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NON-LINEAR INSTABILITY AND PLASTIC COLLAPSE OF PARTIALLY LOADED SPHERICAL SHELLS*

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Based on a two-dimensional discretization scheme, a finite-difference energy method is utilized for the non-linear, elastic and elastic-plastic collapse analysis of spherical shells under external pressures acting asymmetrically on a given fraction of the shell surface. The results are compared with some of the limited experimental data and theoretical predictions previously available.

1. INTRODUCTION

The stability of non-uniformly loaded shells is not well understood. Since the original experiments of KLÖPPEL and ROOS [1] on the snap-through of partially loaded spherical shells a few theoretical works have appeared in which similar problems of non-linear instability are solved [2—4]. The non-linear response of spherical shells due to static and dynamic area loads of arbitrary location has also been investigated [5, 6]. The influence of plasticity for such asymmetric loading problems, however, has received very little attention [7]

In the present paper, a method which has recently become available [8, 9] is utilized in order to obtain numerical solutions for partially loaded spherical shells in both the elastic and elastic-plastic cases. In the elastic case, the results are compared with the available experimental data [1] and other theoretical solutions [3, 4]. In the elastic-plastic case, a comparison with another solution is presented only for a uniformly loaded shell [10], related experimental data, for partially loaded shells [7], can also be used to evaluate this method further.

Unevenly distributed external overpressures are currently of interest in the structural safety analysis of the spherical containment vessel of certain nuclear power plants. Such loads may be caused by pressure build-up in the spaces between the containment vessel and the surrounding reactor building following, for instance, the rupture of a pipe. The transient nature of such an incident, usually accompanied by asymmetric moving loads, is not considered here; it is, however, the subject of a continuing investigation of the effects of dynamic loads [11—14].

2. SUMMARY OF METHOD

The method of analysis is based on the non-linear shell theory of MARLOWE and FLÜGGE [15]. It is assumed that the rotation about the normal is of the same order of magnitude as the square root of a typical middle surface strain and that the out-

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of-plane rotation can be moderately large. The kinematical relations of this theory are summarized by ALMROTH, BROGAN, and MARLOWE [16].

The plasticity theory utilized in the analysis follows the approach of BESSELING [17]. It is assumed that the material of the shell consists of several perfectly plastic constituents with identical elastic properties but with different yield limits. This model can thus represent strain-hardening effects in the original material by means of a piecewise linear stress-strain relation.

The numerical solution is based on a two-dimensional finite difference approximation of the energy expression. A system of non-linear algebraic equations is obtained when the energy is rendered stationary. These equations are then solved by means of a modified Newton-Raphson method of iteration for increasing values of the load parameter. The collapse load of the shell is determined as a maximum in the non-linear load-deflection curve. The procedure is further described by BUSHNELL, ALMROTH and BROGAN [18].

3. NUMERICAL RESULTS

The series of tests performed by KLÖPPEL and ROOS [1] is considered first. In these experiments, thin spherical shells with and without a cylindrical skirt were subjected to external pressure distributed over half of the shell until snap-through occurred. The following empirical relation

$$(3.1) \qquad p_c = 0.295(1-0.1\bar{\alpha})ET^2/R^2, \quad \bar{\alpha} \equiv (\alpha^\circ - 10^\circ)/10^\circ, \quad 15^\circ \leqslant \alpha \leqslant 60^\circ,$$

where p_c is the collapse pressure, R the radius of the shell, T the wall thickness, α half the shell angle, and E the Young's modulus, has been suggested [1] as representing the response of spherical shells to such asymmetric loads. In one case, stresses were also recorded. A comparison of numerical results obtained for this case and the experimental data is given in Fig. 1, where the meridional stress σ_x and the circumferential stress σ_y at the inner surface of the shell for stations along the *aa'* meridian are shown. The calculations are for a simply-supported shell, with $R=52$ cm, $T=0.059$ cm, $\alpha=22.6°$, $E=2.1\times10^6$ kg/cm^2, $\nu=0.3$, and $q=0.35$ kg/cm^2, where ν denotes Poisson's ratio and q the external pressure.

For two values of the angle α, calculated collapse loads are compared with the predictions of the Eq. (3.1) over a range of R/T in Fig. 2. For $\alpha=53.1°$, points falling above and below the curve representing the Eq. (3.1) are for clamped and simply-supported shells, respectively. Similarly, for $\alpha=22.6°$, points above the curve are for shells whose edges are allowed only meridional displacements, while points below are again for simply-supported shells. Therefore, the Eq. (3.1) appears to be valid for shells with boundary conditions of an intermediate nature, as was probably the case in the experimental investigation.

We turn next to other numerical solutions available for this problem. PERRONE and KAO [4] utilized the Marguerre shell theory as formulated by FAMILI and ARCHER [2] and a non-linear finite-difference relaxation method of solution. Their results and those obtained here for pressures distributed over a quarter as well

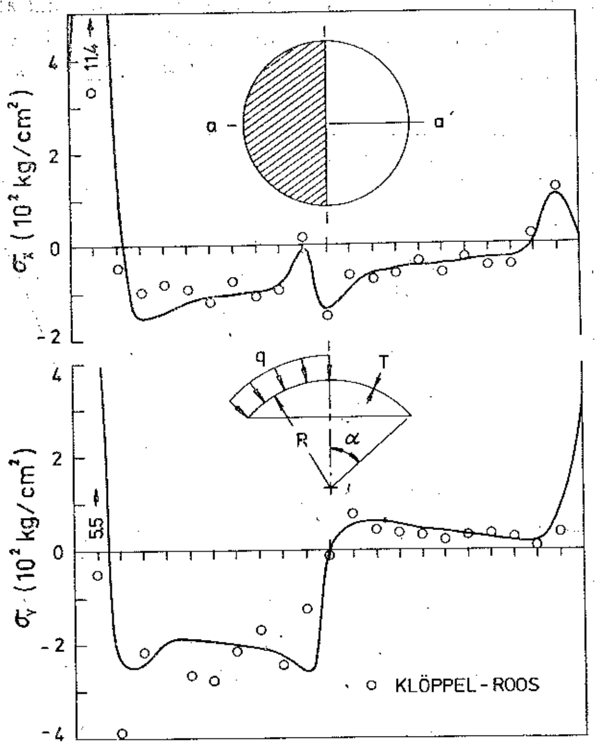

Fig. 1. Stresses along the *aa′* meridian.

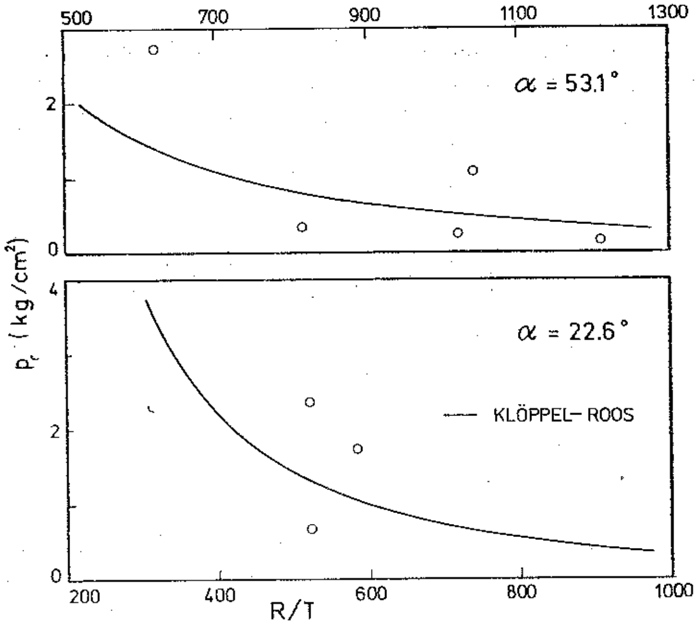

Fig. 2. Collapse pressure versus radius to thickness ratio.

as half of the shell are compared in Fig. 3, where, for stations along the *bb′* meridian, the ratio $\bar{w}$ of deflection to shell thickness is shown at load levels corresponding to collapse. The calculations are for a clamped shell, with $R=907.2$ cm, $T=1$ cm, $\alpha=6.3°$, $E=2.1\times10^6$ kg/cm^2, and $\nu=0.3$. BALL [3] solved the same half-loaded shell problem by using the Sanders shell theory and a combined series of Fourier finite-difference approach; the results are reproduced in Fig. 3. Introducing the notation

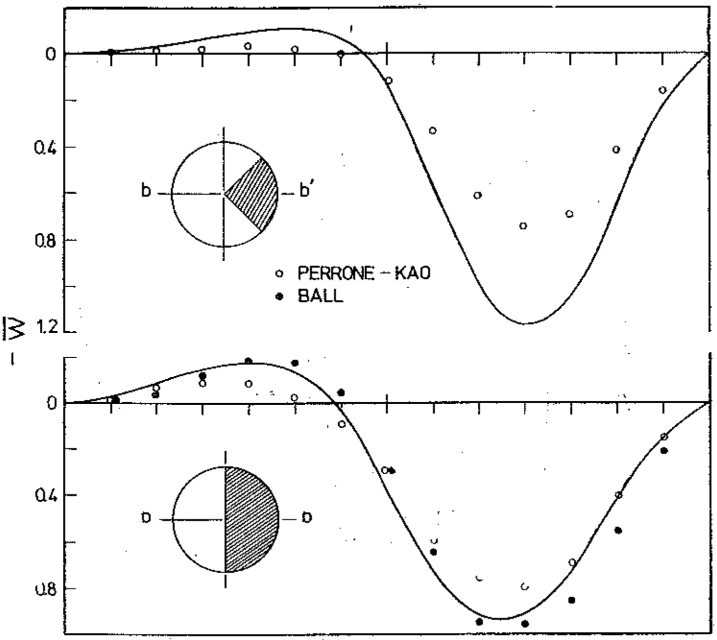

Fig. 3. Deflections along the *bb′* meridian.

$$(3.2) \qquad p\equiv qR^2\sqrt{3(1-\nu^2)}\,/2ET^2,$$

where p is a dimensionless external pressure, PERRONE and KAO [4] calculated that, at collapse, $p=0.56$ for the quarter-loaded shell, while a value of $p=0.54$ was found here. Similarly, for the half-loaded shell, PERRONE and KAO [4] and BALL [3] report that $p=0.665$, compared to $p=0.55$ calculated here. Thus it appears that the present method provides a conservative theoretical estimate of collapse strength in this case.

For thicker shells under external pressure, the plastic behaviour of the material of the shell influences the collapse load significantly. Such a case has recently been considered by HARTZMAN [10]. The corresponding geometry and estimated piecewise linear stress (σ)-strain (ε) behaviour of the material of the shell are shown in Fig. 4. For a uniformly loaded spherical shell, the plastic collapse pressure calculated here is compared in Fig. 5 with the value obtained by HARTZMAN [9] using the finite-element method. The number of integration stations through the shell thickness is denoted by N. The calculations are for a clamped shell, with $R=18.1$ cm, $T=0.635$ cm, $\alpha=41.5°$, and $\nu=0.3$. For $N=5$, the calculated collapse load of 442.5 kg/cm^2 compares well with the value of 434.5 kg/cm^2 found by Hartzman [10]. Despite the slightly unconservative character of the results obtained by the present method for this case, the ability of the theory to predict the collapse of rather thick shells has been demonstrated. Accordingly, for the same geometry and material response, the plastic collapse analysis of half- and quarter-loaded shells was also carried out. No previous solution of such problems appears to be available. The results are shown in Fig. 6 in the form of load-deflection diagrams for the three cases indicated; δ denotes the deflection of point A located on the bb' meridian

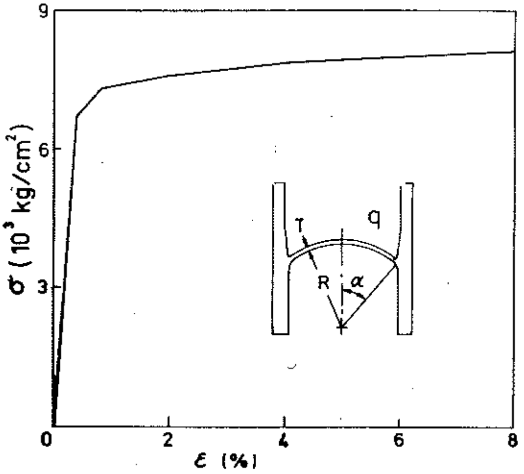

Fig. 4. Shell geometry and material response used in plastic collapse analysis.

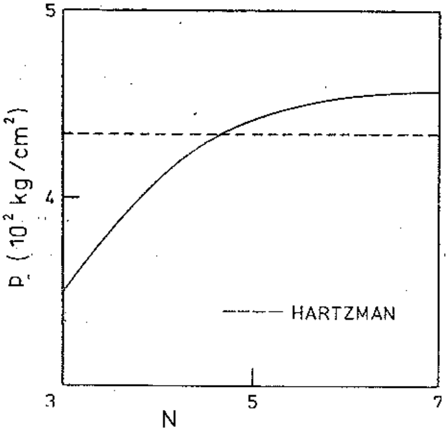

Fig. 5. Plastic collapse pressure versus number of integration stations through shell thickness; uniformly loaded shell.

(shown in Fig. 3) at $\beta = 20.8°$. The calculations are for $N=5$. The influence upon the non-linear response of the shell of the manner in which the loading is distributed over the shell surface is evident.

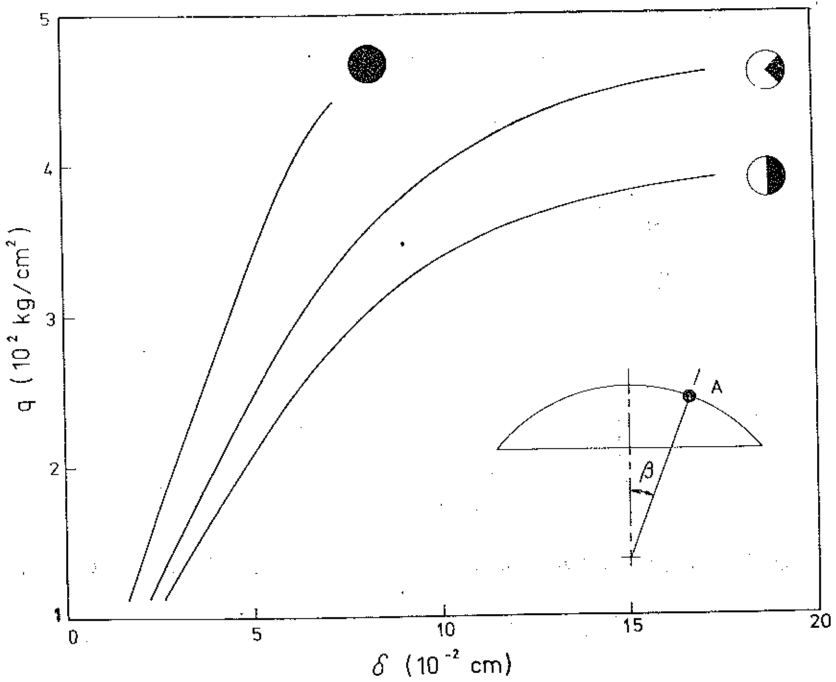

Fig. 6. Load-deflection behaviour of uniformly and partially loaded elastic-plastic shells.

4. Conclusion

The present method is a useful tool for the study of non-linear collapse phenomena in shells, including the effects of plasticity. Considerable progress in understanding such phenomena appears possible, since many cases which would otherwise be theoretically untractable (shells with stiffeners and cutouts, for instance) can now be resolved numerically [19]. Therefore, renewed interest in more general non-linear shell theories [20] also seems warranted in cases involving large displacement gradients.

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STRESZCZENIE

NIELINIOWA NIESTATECZNOŚĆ I PLASTYCZNE ZNISZCZENIE CZĘŚCIOWO OBCIĄŻONYCH POWŁOK KULISTYCH

Opierając się na dwuwymiarowym schemacie dyskretyzacji wykorzystano metodę energetyczną dla różnic skończonych celem analizy zniszczenia nieliniowych, sprężystych i sprężysto-plastycznych powłok kulistych poddanych ciśnieniu zewnętrznemu działającemu osiowo-symetrycznie na daną część powierzchni powłoki. Wyniki porównano z nielicznymi danymi doświadczalnymi i otrzymanymi uprzednio danymi teoretycznymi.

Резюме

НЕЛИНЕЙНАЯ НЕУСТОЙЧИВОСТЬ И ПЛАСТИЧЕСКОЕ РАЗРУШЕНИЕ ЧАСТИЧНО НАГРУЖЕННЫХ СФЕРИЧЕСКИХ ОБОЛОЧЕК

Базируя на двумерной схеме дискретизации использован энергетический метод для конечных разностей с целью анализа разрушения нелинейных, упругих и упруго пластических сферических оболочек подвергнутых внешнему давлению действующему осесимметрично на данную часть поверхности оболочки. Результаты сравнены с немногочисленными экспериментальными данными и с полученными раньше теоретическими данными.

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