "type 1" energy analysis [3] is

(7.2)
$$\dot{\varepsilon}_\phi=\left[\frac{3}{4}S^2\right]^{(n-1)/2}\frac{3}{4}S-\frac{\dot{\varepsilon}_l}{2},\qquad \dot{\varepsilon}_{\phi l}=[3W_1^2]^{(n-1)/2}\frac{3}{2}W_1,$$

which clearly neglects the effect of the torque T and $\rho(\Phi)$ on the circumferential strain rate, and the effect of the pressure P and $\rho(\Phi)$ on the shear strain rate. This would account for the discrepancies at low values of β between a "type 1" analysis and the "exact" analysis. Nevertheless, for large β values it would seem that assumptions (7.2) as opposed to (7.1) are reasonable approximations.

8. An alternative approach

The deformation results would suggest the following alternative approach to the problem based on assumed strain rates. Suppose that the strain rate field is of the form

(8.1)
$$\dot{\varepsilon}_l=(r\dot{\gamma}\sin\Phi)/L,\qquad \dot{\varepsilon}_\phi=\psi_1(\Phi)-\frac{\dot{\varepsilon}_l}{2},\qquad \dot{\varepsilon}_{\phi l}=\psi_2(\Phi),$$

where ε_l is clearly "exact" and $\psi_1(\Phi)$, $\psi_2(\Phi)$ are functions which depend on the loading parameters.

The applied pressure and torque loadings are known to be isostatic and can be obtained directly from equilibrium as before

(8.2)
$$\sigma_\phi=\frac{Pr}{2h},\qquad \sigma_{\phi l}=\frac{T}{4\pi hr^2}.$$

However, it is possible to express the stresses in terms of the assumed strain rate field (8.1) (Appendix II eqs (d), (f)); if a comparison is made with (8.2) there results

$$\frac{\psi_1}{\frac{3}{4}\left(\frac{Pr}{2h}\right)}=\frac{\psi_2}{\frac{3}{2}\left(\frac{T}{4\pi hr^2}\right)}=\psi(\Phi),$$

say, and

$$\Psi^2\left(\frac{3}{4}\Psi^2\beta^2+\left(\frac{K\sin\Phi}{D_0^n}\right)^2\right)^{1/n-1}=1, \tag{8.3}$$

where

$$B_1\,\Psi(\Phi)=\frac{(B_1\sigma_0)^n}{\dot{\varepsilon}_0}\,\psi(\Phi).$$

Similarly the moment equilibrium equation can be written as

$$\int_0^{2\pi}\left(\Psi^{-2(n-1)}-\frac{3}{4}\beta^2\right)^{1/2}\sin\Phi\,d\Phi=1. \tag{8.4}$$

In view of Eqs. (8.3) and (8.4) the problem of determining K for a given β and n can thus be written:

Given β and n, find K such that the equation

$$H(K)=\int_0^{2\pi}\left(\Psi^{-2(n-1)}-\frac{3}{4}\beta^2\right)\sin\Phi\,d\Phi-1=0,$$

is satisfied, with $\Psi(\Phi)$ the solution of the equation

$$\Psi^2\left(\frac{3}{4}\Psi^2\beta^2+\left(\frac{K\sin\Phi}{D_0^n}\right)^2\right)^{1/n-1}=1.$$

In fact this problem has exactly the same solution as the previous one for the evaluation of K. This is easily seen if a function $V(\Phi)$ is defined such that

$$\Psi^2(\Phi)=\left(\frac{3}{4}\beta^2+V^2(\Phi)\right)^{n-1},$$

in which case $V(\Phi)\equiv R(\Phi)$ and $H(K)\equiv G(K)$.

Obviously a "type 1" energy analysis could be performed based on assumptions (8.1) and would constitute a better approximation than previously to the above "exact" analysis. Finally it should be noted that the use of the strain rate field (8.1) instead of (7.2) would remove the inconsistencies mentioned in [3] between an α substitution and a β substitution for combined loading cases: that is, the parameter $\alpha=(\mathcal{S}^{2n}+W^{2n})^{1/2n}$ arises out of a "type 1" analysis for the strain rate field (7.2), yet the β substitution given above comes naturally from (8.1) giving agreement for all analyses.

9. Conclusions

An exact analysis has been presented for a thin cylindrical shell in stationary creep under combined loading in terms of two useful design parameters. Results are expected to be more reliable than previously published analyses (i.e., [1 and 2]).

It is shown that the same charts can be used for alternate parameters $\mathcal{S}=\Pr^3/M$, $W=T/\pi M$ or $(\mathcal{S}^2+W^2)^{1/2}$, independently of the method of analysis, which in itself represents a considerable simplification for design purposes as recommended in [3].

Appendix I

It is found necessary to solve an algebraic equation of the form

$$F(x)=\int_a^b f_1(x,t)\,dt+f_2(x)=0, \tag{a}$$

such that the integrand satisfies the algebraic equation

$$g\big(f_1(x,t)\big)=0. \tag{b}$$

It is possible in some cases to solve (b) analytically. However, in general, this does not seem possible and hence a numerical technique is employed. With an initial approximation $x=x_0$ a first approximation $x=x_1$ is formed

$$x_1=x_0-\frac{F(x_0)}{F'(x_0)},$$

following a Newton/Raphson procedure.

Now $F(x_0)$ requires the evaluation of the integral $\int_a^b f_1(x_0,t)\,dt$; any numerical integration technique (in this case Simpson was used) uses values of the integrand at a number of intermediate points t_j in the closed interval $[a,b]$; such values can be achieved by solving the algebraic equation

$$g\big(f_1(x_0,t_j)\big)=0$$

for $f_1(x_0,t_j)$ (using, say, a Newton/Raphson technique). Similarly from (a), $F'(x_0)$ requires the evaluation of the integral $\int_a^b \frac{\partial}{\partial x} f_1(x,t)|_{x_0}\,dt$, where for intermediate points t_j in $[a,b]$

$$\frac{\partial}{\partial x} f_1(x,t_j)|_{x_0}=\frac{\partial}{\partial x} g\big(f_1(x,t_j)\big)|_{x_0}\div\left.\frac{dg}{df_1}\right|_{x_0,t_j}.$$

In the same manner a second approximation $x=x_2$ can be found. The process continues until it has sufficiently converged.

Appendix II

The material obeys a Norton type isothermal power law given by (1.1), where

$$\bar{\sigma}^2 = \sigma_l^2 - \sigma_l \sigma_\phi + \sigma_\phi^2 + 3\sigma_{\phi l}^2 .$$

If it is assumed that it is also incompressible then the effective strain rate is given by

$$\bar{\dot{\varepsilon}}^2 = \frac{4}{3} (\dot{\varepsilon}_l^2 + \dot{\varepsilon}_l \dot{\varepsilon}_\phi + \dot{\varepsilon}_\phi^2 + \dot{\varepsilon}_{\phi l}^2) .$$

Then for a von Mises flow rule the strain-stress relations are

(a) $$\frac{\dot{\varepsilon}_\phi}{\dot{\varepsilon}_0} = \left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1} \left(\sigma_\phi - \frac{\sigma_l}{2}\right) \Big/ \sigma_0 ,$$

(b) $$\frac{\dot{\varepsilon}_l}{\varepsilon_0} = \left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1} \left(\sigma_l - \frac{\sigma_\phi}{2}\right) \Big/ \sigma_0 ,$$

(c) $$\frac{\dot{\varepsilon}_{\phi l}}{\dot{\varepsilon}_0} = \frac{3}{2} \left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1} \frac{\sigma_{\phi l}}{\sigma_0} ;$$

and the stress-strain relations are

(d) $$\frac{\sigma_\phi}{\sigma_0} = \frac{4}{3} \left(\frac{\bar{\dot{\varepsilon}}}{\dot{\varepsilon}_0}\right)^{1/n-1} \left(\dot{\varepsilon}_\phi + \frac{\dot{\varepsilon}_l}{2}\right) \Big/ \dot{\varepsilon}_0 ,$$

(e) $$\frac{\sigma_l}{\sigma_0} = \frac{4}{3} \left(\frac{\bar{\dot{\varepsilon}}}{\dot{\varepsilon}_0}\right)^{1/n-1} \left(\dot{\varepsilon}_l + \frac{\dot{\varepsilon}_\phi}{2}\right) \Big/ \dot{\varepsilon}_0 ,$$

(f) $$\frac{\sigma_{\phi l}}{\sigma_0} = \frac{4}{3} \left(\frac{\bar{\dot{\varepsilon}}}{\dot{\varepsilon}_0}\right)^{1/n-1} \frac{\dot{\varepsilon}_{\phi l}}{2} .$$

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Streszczenie

O ŚCISŁEJ ANALIZIE PROBLEMU USTALONEGO PEŁZANIA RURY CIENKOŚCIENNEJ W ZŁOŻONYM STANIE OBCIĄŻENIA

Przeprowadzono ścisłą analizę procesu pełzania długiej, cienkościennej rury w kształcie walca kołowego poddanej różnym obciążeniom. Wyniki przedstawiono w postaci specjalnego wykresu, z którego można korzystać przy różnych kombinacjach obciążeń, jak również w postaci tabeli współczynników podatności. Przedyskutowano poprzednie rozwiązania przybliżone tego samego problemu.

Резюме

О ТОЧНОМ АНАЛИЗЕ ПРОБЛЕМЫ УСТАНОВИВЩЕЙСЯ ПОЛЗУЧЕСТИ ТОНКОСТЕННОЙ ТРУБЫ В СЛОЖНОМ НАГРУЖЕННОМ СОСТОЯНИИ

Проведен точный анализ процесса ползучести длинной, тонкостенной трубы, в форме кругового цилиндра, подвергнутой разным нагрузкам. Результаты представлены так в виде специальной диаграммы, которой можно пользоваться при разных комбинациях нагрузок, как и в виде таблицы коэффициентов податливости. Обсуждены предущие проближенные решения этой же самой проблемы.

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