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ON IDENTIFICATION OF THE SYSTEMS WITH SPACE-DISTRIBUTED PARAMETERS

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We consider the problem of the state identification of the deterministic systems with space-distributed parameters, described by differential or integral equations. Formulation of the problem was based on the model of dynamics and observations of the systems in question as well as on the proposed error functional. Conditions for the optimal solution were discussed with the use of dynamic programming and maximum principle.

1. Introduction

One of the basic prerequisites for the analytical design of dynamic systems in advancing science and technology is the establishment of an adequate mathematical model of the system. The derivation of such a model generally requires considerable physical insight. But in the most of practical situations because of complications of physical phenomena, presence of the external disturbances as well as a result of measurements errors such as inaccuracy of the measuring devices, it is often not sufficient to describe the state of the system. Here, one can apply the state identification method presented below for the case of deterministic space-distributed parameter systems. Certain functional equations of dynamic programming approach and the maximum principle associated with the optimal solution of the problem for systems in differential or integral form will be discussed.

2. General statement of the problem

Consider a space-distributed parameter systems defined on a fixed spatial domain Ω, an open, connected subset of an M-dimensional Euclidean space E_M. We shall denote the boundary of Ω by $\partial\Omega$, the closure of Ω by $\bar{\Omega}=\Omega\cup\partial\Omega$ and the spatial coordinate vector by $X=[x_1, x_2, ..., x_M]$. The state of such systems at any fixed time $t\in[t_0, T]$, where $[t_0, T]$ is the interval of the system observation, can generally be specified by the real-valued vector function $U(t, X)=[u_1(t, X), u_2(t, X), ..., u_N(t, X)]$, defined for all $X\in\bar{\Omega}$ from specified state function space $\Gamma(\bar{\Omega})=\Gamma_1(\bar{\Omega})\times...\times\Gamma_N(\bar{\Omega})$. We assumed that the state function $U(t, X)$ is not directly observable. We can observe only certain prescribed functions of U, the output of the system denoted by the real-valued vector function $Z(t, X)=[z_1(t, X), z_2(t, X), ..., z_L(t, X)]$ from specified output function space $\mathcal{V}(\bar{\Omega})=\mathcal{V}_1(\bar{\Omega})\times...\times\mathcal{V}_L(\bar{\Omega})$, where $1\leqslant L\leqslant N$. It can be seen as the result of a zero-memory continuous spatially dependent transformations of the state function from $\Gamma(\bar{\Omega})$ into $\mathscr{V}(\bar{\Omega})$. The inaccuraces of the mathematical description of the dynamic systems considered as well as observations will be characterized by real-valued vector functions $F_{\bar{\Omega}}^{\Gamma}(t, X)=[f_1^{\Gamma}(t, X), f_2^{\Gamma}(t, X), \ldots, f_K^{\Gamma}(t, X)]$ and $F_{\bar{\Omega}}^{\mathscr{V}}(t, X)=[f_1^{\mathscr{V}}(t, X), f_2^{\mathscr{V}}(t, X), \ldots, f_P^{\mathscr{V}}(t, X)]$ for $X \in \bar{\Omega}$, and $t \in [t_0, T]$, which are distributed over all or certain subsets of $\bar{\Omega}$, defined as dynamic and observation error functions, respectively. It is convenient to classify these error functions as follows:

(i) local error functions $F_{\Omega}^{\Gamma}(t, X)$ and $F_{\Omega}^{\mathscr{V}}(t, X)$ for $X \in \Omega$,

(ii) boundary error functions $F_{\partial\Omega}^{\Gamma}(t, X)$ and $F_{\partial\Omega}^{\mathscr{V}}(t, X)$ for $X \in \partial\Omega$,

(iii) initial error functions $F_{\bar{\Omega}}^{\Gamma}(t, X)$ and $F_{\bar{\Omega}}^{\mathscr{V}}(t, X)$ for $X \in \bar{\Omega}$ and $t=t_0$.

In the case where both local and boundary error functions are present, certain compatibility conditions may have to be satisfied in the neighbourhood of domain boundary as well as at the initial time t_0.

The dynamic behaviour of considered space-distributed parameter systems at any point X and time t can be described by:

(a) partial differential equations of the vector form:

$$\frac{\partial U(t, X)}{\partial t} = \mathscr{H}[U(t, X), F_{\Omega}^{\Gamma}(t, X)], \qquad X \in \Omega, \qquad t \in [t_0 T], \tag{2.1}$$

$$\mathscr{G}[U(t, X), F_{\partial\Omega}^{\Gamma}(t, X)]=0, \qquad X \in \partial\Omega, \quad t \in [t_0, T], \tag{2.2}$$

$$U(t_0, X)=\mathscr{J}[U_0(X), F_{\bar{\Omega}}^{\Gamma}(t_0, X)], \qquad X \in \bar{\Omega}, \qquad t=t_0, \tag{2.3}$$

where $\mathscr{H}=[h_1, h_2, \ldots, h_N]$ is specified spatial differential operator whose parameters may depend upon X and t; $\mathscr{G}=[g_1, g_2, \ldots, g_R]$ is specified spatial differential operator of the boundary conditions whose parameters may depend upon X and t; $\mathscr{J}=[j_1, j_2, \ldots, j_S]$ is specified spatial operator of initial conditions at X and $t=t_0$; $U_0(X)$ is a vector function of the *a priori* estimate of the state at $t=t_0$ in $\Gamma(\bar{\Omega})$,

(b) integral equations of the form:

$$U(t, X)= \int_{\bar{\Omega}} K_0[t, X, X', U_0(X'), F_{\bar{\Omega}}^{\Gamma}(t_0, X')]\, dX' + \int_{t_0}^{t} \int_{\bar{\Omega}} K_1[t, t', X, X', U(t', X'), F_{\bar{\Omega}}^{\Gamma}(t', X')]\, dX'\, dt', \tag{2.4}$$

where K_0 and K_1 are specified vector valued functions of their arguments and K_0 has the property that

$$\int_{\bar{\Omega}} K_0[t_0, X, X', U_0(X'), F_{\bar{\Omega}}^{\Gamma}(t_0, X')]\, dX' = \mathscr{J}[U_0(X), F_{\bar{\Omega}}^{\Gamma}(t_0, X)], \qquad X \in \bar{\Omega}. \tag{2.5}$$

The observations action of dynamic behaviour of considered space-distributed parameter systems is assumed to be achieved in the following manner:

$$Z(t, X)=\mathscr{M}[U(t, X), F_{\bar{\Omega}}^{\mathscr{V}}(t, X)], \qquad X \in \bar{\Omega}, \quad t \in [t_0, T], \tag{2.6}$$

where $\mathscr{M}=[m_1, m_2, \ldots, m_J]$ is specified space-time operator whose parameters may depend upon X and t, and $Z(t, X)$ is an output vector function.

From the physical point of view it is sometimes necessary to impose on the sought for state as well as error functions estimates additional constraints, in general inequalities constraints, of the type:

$$\mathscr{B}_i\,[U(t,X), F^{\Gamma}_{\Omega}(t,X), F^{\Psi}_{\Omega}(t,X)] \geq 0, \quad i=1,2,\ldots,N_C, \tag{2.7}$$

where $\mathscr{B}_i$ are specified vector functions or functionals of their arguments.

The basis of selecting the estimates of the state as well as dynamic and observation error functions associated with a given partial differential or integral equations, that is, the criterion of optimality, is that of "least squares". We assumed the following form of the error functional:

$$\beta\,[U, F^{\Gamma}_{\Omega}, F^{\Psi}_{\Omega}] = \|\mathscr{Z}(t,X) - Z(t,X)\|^2, \quad X \in \bar{\Omega},\; t \in [t_0, T], \tag{2.8}$$

where $\mathscr{Z}(t,X) = [\mathscr{z}_1, \mathscr{z}_2, \ldots, \mathscr{z}_I]$ is continuous real-valued vector function from specified measurement space $\mathscr{W}(\bar{\Omega}) = \mathscr{W}_1(\bar{\Omega}) \times \ldots \times \mathscr{W}_I(\bar{\Omega})$, and $\|(\cdot)\|^2$ is some appropriate squared metric.

Although the problem considered is deterministic one, that is if knowledge of the statistics of the random disturbances which may be associated with the systems and/or measurements is available, this knowledge of the statistics can be used to advantage in choosing some weighting matrix defined then in error functional. Also in the practical situations it is not always possible to indicate spatial profile of $\mathscr{Z}(t,X)$ in the continuous manner but we shall consider the problem associated with the above error functional. Next, we can formally report the results obtained for the case of point measurements in space domain.

The optimal state estimation problem may now be summarized as follows. It is necessary to find the estimates U, F^{Γ}_{Ω}, F^{Ψ}_{Ω} satisfying the Eqs. (2.1), (2.2), (2.3) or (2.4), (2.5), and (2.6), with respect to the Ineq. (2.7) such that the error functional, the Eq. (2.8), is minimized for the given measurement function $\mathscr{Z}(t,X)$. Thus, the determined estimates will obviously be optimal in the sense of criterion (2.8).

3. Identification of the systems in differential form

We will discuss the conditions of optimal solution associated with state identification of particular class of the systems in differential form. We shall assume the same form of the operator $\mathscr{H}$ as in the Eq. (2.1), that is, non-linear spatial differential operator acting on U and a dynamic error function F^{Γ}_{Ω}:

$$\frac{\partial U(t,X)}{\partial t} = \mathscr{H}\,[U(t,X), F^{\Gamma}_{\Omega}(t,X)], \quad X \in \Omega,\; t \in [t_0, T]. \tag{3.1}$$

The boundary conditions given by the Eq. (2.2) will be restricted to the form:

$$U(t,X) = 0, \quad X \in \partial\Omega,\; t \in [t_0, T]. \tag{3.2}$$

The initial condition given by the Eq. (2.3) receives the form:

$$U(t_0, X) = U_0(X), \quad X \in \bar{\Omega},\; t = t_0. \tag{3.3}$$

We also assume that the operator $\mathscr{M}$ in the Eq. (2.6) is linear and consider the output transformation of the type:

$$(3.4)\qquad Z(t,X)=M(t,X)\,U(t,X),\qquad X\in\bar{\Omega},\ t\in[t_0,T],$$

where M is $(J\times N)$-dimensional space and time dependent transforming matrix.

We consider the following error functional:

$$(3.5)\qquad \beta=\int_{t_0}^{T}\int_{\bar{\Omega}}[\mathscr{Z}(t,X)-Z(t,X)]^{\mathrm{Tr}}[\mathscr{Z}(t,X)-Z(t,X)]\,dX\,dt=$$
$$=\int_{t_0}^{T}\int_{\bar{\Omega}}[\mathscr{Z}(t,X)-M(t,X)\,U(t,X)]^{\mathrm{Tr}}[\mathscr{Z}(t,X)-M(t,X)\,U(t,X)]\,dX\,dt,$$

where Tr denotes the transpose of the matrix in brackets.

The condition of optimal solution will be obtained in the form of certain functional equations, using dynamic programming [1] approach given in [2]. We assume that the solution to the Eq. (3.1), denoted by $U_{F^{\Gamma}_{\Omega}}$, exists in the time inferval $[t_0, T]$ and, for sufficiently small time increment Δ, can be written as:

$$(3.6)\qquad U_{F^{\Gamma}_{\Omega}}[t_0+\Delta,X;U_0(X),t_0]\approx U_0(X)+\Delta\{\mathscr{H}[U(t,X)F^{\Gamma}_{\Omega}(t,X)]\}+O(\Delta),$$

where $O(\Delta)$ is an infinitesimal quantity of higher order than Δ. Denote

$$(3.7)\qquad \pi[U_0(X),\tau]=\min_{F^{\Gamma}_{\Omega}}\beta,\qquad \text{where } \tau=T-t.$$

Applying the principle of optimality, we have

$$(3.8)\qquad \pi[U_0(X),\tau]=\min_{F^{\Gamma}_{\Omega}}\Big\{\int_{t_0}^{t_0+\Delta}\int_{\bar{\Omega}}[\mathscr{Z}(t,X)-M(t,X)\,U_{F^{\Gamma}_{\Omega}}[t,X;U_0(X),t_0]]^{\mathrm{Tr}}\times$$
$$\times[\mathscr{Z}(t,X)-M(t,X)\,U_{F^{\Gamma}_{\Omega}}[t,X;U_0(X),t_0]]+\pi[U(t_0+\Delta,X),\tau-\Delta]\Big\}.$$

We expand $\pi[U(t_0+\Delta,X),\tau-\Delta]$ about U_0 and τ as follows:

$$(3.9)\qquad \pi[U(t_0+\Delta,X),\tau-\Delta]\approx\pi[U_0(X),\tau]+\Delta\int_{\bar{\Omega}}\left(\frac{\delta\pi[U_0(X),\tau]}{\delta U_0(X)}\right)^{\mathrm{Tr}}\times$$
$$\times\mathscr{H}[U_0(X),F^{\Gamma}_{\Omega}(t_0,X)]\,dX-\frac{\partial\pi[U_0(X),\tau]}{\partial\tau}+O'(\Delta),$$

where $\delta(\cdot)/\delta(\cdot)$ denotes a functional partial derivative of the functional π with respect to vector function U_0 at a point $X\in\Omega$. Using the approximation

$$(3.10)\qquad \int_{t_0}^{t_0+\Delta}\int_{\Omega}[\mathscr{Z}(t,X)-M(t,X)\,U_{F^{\Gamma}_{\Omega}}[t,X;U_0(X),t_0]]^{\mathrm{Tr}}[\mathscr{Z}(t,X)-$$
$$-M(t,X)\,U_{F^{\Gamma}_{\Omega}}[t,X;U_0(X),t_0]]\,dX\,dt\approx\Delta\int_{\Omega}[\mathscr{Z}(t_0,X)-$$
$$-M(t_0,X)\,U_0(X)]^{\mathrm{Tr}}[\mathscr{Z}(t_0,X)-M(t_0,X)\,U_0(X)]\,dX+O''(\Delta),$$

and substituting the Eq. (3.9) into the Eq. (3.8) taking the limit as $\Delta \to 0$, and remarking that result must hold for all $t \in [t_0, T]$, we obtain the following partial differential-integral equation:

$$(3.11) \quad \frac{\partial \pi [U(t, X), \tau]}{\partial \tau} = \min_{F_\Omega^\Gamma} \int_\Omega \left\{ \left(\frac{\delta \pi [U(t, X), \tau]}{\delta U(t, X)} \right)^{\mathrm{Tr}} \mathscr{H} [U(t, X), F_\Omega^\Gamma (t, X)] + \right.$$
$$\left. + [\mathscr{Z}(t, X) - M(t, X) U(t, X)]^{\mathrm{Tr}} [\mathscr{Z}(t, X) - M(t, X) U(t, X)] \right\} dX,$$

with the initial condition

$$(3.12) \quad \pi [U(T, X), 0] = 0.$$

By introducing the Hamiltonian defined by

$$(3.13) \quad H(U, P, t) = \langle P, Q \rangle_\Omega = \int_\Omega \sum_{i=1}^{N+1} p_i q_i \, dX,$$

where $\langle P, Q \rangle_\Omega$ is the inner product in $\Gamma(\Omega)$ and

$$(3.14) \quad P = [p_1, p_2, \ldots, p_{N+1}] = \left[\frac{\delta \pi [U(t, X), \tau]}{\delta U(t, X)}, 1 \right],$$

$$(3.15) \quad Q = [q_1, q_2, \ldots, q_{N+1}] = [\mathscr{H}^{\mathrm{Tr}} [U(t, X), F_\Omega^\Gamma (t, X)], [\mathscr{Z}(t, X) -$$
$$- M(t, X) U(t, X)]^{\mathrm{Tr}} [\mathscr{Z}(t, X) - M(t, X) U(t, X)]],$$

we obtain simplified form of the Eq. (3.11) as follows:

$$(3.16) \quad \frac{\partial \pi [U(t, X), \tau]}{\partial \tau} = \min_{F_\Omega^\Gamma} H(U, P, t) = H^0 (U, P, t),$$

where H^0 is the minimum of Hamiltonian H with respect to F_Ω^Γ. It can be shown [2], if the solution of the Eq. (3.16) is regular that the optimum solution of the problem is the solution of the Hamiltonian canonical equations of the form

$$(3.17) \quad \frac{\partial U(t, X)}{\partial t} = \frac{\delta H^0 (U, P, t)}{\delta U(t, X)}, \qquad X \in \Omega, \quad t \in [t_0, T],$$

$$(3.18) \quad \frac{\partial P(t, X)}{\partial t} = - \frac{\delta H^0 (U, P, t)}{\delta U(t, X)},$$

where $\delta(\cdot)/\delta(\cdot)$ denotes functional partial derivative, with initial condition

$$(3.19) \quad U(t_0, X) = U_0(X), \qquad X \in \Omega,$$

and terminal condition at time T

$$(3.20) \quad P(T, X) = [0, 1]$$

because $U(t, X)$ at $t = T$ is free.

It can be mentioned that the boundary conditions, the Eq. (3.2), are taken care of by restricting the domain of the operator acting on U. The Eqs. (3.17) to (3.20), which consist of a two-point boundary value problem in a function space, are the necessary condition of the optimal solution of the identification problem for the dynamic systems described by the Eqs. (3.1) to (3.4) with error functional given by the Eq. (3.5).

4. Identification of the systems in integral form

We shall consider a class of dynamic systems described by the Eqs. (2.4) where $U_0(X)$ is taken to be zero, without loss of generality, and with output transformation of the type given by the Eq. (3.4). The error functional will be taken in the form of the Eq. (3.5). Extending formally the outline of the proof given in [2], we obtain necessary condition of optimal solution as a particular case of maximum principle [3].

To simplify the notations, we denote (t, X) by S and $[t_0, T]\times\bar{\Omega}$ by $\mathscr{E}$. We shall consider the functional

$$\beta' = c_0 \int_{\mathscr{E}} [\mathscr{Z}(S) - M(S)U(S)]^{\mathrm{Tr}} [\mathscr{Z}(S) - M(S)U(S)]\, d\mathscr{E}. \tag{4.1}$$

Let S be a regular point in the domain $\mathscr{E}$, and Δ_ε a small region surrounding $\tilde{S}$ with volume ε such that $\varepsilon \to 0$ as the diameter of $\Delta_\varepsilon \to 0$. We shall introduce a perturbed function $\hat{F}^{\Gamma}_{\bar{\Omega}}$ defined about the optimal ${}^0F^{\Gamma}_{\bar{\Omega}}$ as

$$\hat{F}^{\Gamma}_{\bar{\Omega}}(S) = \begin{cases} {}^0F^{\Gamma}_{\bar{\Omega}}(S) & \text{for all} \quad S \in \mathscr{E} - \Delta_\varepsilon, \\ F^{\Gamma *}_{\bar{\Omega}} & \text{for all} \quad S \in \Delta_\varepsilon \end{cases} \tag{4.2}$$

and corresponding to the state function as $U(S)$ and $U^0(S)$, respectively. Because the function under the integral in the Eq. (4.1) has continuous first partial derivatives with respect to $U(S)$, we can compute the value of β' with the perturbed function $\hat{F}^{\Gamma}_{\bar{\Omega}}$ as

$$\hat{\beta}' = c_0 \int_{\mathscr{E}} \{[\mathscr{Z}(S) - M(S)U^0(S)]^{\mathrm{Tr}} [\mathscr{Z}(S) - M(S)U^0(S)] - \\ - [\mathscr{Z}(S) - M(S)U^0(S)] M(S) \delta U(S)\}\, d\mathscr{E} \tag{4.3}$$

because U depends on $\hat{F}^{\Gamma}_{\bar{\Omega}}$. Next, from the Eq. (2.4), the increment $\delta U(S)$ satisfies, with an accuracy up to small quantities of higher order than ε, a nonhomogeneous Fredholm integral equation, linear in $\delta U(S)$:

$$\delta U(S) = \varepsilon \{K_1[S, \tilde{S}, U^0(\tilde{S}), F^{\Gamma *}_{\bar{\Omega}}] - K_1[S, \tilde{S}, U^0(\tilde{S}), {}^0F^{\Gamma}_{\bar{\Omega}}(\tilde{S})]\} + \\ + \int_{\mathscr{E}} \frac{\partial K_1(S, S', U^0, {}^0F^{\Gamma}_{\bar{\Omega}})}{\partial U^0} \delta U(S')\, d\mathscr{E}', \tag{4.4}$$

where $\partial K_1/\partial U^0$ is a matrix with elements $\partial K_{1i}/\partial u^0_j$, $i, j = 1, 2, \ldots, N$. The solution to the Eq. (4.4) can be written as

$$\delta U(S) = \varepsilon \Big\{K_1[S, \tilde{S}, U^0(\tilde{S}), F^{\Gamma *}_{\bar{\Omega}}] - K_1[S, \tilde{S}, U^0(\tilde{S}), {}^0F^{\Gamma}_{\bar{\Omega}}(\tilde{S})]\Big\} - \\ - \int_{\mathscr{E}} W(S, S')[K_1[S', \tilde{S}, U^0(\tilde{S}), F^{\Gamma *}_{\bar{\Omega}}] - K_1[S', \tilde{S}, U^0(\tilde{S}), {}^0F^{\Gamma}_{\bar{\Omega}}(\tilde{S})]]\, d\mathscr{E}', \tag{4.5}$$

where the kernel $W(S, S')$ satisfies the following integral equation:

$$(4.6) \qquad W(S, S') + \frac{\partial K_1 [S, S', U(S), F^{\Gamma}_{\bar{\Omega}}(S')]}{\partial U} = \int_{\mathscr{E}} W(S, S') \frac{\partial K_1 [S, S', U(S'), F^{\Gamma}_{\bar{\Omega}}(S')]}{\partial U} d\mathscr{E}'.$$

Substituting the Eq. (4.5) into the Eq. (4.3) and calculating the difference between $\hat{\beta}'$ and β' corresponding to ${}^{0}F^{\Gamma}_{\bar{\Omega}}$, we have:

$$(4.7) \qquad \Delta\beta' = \hat{\beta}' - c_0 \int_{\mathscr{E}} [\mathscr{Z}(S) - M(S)\, U^0(S)]^{\mathrm{Tr}} [\mathscr{Z}(S) - M(S)\, U^0(S)]\, d\mathscr{E} = -c_0 \int_{\mathscr{E}} \{[\mathscr{Z}(S) - M(S)\, U^0(S)]\, M(S)\, \delta U(S)\}\, d\mathscr{E} = \varepsilon' [\theta(\tilde{S}, F^{\Gamma *}_{\bar{\Omega}}) - \theta(\tilde{S}, {}^{0}F^{\Gamma}_{\bar{\Omega}})],$$

where $\varepsilon' > 0$, and the function θ is defined by:

$$(4.8) \qquad \theta(S, F^{\Gamma}_{\bar{\Omega}}) = c_0 [\mathscr{Z}(S) - M(S)\, U(S)]^{\mathrm{Tr}} [\mathscr{Z}(S) - M(S)\, U(S)] - 2c_0 \int_{\mathscr{E}} [\mathscr{Z}(S) - M(S)\, U(S)]\, M(S) \Big\{ K_1 [S'', S, U(S), F^{\Gamma}_{\bar{\Omega}}(S)] - \int_{\mathscr{E}} W(S'', S') K_1 [S', S, U(S) F^{\Gamma}_{\bar{\Omega}}(S)]\, d\mathscr{E}' \Big\}\, d\mathscr{E}''.$$

If we set in the Eq. (4.7) $c_0 = -1$, then $\Delta\beta'$ must be nonpositive about the optimum β'. Thus

$$(4.9) \qquad \theta(\tilde{S}, {}^{0}F^{\Gamma}_{\bar{\Omega}}) \geq \theta(\tilde{S}, F^{\Gamma *}_{\bar{\Omega}}).$$

Since the inequality (4.9) is valid for any $F^{\Gamma *}_{\bar{\Omega}}$, then $\theta(\tilde{S}, F^{\Gamma}_{\bar{\Omega}})$ attains a maximum with respect to $F^{\Gamma}_{\bar{\Omega}}$ for fixed $\tilde{S}$, that is, for almost all $\tilde{S} \in \mathscr{E}$ we have:

$$(4.10) \qquad \theta(S, {}^{0}F^{\Gamma}_{\bar{\Omega}}) = \sup_{F^{\Gamma}_{\bar{\Omega}}} \theta(S, F^{\Gamma}_{\bar{\Omega}}).$$

Now, we can summarize the results as follows. The optimal solution of the problem for the dynamic systems in the integral form given by the Eq. (2.4), at $U_0(X) = 0$, with respect to error functional, the Eq. (3.5) is the dynamic error function ${}^{0}F^{\Gamma}_{\bar{\Omega}}$ which maximizes the function $\theta(S, F^{\Gamma}_{\bar{\Omega}})$ defined by the Eq. (4.8), according to the Eq. (4.10). Optimal estimate of the state function $U(S)$ we can next obtain from the Eq. (2.4).

5. Examples of the application

To illustrate the basic features of the above formalism, we shall consider two simple examples. The first is the case of the dynamic system in the differential form described by following diffusion scalar equation in one spatial coordinate x given by:

$$(5.1) \qquad \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + f^{\Gamma}_{\Omega}(t, x), \qquad x \in [0, 1],\ t \in [0, T],$$

with boundary condition $u(t,0)=u(t,1)=0$ and initial condition $u(t_0,x)=u_0(x)$, when $f_\Omega^\Gamma(t,x)$ is dynamic error function. The observation is given by

$$z(t,x)=\alpha u(t,x), \quad \alpha>0,\ x\in[0,1],\ t\in[0,T], \tag{5.2}$$

where α is proportionality constant.

As an error functional we choose

$$\beta=\int_0^T\int_0^1 \gamma\,[\mathscr{z}(t,x)-z(t,x)]^2\,dx\,dt, \tag{5.3}$$

where γ is the weighting coefficient.

In view of the Eq. (3.11), the corresponding functional equation is as follows:

$$\frac{\partial\pi[u(t,x),\tau]}{\partial\tau}=\min_{f_\Omega^\Gamma}\int_0^1\left\{\frac{\delta\pi[u(t,x),\tau]}{\delta u(t,x)}\left[\frac{\partial^2 u(t,x)}{\partial x^2}+ +f_\Omega^\Gamma(t,x)\right]+\gamma\,[\mathscr{z}(t,x)-\alpha u(t,x)]\right\}dx, \tag{5.4}$$

where $\tau=T-t$, with initial condition

$$\pi[u(T,x),0]=0. \tag{5.5}$$

If $\delta\pi/\delta u\neq 0$, then it is evident that the integral in the Eq. (5.4) will take on its minimum with respect to $f_\Omega^\Gamma(t,x)=-\infty\cdot\operatorname{sgn}(\delta\pi/\delta u)$. Then it is necessary to restrict the magnitude of estimated dynamic error function as follows:

$$|f_\Omega^\Gamma(t,x)|\leqslant F_0(t,x), \tag{5.6}$$

where $F_0(t,x)$ is maximum admissible value of dynamic error function for given t and x. At that time

$$f_\Omega^\Gamma(t,x)=-F_0(t,x)\operatorname{sgn}\left\{\frac{\delta\pi[u(t,x),\tau]}{\delta u(t,x)}\right\}. \tag{5.7}$$

From the Eq. (5.4) we obtain:

$$\frac{\partial\pi[u(t,x),\tau]}{\partial\tau}=\int_0^1\left\{\frac{\delta\pi[u(t,x),\tau]}{\delta u(t,x)}\left[\frac{\partial^2 u(t,x)}{\partial x^2}- -F_0(t,x)\operatorname{sgn}\left\{\frac{\delta\pi}{\delta u}\right\}\right]+\gamma\,[\mathscr{z}(t,x)-\alpha u(t,x)]^2\right\}dx \tag{5.8}$$

with initial condition, the Eq. (5.5).

Now, the Hamiltonian canonical equations are:

$$\frac{\partial u(t,x)}{\partial t}=\frac{\partial^2 u(t,x)}{\partial x^2}-F_0(t,x)\operatorname{sgn}p(t,x), \tag{5.9}$$

$$\frac{\partial p(t,x)}{\partial t}=-\frac{\partial^2 p(t,x)}{\partial x^2}+2\gamma\alpha\,[\mathscr{z}(t,x)-\alpha u(t,x)], \tag{5.10}$$

where

$$p(t, x) = \frac{\delta \pi [u(t, x), \tau]}{\partial u(t, x)}$$

with initial and terminal conditions

$$u(t_0, x) = u_0(x), \tag{5.11}$$

$$p(T, x) = 0 \tag{5.12}$$

and with boundary conditions

$$u(t, 0) = u(t, 1) = p(t, 0) = p(t, 1) = 0. \tag{5.13}$$

The value of state estimate $u(t, x)$ we obtain by numerical solution above two-point boundary value problem in function space directly, or transforming the Eqs. (5.9) and (5.10) with conditions, the Eqs. (5.11) to (5.13), into integral equation as below. Because the Eqs. (5.9) and (5.10) are linear partial differential equations, we can assume that their solutions satisfying the boundary condition, the Eq. (5.13), at any time t can be expressed in the form:

$$u(t, x) = \int_0^1 k_1(t, 0, x, x') u_0(x')\, dx' - \int_0^t \int_0^1 k_1(t, t', x, x') F_0(t', x') \operatorname{sgn} p(t', x')\, dx'\, dt', \tag{5.14}$$

$$p(t, x) = \int_0^1 k_2(t, 0, x, x') p_0(x')\, dx' + 2\gamma\alpha \int_0^t \int_0^1 k_2(t, t', x, x') [\mathscr{z}(t', x') - \alpha u(t', x')]\, dx'\, dt', \tag{5.15}$$

where kernels k_1 and k_2 are Green functions of the type

$$k_1(t, t', x, x') = 2 \sum_{n=1}^{\infty} \exp[-n^2 \pi^2 (t - t')] \sin(n\pi x) \sin(n\pi x'), \tag{5.16}$$

$$k_2(t, t', x, x') = 2 \sum_{n=1}^{\infty} \exp[n^2 \pi^2 (t - t')] \sin(n\pi x) \sin(n\pi x'). \tag{5.17}$$

From terminal condition, the Eq. (5.12), we obtain the relation for $p_0(x)$ as follows:

$$p_0(x) = -2\alpha\gamma\theta^{-1}(T, 0) \int_0^T \int_0^1 k_2(T, t', x, x') [\mathscr{z}(t', x') - \alpha u(t', x')]\, dx'\, dt', \tag{5.18}$$

where $\theta^{-1}(T, 0)$ is the inverse of the operator $\int_0^T k_2(T, 0, x, x')(\cdot)\, dx'$. Substituting the Eqs. (5.15) and (5.18) into the Eq. (5.14), we obtain the relation for state estimate of diffusion system considered in the form of integral equation:

$$u(t, x) = \int_0^1 k_1(t, 0, x, x') u_0(x')\, dx' - \int_0^t \int_0^1 k_1(t, t', x, x') F_0(t', x') \times \tag{5.19}$$

$$\times \operatorname{sgn}\left\{\int_0^1 k_2(t', 0, x', x'')\left[-2\alpha\gamma\theta^{-1}(T, 0)\int_0^T\int_0^1 k_2(T, t'', x', x'''')[\mathscr{z}(t'', x''') - \right.\right.$$

$$\left. - \alpha u(t'', x''')]\, dx'''\, dt''\right] dx'' + 2\alpha\gamma \int_0^t \int_0^1 k_2(t', t'', x', x'')\, [\mathscr{z}(t'', x'') -$$

$$\left. - \alpha u(t'', x'')]\, dx'\, dt'\right\} dx'\, dt'$$

which can be solved by means of some numerical procedure.

The second example is the case of dynamic system in the integral form described by the following linear scalar equation:

$$u(t, x) = \int_0^t k(t, t', x)\, f^{\Gamma}_{\partial\Omega}(t')\, dt', \tag{5.20}$$

where $f^{\Gamma}_{\partial\Omega}$ is a boundary error function which does not vary along the boundary of the region [0, 1], k is a specified Green's function, and the initial condition at $t=0$ is $u_0(x)=0$.

The observation is given by the Eq. (5.2) and as an error functional we choose the index given by the Eq. (5.3). In addition, the boundary error function is constrained by:

$$|f^{\Gamma}_{\partial\Omega}(t)| \leqslant F_0, \tag{5.21}$$

where F_0 is the maximum admissible value of a boundary error function. The function corresponding to θ in the Eq. (4.8) is as follows:

$$\theta(t, x, f^{\Gamma}_{\partial\Omega}) = c_0\, [\mathscr{z}(t, x) - \alpha u(t, x)]^2 - \tag{5.22}$$

$$-2c_0\, \gamma\alpha \int_0^T \int_0^1 [\mathscr{z}(t, x) - \alpha u(t, x)]\, k(T, t, x)\, f^{\Gamma}_{\partial\Omega}(t)\, dx\, dt.$$

Since $c_0 = -1$, then the maximum of θ with respect to $f^{\Gamma}_{\partial\Omega}(t)$, subject to the constraint Eq. (5.21), is attained when

$$^{0}f^{\Gamma}_{\partial\Omega}(t) = F_0 \operatorname{sgn}\left\{\int_0^1 [\mathscr{z}(t, x) - \alpha u(t, x)]\, k(T, t, x)\, dx\right\}. \tag{5.23}$$

Next, the optimal state estimate we obtain from the Eq. (5.20) with $^{0}f^{\Gamma}_{\partial\Omega}$ given by the Eq. (5.23)

$$u(t, x) = \int_0^t k(t, t', x)\, F_0 \operatorname{sgn}\left\{\int_0^1 [\mathscr{z}(t', x) - \alpha u(t', x)]\, k(T, t', x)\, dx\right\} dt'. \tag{5.24}$$

For particullar case of the diffusion system with boundary conditions $u(t, 0)=0$ and $u(t, 1)=f^{\Gamma}_{\partial\Omega}(t)$, the Green's function k is given by:

$$k(t, t', x)=\sum_{n=1}^{\infty} 2\pi(-1)^{n+1}\, n \sin(n\pi x) \exp[-n^2 \pi^2 (t-t')]. \tag{5.25}$$

From the physical point of view is interesting the constraint of the magnitude of dynamic error function (in the first example) and boundary error function (in the second example) estimates, given by the Eqs. (5.7) and (5.27), respectively. It is seen that in the linear case it is not possible to obtain more precisely error function estimates but only their maximum value, if we used dynamic programming approach or maximum principle.

6. Conclusion

The method of identification of the space-distributed parameter systems presented in this paper is based on the mathematical formalism known as dynamic optimization which is used in the theory of optimal control. Thus, the methods of dynamic optimization can be used to the solution of the problem under consideration. General condition of the optimal state estimation derived above can be applied to the solution of a large number of problems. But it must be underlined that the semi-analytical solutions we can obtain for simple cases, for example, linear homogeneous systems described by a few of differential or intergral equations. In other situations effective approximation schemes and computational procedures must be devised [4].

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Streszczenie

O IDENTYFIKACJI UKŁADÓW O STAŁYCH ROZŁOŻONYCH W PRZESTRZENI

Rozważany jest problem identyfikacji stanu układów deterministycznych o stałych rozłożonych w przestrzeni, opisywanych równaniami różniczkowymi lub całkowymi. Problem został sformułowany na podstawie modelu dynamiki i obserwacji rozważanych układów, jak również przy wykorzystaniu wprowadzonego wskaźnika jakości identyfikacji. Przedyskutowano warunki optymalnego rozwiązania wykorzystując metodę programowania dynamicznego i zasadę maksimum.

Резюме

ОБ ИДЕНТИФИКАЦИИ СИСТЕМ С ПРОСТРАНСТВЕННО РАСПРЕДЕЛЕННЫМИ ПАРАМЕТРАМИ

В настоящей работе рассматривается проблема идентификации состояния детерминистических систем с пространственно распределенными параметрами, описываемых дифференциальными или интегральными уравнениями. Формулировка упомянутой проблемы дается на основе модели динамики и наблюдений рассматриваемых систем, а также на основе предложенного показателя качества. Приводятся условия оптимального решения, используя подход динамического программирования и принцип максимума.

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