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EXTREMUM AND VARIATIONAL PRINCIPLES IN PLASTICITY (*)

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Extrema of functions (functionals) W are for numerical reasons frequently expressed in the weaker variational form $\delta W=0$. The relationship between both is discussed. In most theories W denotes somewhat like work, and $\delta W=0$ the principle of virtual work. In the Sec. 2 the variational and extremum principles for rigid plastic materials are considered. The classical dual Haar-von Kármán-Sadowski-Phillips-Hill upper and lower bound principles are given in general form which is not restricted to specific boundary conditions, or to incompressibility, rate-independence, homogeneity, or isotropy. They have become one of the strongest tools for applying theory to practical problems. This is illustrated by a series of examples taken from structural mechanics, metal forming technology and soil mechanics to show also some recently studied features concerning surface fraction, action of volume forces, and volume compression, or extension, respectively. The Sec. 3 concerns the static problems for elastic-plastic material. Starting from the Cotterill-Castigliano principles for elasticity, PRAGER, HODGE, GREENBERG and BAUER dexeloped similar principles for perfectly-plastic or strain-hardening materials. There are, however, only few numerical applications. Recently it has even been shown that utmost caution is necessary to avoid systematic errors. In the Sec. 4 the rate-dependent or dynamic plasticity is discussed. Besides general principles of mechanics like Hamilton's principle there are a few specific ones partly related to the theorem of work and energy which allow to estimate the magnitude of total deformation, or other quantities. The Sec. 5 is devoted to some generalizations and applications of principles discussed. As in elasticity, attempts can be made to apply directly the virtual work principle $\delta W=0$ in order to obtain pointwise information on the unknown solution looked for. Also more general materials may be considered like those having a non-associated flow rule. Most general principles are, however, closely connected to the method of weighted residuals only.

1. INTRODUCTION

1.1. *Principle of virtual work*

This paper will report on extremum and variational principles which have a proper physical background in the sense that they are in some way correlated to the principle of virtual work. We refer to an earlier review by HODGE [59] as well as to considerations given in the textbook [1], the notation of which will also be applied in this paper, and use additionally some operator expressions which are in a simplified manner taken from [2]. Then the well-known statical principle of virtual work may be stated as

$$\delta W_{\mathrm{in}}=\delta W_{\mathrm{ex}}, \tag{1.1}$$

where δW_{in} represents the internal, and δW_{ex} the external virtual work done at the considered (system of) continuous bodies. Mathematically speaking, δW_{in} and δW_{ex} are bilinear functions or functionals,

$$\delta W_{\mathrm{in}}=A_{\mathrm{in}}(Q; \delta q), \quad \delta W_{\mathrm{ex}}=A_{\mathrm{ex}}(F, \delta x),$$

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of the state of (generalized, internal) "stress" Q, (generalized, internal) virtual "strain" δq, (generalized, external) "load" F, and (generalized, external) virtual "displacement" δx, respectively. The quantities Q, δq, F, δx may be considered as points of given linear spaces, as any two different states $\overset{1}{Q}$, $\overset{2}{Q}$, etc. can be superimposed according to $\alpha\overset{1}{Q}+\beta\overset{2}{Q}$, α and β being real numbers. So, for instance, in a classical three-dimensional continuum (finite body with a piecewise smooth, closed surface) Q would represent the state of Cauchy stress σ_{jk} (piecewise continuously differentiable with respect to space coordinates), δq the state of linear virtual strain $\delta\varepsilon^{jk}$ (piecewise continuous), F the state of surface tractions T_j as well as of volume forces p_j (both piecewise continuous but allowing isolated Dirac δ-distributions, i.e. single loads), and δx^j the overall state of point displacements (continuously differentiable with respect to space coordinates, apart from isolated jump surfaces). Accordingly, for a straight Bernoulli beam undergoing pure bending we may adopt Q as the mathematically sufficiently regular distribution of bending moment M, δq of virtual curvature $\delta\kappa$, while F comprehends the distribution of external transversal forces p (including single loads), and δx represents the related vertical displacement δy.

For the three-dimensional body (volume Ω, surface B) we have (1)

$$A_{\text{in}}=\int^{\Omega}\sigma_{jk}\,\delta\varepsilon^{jk}\,d\Omega+\int^{S}\overset{S}{T}_j\,\Delta\delta x^j\,dS\,,$$

where the displacements are assumed continuous except for single surfaces S (tractions $\overset{S}{T}_j$) across which a jump $\Delta\delta x^j$ may occur, while

$$A_{\text{ex}}=\int^{\Omega}p_j\,\delta x^j\,d\Omega+\int^{B}T_j\,\delta x^j\,dB\,.$$

Accordingly, for the beam of length l (axial coordinate ξ, $0\leqslant\xi\leqslant l$) it holds that

$$A_{\text{in}}=\int_0^l M\delta\kappa d\xi\,,\qquad A_{\text{ex}}=\int_0^l p\delta y d\xi\,,$$

and we shall generally assume in what follows that the function(al)s A_{in}, A_{ex} are for the considered problem known in advance.

For the principle of virtual work to hold, the following two relations have to be fulfilled:

"compatibility"

$$\delta q=C\delta x\,, \tag{1.2}$$

and "equilibrium"

$$F=LQ \tag{1.3}$$

(1) Summation over pairs of equal subscripts, one being an upper, the other a lower one (RICCI). The instantaneous configuration is considered.

using appropriate linear operators C and L. E.g., they are defined for the three-dimensional continuous body by

$$(1.4)\qquad C:\quad \delta\varepsilon^{jk}=\frac{1}{2}(\delta x^{j}|^{k}+\delta x^{k}|^{j}),$$

$$L:\quad \begin{cases} p_j=-\sigma_{kj}|^{k} & \text{in } \Omega,\\ T_j=\sigma_{kj}n^{k} & \text{on } B,\end{cases}$$

where n^k is the outward surface normal on B, and $|^k$ denotes (contra-variant components of) co-variant differentiation with respect to the geometric coordinates ξ^k. Accordingly for the beam,

$$C:\quad \delta\kappa=-\frac{\partial^2\,\delta y}{\partial\xi^2},$$

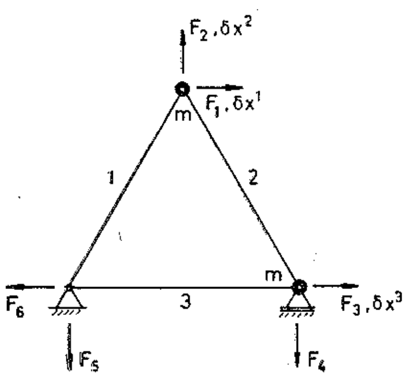
$$L:\quad p=-\frac{\partial^2 M}{\partial\xi^2}.$$

We shall assume in what follows that C and L are pre-known for each considered problem. Note that the inverse operators need be neither everywhere defined nor unique. Therefore, we call

$$\delta q \text{ ``compatible'' if, and only if,}\quad \delta x=C^{-1}\,\delta q,$$

$$F \text{ ``self-equilibrated'' if, and only if,}\quad Q=L^{-1}F$$

may be formed in the sense that at least one state δx or Q exists, so that the Eq. (1.2) or (1.3) holds, respectively.



m point masses (for kinetics)
l_j length
A_j cross sectional area
Q_j force
δq^j longitudinal virtual extension of rod no. j

C: $\delta q^1=\frac{1}{2}(\delta x^1+\sqrt{3}\,\delta x^2)$, $\delta q^2=\frac{1}{2}(\delta x^3-\delta x^1+\sqrt{3}\,\delta x^2)$, $\delta q^3=\delta x^3$

L: $F_1=\frac{1}{2}(Q_1-Q_2)$, $F_2=\frac{\sqrt{3}}{2}(Q_1+Q_2)$, $F_3=(Q_3+\frac{Q_2}{2})$, $F_4=\frac{\sqrt{3}}{2}Q_2$, $F_5=\frac{\sqrt{3}}{2}Q_1$, $F_6=(Q_3+\frac{Q_1}{2})$

Fig. 1. Plane truss.

To illustrate the above relations as well as some of the results to follow in a most simple manner, we need consider only a discrete system like the truss of Fig. 1 which is to serve us as a standard example rather than a continuum. The virtual work is then given by the bilinear functions

$$A_{\text{ex}}(F, \delta x) = F_j\, \delta x^j, \qquad A_{\text{in}}(Q, \delta q) = Q_j\, \delta q^j . \tag{1.5}$$

In Fig. 1, the considered truss is even statically determined which means that for each system of applicable loads F_1, F_2, F_3 there are unique reactions F_4, F_5, F_6, so that F becomes self-equilibrated, and L^{-1} is then uniquely defined according to

$$\begin{aligned} L^{-1}:\quad & Q_1 = F_1 + \frac{F_2}{\sqrt{3}}, \qquad Q_2 = -F_1 + \frac{F_2}{\sqrt{3}}, \qquad Q_3 = \frac{F_1}{2} - \frac{F_2}{2\sqrt{3}} + F_3, \\ \text{if}\quad & F_4 = -\frac{\sqrt{3}}{2} F_1 + \frac{1}{2} F_2, \qquad F_5 = \frac{\sqrt{3}}{2} F_1 + \frac{1}{2} F_2, \qquad F_6 = F_1 + F_3 . \end{aligned} \tag{1.6}$$

In the same way each state δq is compatible, so that

$$\begin{aligned} C^{-1}:\qquad & \delta x^1 = \delta q^1 - \delta q^2 + \frac{1}{2}\, \delta q^3 , \\ \delta x^2 = \frac{1}{\sqrt{3}} & \left[\delta q^1 + \delta q^2 - \frac{1}{2}\, \delta q^3 \right], \qquad \delta x^3 = \delta q^3 . \end{aligned} \tag{1.7}$$

Though virtual kinematical quantities are in general considered as variations independent of time t, the foregoing analysis remains correct if δ is replaced by the "incremental" differential d with respect to time t, or simply by a dot $(\dot{\ })$. Thus we may replace δx, δq by the state of point velocities or strain rates

$$\dot{x} = v , \qquad \dot{q} = \lambda ,$$

respectively (belonging, apart from their physical dimensions, to the same linear spaces as δx, δq do), and obtain the rate-of-work balance

$$\dot{W}_{\text{in}} = \dot{W}_{\text{ex}} , \tag{1.8}$$

or, after an integration, the work balance itself

$$W_{\text{in}} = W_{\text{ex}} \tag{1.9}$$

(preservation-of-work theorem), where

$$\dot{W}_{\text{in}} = A_{\text{in}}(Q, \lambda) , \qquad \dot{W}_{\text{ex}} = A_{\text{ex}}(F, v) \tag{1.10}$$

holds while the conditions of compatibility (1.2) and equilibrium (1.3) become

$$\lambda = Cv , \qquad F = LQ , \tag{1.11}$$

respectively. We may generalize the virtual work theorem in a straightforward manner also to kinetics, by adding simply (symbolically) the inertial terms $-\rho\dot{v}$ to the external loads F or, more specifically, to the volume forces p.

1.2. True and admissible states

While the preceding relations are independent of material, we must now introduce the "constitutive law", and at first do this with regard to some rigid-plastic, or viscous body by relating strain rates λ and stresses Q according to

$$Q=H\lambda, \quad \lambda=H^{-1}Q. \tag{1.12}$$

The "constitutive operator" H as well as its inverse H^{-1} may have a bounded domain only, and need by no means be single-valued(²). Consider as an example the truss of Fig. 1 again, and assume that $Y_i>0$ denotes the uniaxial yield stresses of the rods $i=1, 2, 3$. Then H, H^{-1} are defined by the constitutive relations

$$Q_j=A_j Y_j \,\mathrm{SGN}\,\lambda_j, \qquad \lambda_j=\lambda\left(\frac{Q_j}{A_j Y_j}\right), \tag{1.13}$$

in which the constitutive functions

$$\mathrm{SGN}\alpha=\begin{cases}1 & \text{if } \alpha>0\\ \beta & \text{if } \alpha=0, \quad \text{where } -1\leqslant\beta\leqslant 1,\\ -1 & \text{if } \alpha<0\end{cases}$$

$$\lambda(\alpha)\begin{cases}\geqslant 0 & \text{if } \alpha=1\\ =0 & \text{if } -1<\alpha<1\\ \leqslant 0 & \text{if } \alpha=-1\end{cases} \tag{1.14}$$

are indeed multi-valued. SGNα differs from the conventional sign-function sgn α just by its multi-valuedness at $\alpha=0$ but possesses an unbounded domain $-\infty<\alpha<\infty$, while the bounded domain of $\lambda(\alpha)$ is given by $-1\leqslant\alpha\leqslant 1$ (Fig. 2).

Now, for each problem to be considered we shall assume explicitly that there exists a

$$\text{"true state"} \quad \hat{F},\ \hat{v},\ \hat{Q},\ \hat{\lambda} \tag{1.15}$$

which obeys compatibility and equilibrium (1.11) as well as the constitutive law (1.12), besides given boundary conditions not to be examined here. It will be compared with varying "kinematically admissible states"

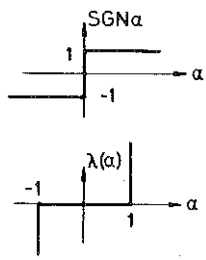


Fig. 2. Constitutive functions for rigid-plastic trusses.

$$\overset{*}{v}=C^{-1}\overset{*}{\lambda}, \quad \overset{*}{Q}=H\overset{*}{\lambda}, \quad F=L\overset{*}{Q}, \tag{1.16}$$

the strain rate $\overset{*}{\lambda}$ of which must therefore be compatible, and belong to the domain of the constitutive operator H. Besides, the true state will be compared as well with varying "statically admissible states"

$$\overset{0}{\lambda}=H^{-1}\overset{0}{Q}, \quad \overset{0}{F}=L\overset{0}{Q}, \tag{1.17}$$

(²) This is the reason why theorems stating uniqueness in plasticity require further restrictive assumptions to be made which were sometimes not realistic; cf. also DRUCKER [108].

in which the stress $\overset{0}{Q}$ belongs to the domain of the inverse constitutive operator H^{-1}. The quantity $\overset{0}{v}=C^{-1}\overset{0}{\lambda}$ will in general not be formed so that $\overset{0}{\lambda}$ need not be compatible. Note that the definitions of admissibility used here were not at all related to any boundary conditions [4] and are therefore more general in application [3, 5] than those generally presented in literature (cf. [59]). Considering again the standard example of our truss (Fig. 1), we have, according to the Eqs. (1.13), (1.14), that each triple of "rates" $\overset{*}{\lambda}_1, \overset{*}{\lambda}_2, \overset{*}{\lambda}_3$ defines [because of its compatibility (1.7)] always an admissible state, while "stresses" $\overset{0}{Q}_1, \overset{0}{Q}_2, \overset{0}{Q}_3$ are admissible if, and only if,

$$-1 \leqslant \frac{\overset{0}{Q}_j}{A_j Y_j} \leqslant 1 .$$

1.3. Dummy rate, and dummy load methods

They are based on the following two equations being an immediate consequence of the rate-of-work balance (1.8) in combination with (1.10), (1.11), (1.12), which equations are especially fulfilled by the true state $\hat{F}$, $\hat{v}$. The methods hold for any varying "dummy" state of velocity v or any self-equilibrated varying "dummy" state of load F according to

I) dummy rate method:

$$A_{\text{in}}(HC\hat{v}, Cv) = A_{\text{ex}}(\hat{F}, v) , \tag{1.18}$$

II) dummy load method:

$$A_{\text{in}}(L^{-1} F, H^{-1} L^{-1} \hat{F}) = A_{\text{ex}}(F, \hat{v}) . \tag{1.19}$$

Both represent a strong tool for dealing with practical problems, but are in literature in the first place formulated with regard to elastic bodies, though they then form an approximation valid for small (linear) strains only rather than for rigid-plastic or viscous bodies where they hold rigorously [1].

Application will be illustrated using our standard example of the truss (Fig. 1) in combination with the Eqs. (1.6), (1.13). Substituting the dummy states $v^1 \neq 0$, $v^2 = v^3 = 0$, or $F_1 \neq 0$, $F_2 = F_3 = 0$, $F_4 = -\frac{\sqrt{3}}{2} F_1$, $F_5 = \frac{\sqrt{3}}{2} F_1$, $F_6 = F_1$ (self-equilibrated) respectively, we obtain the explicit expressions for the true force $\hat{F}_1$, and the true velocity $\hat{v}^1$, according to:

$$\hat{F}_1 = \frac{1}{2} A_1 Y_1 \,\text{SGN}(\hat{v}^1 + \sqrt{3}\,\hat{v}^2) - \frac{1}{2} A_2 Y_2 \,\text{SGN}(\hat{v}^3 - \hat{v}^1 + \sqrt{3}\,\hat{v}^2) , \tag{1.20}$$

$$\hat{v}^1 = \lambda \left\{ \frac{1}{A_1 Y_1} \left[\hat{F}_1 + \frac{\hat{F}_2}{\sqrt{3}} \right] \right\} - \lambda \left\{ \frac{1}{A_2 Y_2} \left[-\hat{F}_1 + \frac{\hat{F}_2}{\sqrt{3}} \right] \right\} + \frac{1}{2} \lambda \left\{ \frac{1}{A_3 Y_3} \left[\hat{F}_3 + \frac{\hat{F}_1}{2} - \frac{\hat{F}_2}{2\sqrt{3}} \right] \right\} . \tag{1.21}$$

They are, as pointed out before, by no means unique and may be completed by similar formulae for $\hat{F}_2$, $\hat{F}_3$, $\hat{v}^2$, $\hat{v}^3$, if the dummy quantities were, as shown above, chosen in an appropriate manner. But this choice is an applicational procedure only, while the principles (1.18), (1.19) defining the methods remain valid no matter how the dummy quantities have been chosen.

1.4. Variational versus extremum principles

The principle of virtual work (1.1), being the background of all the items treated above, may be stated as

$$\delta W = 0, \tag{1.22}$$

W being e.g. the functional $W = W_{ex} - W_{in}$. The Eq. (1.22) represents a "variational principle" only; the functional W may assume an extremum (e.g. a minumum), or a stationary value (e.g. a point of inflection) only. Both possibilities are illustrated in Fig. 3 using for the sake of simplicity a function $W(x)$ rather than a functional.

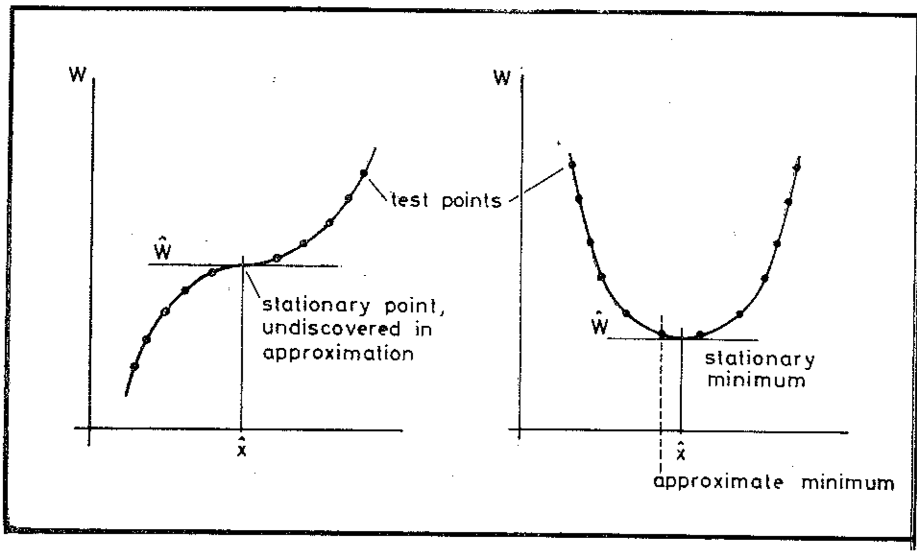


Fig. 3. Step-by-step search for a stationary point.

If the solution $\hat{x}$ of the Eq. (1.22) should, for more complicated problems, be calculated numerically, then a finite process like a step-by-step search must be carried out. It can intuitively be seen that it would yield approximate (or even converging) values only in the case of an extremum, and this is the first reason for proper extremum principles to be preferred.

The second argument would be that extremum principles such as

$$W \geqslant \hat{W}, \qquad W \to W_{min} \tag{1.23}$$

may deliver, by virtue of the inequality (1.22), also bounds for other interesting quantities like true loads, displacements, etc.

Finally, there are historical (philosophical rather than practical) arguments for establishing extremum principles. Some of them have been reviewed in [1].

As a consequence, let us pass on to the extremum principles themselves.

2. Statics of rigid-plastic bodies

2.1. General

According to the Eqs. (1.10), (1.5), the true internal rate-of-work of a truss like that in Fig. 1 becomes $A_{\text{in}}(\hat{Q}, \hat{\lambda}) = |\hat{Q}_j|\,|\hat{\lambda}^j|$, where the amounts alone need be considered as, according to (1.13), the signs of $\hat{Q}_j$ and $\hat{\lambda}^j$ are equal. Only those terms contribute for which $\hat{\lambda}^j \neq 0$ holds, so that according to (1.13) and $(1.14)_1$, $|\hat{Q}_j|$ is maximal: $|\hat{Q}_j| \geqslant |\overset{0}{Q}_j|$ if $\overset{0}{Q}_j$ denotes any admissible state. It follows that $A_{\text{in}}(\hat{Q}, \hat{\lambda}) \geqslant A_{\text{in}}(\overset{0}{Q}, \hat{\lambda})$. Consequently, $A_{\text{in}}(\overset{*}{Q}, \overset{*}{\lambda}) \geqslant A_{\text{in}}(\hat{Q}, \overset{*}{\lambda})$ since any kinematically admissible state $\overset{*}{Q}, \overset{*}{\lambda}$ may, according to (1.16), be considered as true, while $\hat{Q}$ forms always a statically admissible state. So using the Eqs. (1.8), (1.10) we obtain two conjugate basic extremum theorems:

$$A_{\text{ex}}(\overset{0}{F}, \hat{v}) \leqslant A_{\text{ex}}(\hat{F}, \hat{v}), \qquad A_{\text{ex}}(\hat{F}, \overset{*}{v}) \leqslant A_{\text{in}}(\overset{*}{Q}, \overset{*}{\lambda}), \tag{2.1}$$

the first of which is generally referred to as the "lower-bound theorem", while the second one is called the "upper-bound theorem".

If for the sake of simplicity we assume our standard truss of Fig. 1 to have in each of its rods equal cross sectional areas $A_1 = A_2 = A_3 = A$ as well as equal yield limits $Y_1 = Y_2 = Y_3 = Y > 0$, and if it is externally loaded by $\hat{F}_2 > 0$ only while $\hat{F}_1 = \hat{F}_3 = 0$, then admissible states may be set up as follows, using (1.16), (1.17), Fig. 1, and (1.13), (1.14):

$$\begin{gathered} \overset{0}{Q}_1 = \overset{0}{Q}_2 = YA, \quad \overset{0}{Q}_3 = \frac{1}{2} YA, \quad \overset{0}{F}_1 = \overset{0}{F}_3 = 0, \quad \overset{0}{F}_2 = \sqrt{3}\, Y_A; \\ \overset{*}{v}^1 = \overset{*}{\lambda}/2, \quad \overset{*}{v}^2 = \lambda/2\sqrt{3}, \quad \overset{*}{v}^3 = -\overset{*}{\lambda}, \quad \overset{*}{\lambda}^1 = \overset{*}{\lambda}^2 = 0, \quad \overset{*}{\lambda}^3 = -\overset{*}{\lambda}. \end{gathered} \tag{2.2}$$

Thus we obtain from the lower and upper bound theorems actually lower and upper bounds of the true applied load $\hat{F}_2$ according to

$$\sqrt{3}\, YA \leqslant \hat{F}_2 \leqslant 2\sqrt{3}\, YA. \tag{2.3}$$

Note again that the theorems (2.1) themselves hold independently of any boundary conditions, which should be considered, particularly when a special problem such as the one above is to be solved, in order to get the specific information wanted.

A continuum considered as some limit case of more and more branched trusses which form fine nets, the meshes of which tend to zero, obviously preserves the validity of the upper and lower bound theorems. It can be shown [7] to be identical with a "standard" rigid-plastic body. This shall be defined as usual by a convex yield surface (with the origin in its interior) in the stress-space, and the so-called "normality rule" by means of which the theorems could directly be proved (cf. [6]). They have, as stated here, but in contrast with general opinion, again nothing to do with any boundary conditions [3, 4] (which need be regarded, as before, for practical applications only), and are often correlated with the names of HAAR-v. KÁRMÁN (1909), SADOWSKI (1943), PHILLIPIDIS (1948) (3), or HILL (1948). Of course, no further restrictions such as isotropy or homogeneity are necessary [73].

2.2. *Applications*

In civil engineering applications, it would be "safe" to look for a lower bound of the collapse load, while an (additional) upper bound gives information about the degree of approximation. LANCE and SOECHTING derive displacement bounds instead [67]. RADENKOVIC and NGUYEN [2] show how to reduce systematically the procedure for continuous structures to discrete systems if the number of load parameters is finite. Thus the example given above using our standard truss proves to be fairly general, so that we renounce quoting further literature, the more so as, for structures undergoing small deformations, elastic-plastic bodies rather than rigid-plastic ones should be considered.

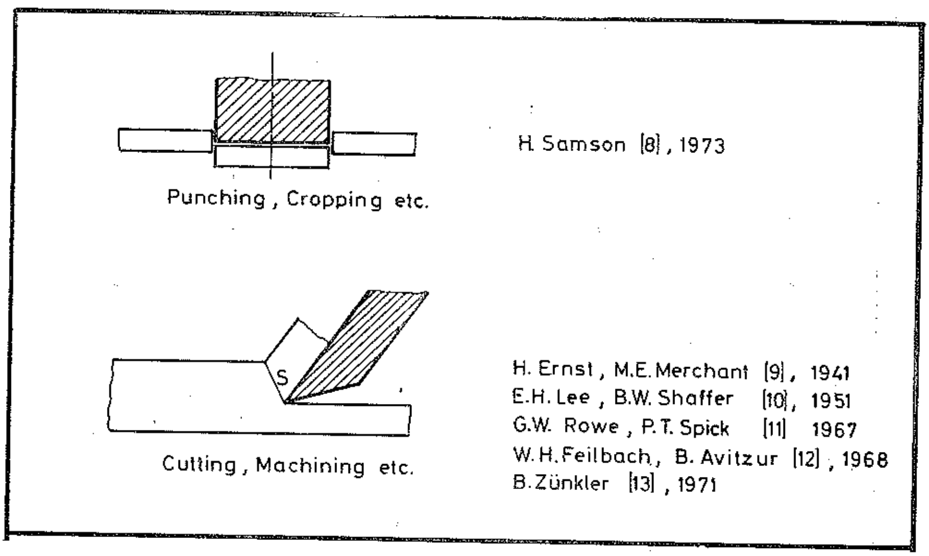


Fig. 4. Metal separating (Review).

Instead, Figs. 4–7 give a review of literature on metal forming applications (large deformations), and on soil mechanics under the (questionable) assumption that soil (and rock) behaves as a rigid-plastic body as well. So far, this review has

(3) Now A. PHILLIPS.

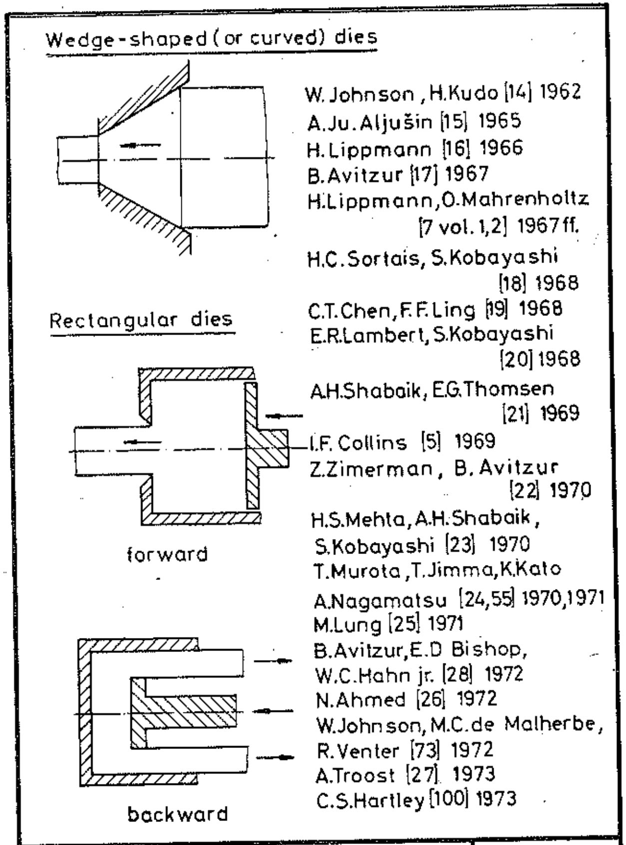


Fig. 5. Drawing and extrusion (Review).

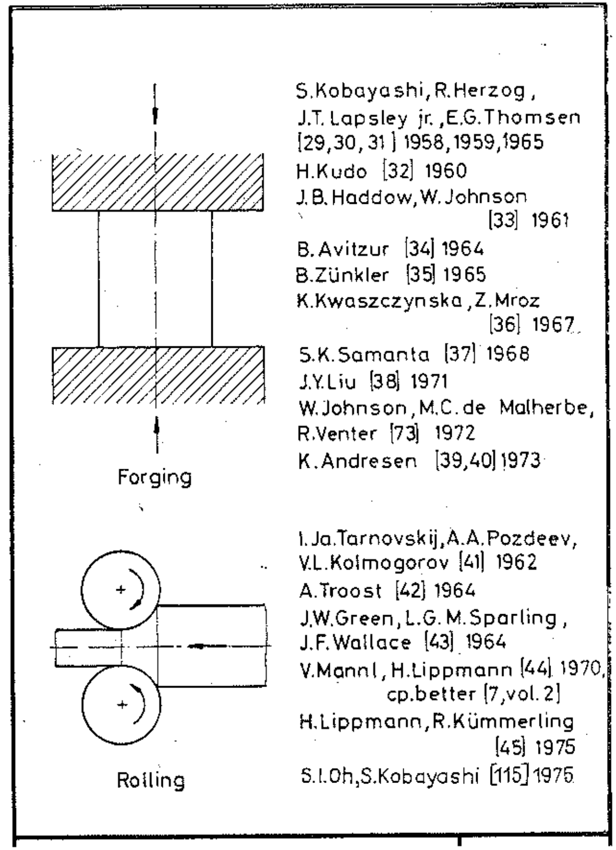


Fig. 6. Metal processing under pressure (Review).

claimed to be neither complete nor critical; there have even been considered papers, the authors of which made use of this (more or less modified) method without being aware that it had to do with the upper and lower bound theorems. Let us now deal with a few special items which deserve comment.

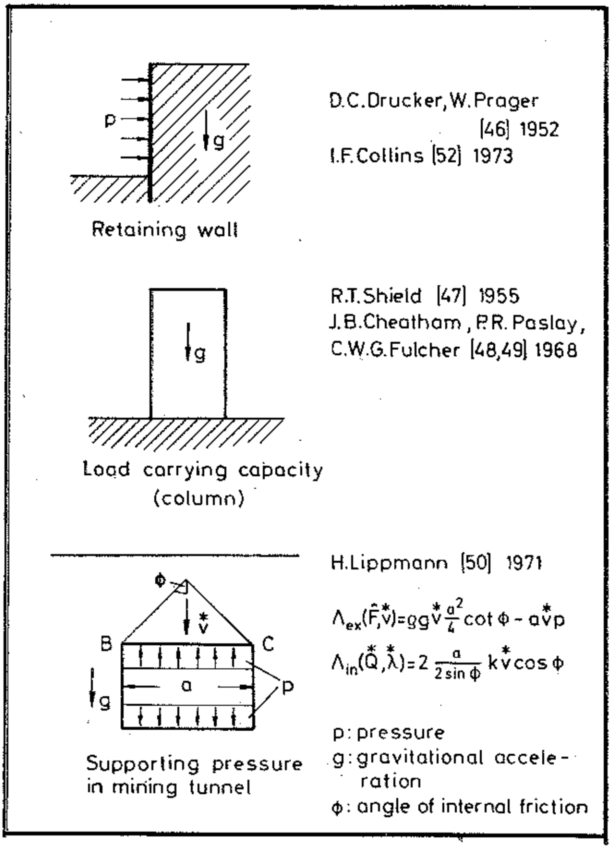


Fig. 7. Soil mechanics (Review).

When varying the admissible fields in order to obtain optimal bounds, one must neither vary the shape of the considered body (see, however, [102]), nor the local distribution of material parameters (like yield stress Y) in it. The first mistake may happen if, e.g., the inclination of the shear zone S in Fig. 4 (cutting) is varied [9]. Similar errors occurred in [11, 12, 13]. It is a basic handicap of the method, that the "true" shape of the body (i.e. one for which a complete theoretical solution may be assumed to exist) is rarely known in advance. So the sharp edge of material in cutting looks as questionable as the straight entry and exit of material (without any lips forming) in drawing, extrusion and rolling. One can only hope that small deviations in geometry do not affect the calculated bounds more than is admissible. Occasionally, the unknown boundary may be approximated by an iterative procedure [45], [115].

Erroneous variations of the Y-distribution could occur if Y depends on λ itself ("rate dependent" material). To avoid this difficulty one may again try some iterative method [25].

Because of the multi-valuedness of the constitutive operator H one cannot expect unicity of solution, at least not in general. So there is also no proof whether the statically and the kinematically admissible states become automatically true if the upper and lower bounds coincide. Based on this expectation, however, authors have tried to set up multi-parameter approaches in order to obtain approximatively true solutions after the bounds have been numerically optimized (generalized RITZ method [24, 25, 39, 40, 45, 55]). To facilitate this, graphical and computational [53, 56] procedures, or even procedures based on functional analysis [54, 57] for constructing the admissible states, have been described.

It can frequently be read that kinematical admissibility is mainly defined by incompressibility, so that especially velocity discontinuities having the character of pure shear become always admissible. This is correct for the commonly used yield laws of metal plasticity (TRESCA, HUBER-LEVY-MISES-HENCKY, cf. [6, 7]), but not for the compressible behaviour of soils (cf. [46, 57, 50], errors in [48, 49]).

Referring again to the boundary conditions which in general are usually included in the concept of admissibility, we are going to illustrate now that this is not only unnecessary but could even become misleading. Actually, neither Coulomb surface friction nor volume forces have been admitted in the past. The mining-tunnel example in Fig. 7 allows, however, the taking into account, in a very simple manner, the constant downward gravitational field (volume force ρg, ρ being the initially constant density of soil). It was proved in [50] that a velocity distribution as shown in Fig. 7, which equals 0 outside the drawn triangle (base BC) but having one constant vertical component $\overset{*}{v}$ inside, is admissible with respect to the so-called Coulomb yield law which means that deformation takes place in a discontinuous manner, only along the inclined sides of the triangle. Then the external work A_{ex} as given in the figure may immediately be formed, while A_{in} follows from the fact [50] that along discontinuity surfaces work is done only by the shear components $\overset{*}{v}\cos\Phi$ against the internal adhesion $k=\text{const}>0$ of the grains. Thus, we obtain from the upper bound theorem $(2.1)_2$ quite strangely a lower (!) bound of the supporting pressure

$$p \geqslant a\cot\Phi\left\{\frac{1}{4}\rho g a-k\right\}$$

which unfortunately is unsafe, while the lower bound theorem would provide a safe upper (!) bound of p [50].

Coulomb friction has been considered by COLLINS with regard to plane strain drawing [5], and to soils [52]. So if $\hat{T}$ in Fig. 8 denotes the (unknown) true drawing force (incompressible metals), then it is known that both dies react horizontally with $\hat{T}/2$, so that

$$\frac{\hat{T}}{2\left(\cos\alpha+\dfrac{\sin\alpha}{\mu}\right)}$$

acts parallel to them (resultant frictional force; constant coefficient of friction $\mu>0$). Introducing admissible velocities $\overset{*}{v}$ that are constant inside the drawn triangles (φ, ψ arbitrary) but jump in pure shear by $\Delta\overset{*}{v}_A$, $\Delta\overset{*}{v}_B$ along AC, BC just as shown in Fig. 8, we may quite easily write down the upper bound theorem $(2.1)_2$ using a constant shear limit $k>0$, according to:

$$\hat{T}\overset{*}{v}-\frac{\hat{T}\overset{*}{v}_{\mathrm{rel}}}{\cos\alpha+\dfrac{1}{\mu}\sin\alpha}\leqslant 2k\,|AC|\,\Delta\overset{*}{v}_A+2k\,|BC|\,\Delta\overset{*}{v}_B\,, \tag{2.4}$$

from which formula there results in general an upper bound of $\hat{T}$. Note, however, that possible difficulties may arise due to the negative second left-hand term. Furthermore, as $\overset{*}{v}_{\mathrm{rel}}$ has to be assumed constant which need not hold for $\hat{v}_{\mathrm{rel}}$, the

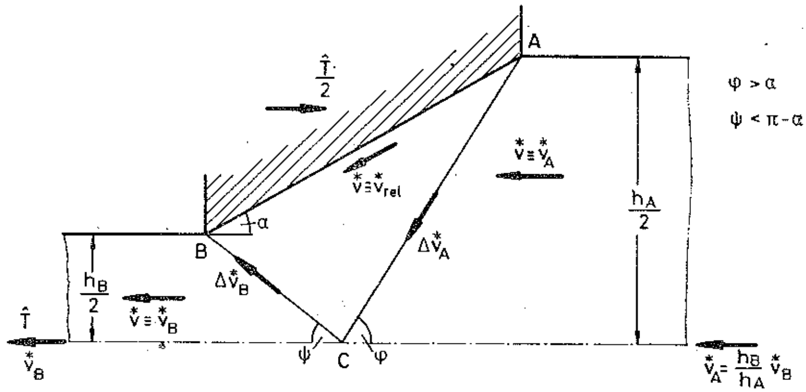


Fig. 8. Plane strain drawing; force $\hat{T}$ per unit width.

method will hardly give us an approximation of the true solution, and the lower-bound theorem does not even provide a bound of $\hat{T}$ at all.

Despite such troubles, the bounding theorems (2.1) form a strong and successful means for approximate calculations with respect to rigid-plastic materials.

3. Statics of elastic-plastic material

3.1. *Elasticity*

Let us first take a glance at the well-known Cotterill (1865) — Castigliano (1873) extremum principles which refer to small (geometrically linearized) elastic strains q, and displacements x which behave kinematically like virtual quantities δq, δx, so that compatibility (1.2)

$$q=Cx\,, \tag{3.1}$$

and consequently the virtual work theorem (1.1) remain valid using q, x instead of δq, δx, i.e.

$$A_{\mathrm{in}}(Q, q)=A_{\mathrm{ex}}(F, x) \tag{3.2}$$

for any two independent states Q, F or q, x, respectively. Considering then a finite, uniquely invertible, sufficiently regular constitutive law,

$$Q = Hq\,, \quad q = H^{-1} Q\,, \tag{3.3}$$

rather than the incremental one (1.12), and re-interpreting the definitions (1.15)–(1.17) of true or admissible states after having replaced v, λ by x, q just in terms of (3.1)–(3.3), the extremum theorems quoted above become [1]:

$$\begin{aligned} U(\overset{*}{q}) - A_{\text{ex}}(\hat{F}, \overset{*}{x}) &\geqslant U(\hat{q}) - A_{\text{ex}}(\hat{F}, \hat{x})\,, \\ V(\overset{0}{Q}) - A_{\text{ex}}(\overset{0}{F}, \hat{x}) &= V(\hat{Q}) - A_{\text{ex}}(\hat{F}, \hat{x})\,. \end{aligned} \tag{3.4}$$

Here, the so-called "elastic potential"

$$U(q) = \int_{\tilde{q}}^{q} A_{\text{in}}(Hq, \delta q) \tag{3.5}$$

is identical with the internal work W_{in}, which therefore has to be assumed independent of the integration path, connecting any arbitrary fixed state $\tilde{q}$ with the current one q. The "complementary potential" $V(Q)$ follows from $U(q)$ for any two states related by the elastic law (3.3), according to

$$U(q) + V(Q) = A_{\text{in}}(Q, q)\,. \tag{3.6}$$

In physically linear elasticity, H forms a linear operator, and

$$U(q) = V(Q) = \frac{1}{2} A_{\text{in}}(Q, q) = \frac{1}{2} A_{\text{in}}(Hq, q) = \frac{1}{2} A_{\text{in}}(Q, H^{-1} Q) \tag{3.7}$$

(cf. [1]) become second order functions of their arguments. Note that an immediate extension of (3.4) to geometrically large deformations cannot hold, as a minimum of the left-hand sides (so-called "potential energy" or "complementary energy", respectively) will then express some statical condition of stability which, quite naturally, need not be valid for unstable structures.

3.2. *Elastic-plastic bodies*

The majority of the elastic-plastic extremum theorems given below, which represent generalizations of (3.4), would therefore expectedly be restricted in validity to small linearized strains as well, even if nowadays they are occasionally set up making a difference between the initial and the instantaneous configuration of the considered body [61, 67, 68, 98].

Consider now any "true" instantaneous state of stress and strain, load and displacement $\hat{Q}, \hat{q}, \hat{T}, \hat{x}$ which is assumed to be prescribed (say, as a result of previous calculations). Only the increments (or rates)

$$Q^{\cdot}, q^{\cdot} = \lambda, \quad F^{\cdot}, x^{\cdot} = v$$

are to be varied. In a continuum, tensorial time differentiation means in general a convective one performed in a moving frame, though sometimes the convective terms become negligeable. More questionable (see below) is the assumption that equilibrium (1.3) is to hold in an unchanged form, also with respect to the rates

$$F^{\bullet}=LQ^{\bullet}. \tag{3.8}$$

The total strain-rate $\lambda=\lambda_e+\lambda_p$ will be formed by superimposing elastic and plastic parts λ_e, λ_p respectively, so that for a strain-hardening, rate-independent material, each increment of stress $dQ=Q^{\bullet}\,dt$ generates an associated increment of strain $dq=\lambda dt$ or,

$$Q^{\bullet}=\bar{H}\lambda\,, \qquad \lambda=\bar{H}^{-1}Q^{\bullet}, \tag{3.9}$$

where the constitutive operator $\bar{H}$ may for every point of the continuum assume two different values depending on whether there is (elastic-plastic) "loading", or (purely elastic) "unloading". Otherwise, $\bar{H}$ and $\bar{H}^{-1}$ are linear and follow directly from differentiating, and superimposing any elasticity law to a "standard" rigid plastic one (cf. Sec. 2.1.). Using (3.9), the concepts of "true", "kinematically admissible", and "statically admissible" incremental states $Q^{\bullet}$, λ, $F^{\bullet}$, v may immediately be transferred to the present situation by replacing in the Eqs. (1.15)–(1.17) the symbols F, Q, H by $F^{\bullet}$, $Q^{\bullet}$, $\bar{H}$, respectively.

Consider now, as an example, again any truss (as in Fig. 1) and form

$$\frac{1}{2}(\overset{*}{Q}{}^{\bullet}_j\overset{*}{\lambda}{}^j-\hat{Q}^{\bullet}_j\hat{\lambda}^j)-(\overset{*}{\lambda}{}^j-\hat{\lambda}^j)\hat{Q}^{\bullet}_j=\frac{1}{2}(\overset{*}{Q}{}^{\bullet}_j\overset{*}{\lambda}{}^j+\hat{Q}^{\bullet}_j\hat{\lambda}^j)-\hat{Q}^{\bullet}_j\overset{*}{\lambda}{}^j\,. \tag{3.10}$$

The first two right-hand terms are, in the event of strain-hardening(4), obviously non negative, independent of the state of loading or unloading, as correlated terms $\dot{Q}_j$, λ bear the same sign. If the true and the admissible states refer to different loading/unloading situations, then $\hat{Q}^{\bullet}_j\overset{*}{\lambda}{}^j\leqslant 0$ (for the considered term j, no summation), and $-\hat{Q}^{\bullet}_j\overset{*}{\lambda}{}^j\geqslant 0$. If, however, both states describe equally loading or unloading, then the constitutive operator $\bar{H}>0$ is the same, so that

$$\frac{1}{2}(\overset{*}{Q}{}_j\overset{*}{\lambda}{}^j+\hat{Q}^{\bullet}_j\hat{\lambda}^j)-\hat{Q}^{\bullet}_j\overset{*}{\lambda}{}^j=\frac{\bar{H}}{2}(\lambda^j-\hat{\lambda}^{*j})^2\geqslant 0\,,$$

where the equality signs hold if, and only if $\hat{\lambda}=\overset{*}{\lambda}$. Thus we have always (because of (1.8), (1.10) using $Q^{\bullet}$, $F^{\bullet}$ rather than Q, F, and the Eqs. (1.5) besides the identity (3.10)) that

$$\frac{1}{2}\Lambda_{\rm in}(\overset{*}{Q}{}^{\bullet},\overset{*}{\lambda})-\Lambda_{\rm ex}(\hat{F}^{\bullet},\overset{*}{v})\geqslant\frac{1}{2}\Lambda_{\rm in}(\hat{Q}^{\bullet},\hat{\lambda})-\Lambda_{\rm ex}(\hat{F}^{\bullet},\hat{v})\,, \tag{3.11}$$

and accordingly

$$\frac{1}{2}\Lambda_{\rm in}(\overset{0}{Q}{}^{\bullet},\overset{0}{\lambda})-\Lambda_{\rm ex}(\overset{0}{F}{}^{\bullet},\hat{v})\geqslant\frac{1}{2}\Lambda_{\rm in}(\hat{Q}^{\bullet},\hat{\lambda})-\Lambda_{\rm ex}(\hat{F}^{\bullet},\hat{v})\,. \tag{3.12}$$

(4) For rods in the stable region (no necking, i.e. not very large strains) are correct. Three--dimensional plastic media are always assumed to be stable in the sense of DRUCKER (cf. [6]).

These are the well-known extremum theorems of PRAGER, HODGE, GREENBERG, BAUER (1946–1948, cf. [6]), valid for elastic-plastic strain-hardening, or, in the limit, for elastic-ideally plastic material in which the minimum of the left-hand sides is reached if, and only if, the admissible states to be varied assume the true ones. Again, boundary conditions have not been involved but may become important for applications. The theorems (3.11), (3.12) have repeatedly been proved (cf. [6]), re-discovered [51, 61, 103, 106], and generalized to nonhomogeneous, non-isotropic [58, 60], visco-thermoplastic [111], or even, in a restricted sense, to strain-softening

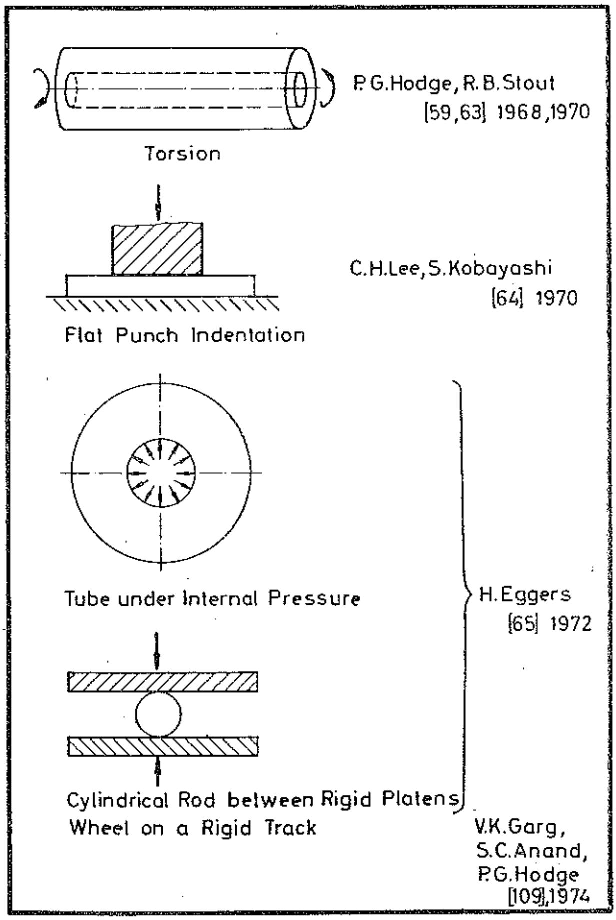


Fig. 9. Elastic-plastic problems (Review).

materials [96, 98]. However, there is one major difficulty. In setting up (3.11), (3.12) the virtual work theorem (1.8), (1.10) was used after replacing Q, F by $Q^{\cdot}$, $F^{\cdot}$. This can be correct as long as the modified equilibrium condition (3.8) holds (besides compatibility $\lambda = Cv$ which is unaffected). Therefore, authors examined the first equilibrium condition (1.4) and demonstrated that in the case of small strains, no large mistake could occur (cf. [6]). However, by a complete counter-example [66] it was shown that the mistake caused by the second condition (1.4) with respect to the first extremum theorem (3.11) may become arbitrarily large, as a differentiation of (1.4) yields, using Cartesian coordinates fixed in space,

$$\hat{T}_j = \hat{\sigma}^{\bullet}_{kj}\hat{n}^k + \hat{\sigma}_{kj}\hat{n}^{\bullet k}, \tag{3.13}$$

so that an additional term appears which contains the unknown true rate-of-deformation $\hat{n}^{\bullet}$ beside the admissible kinematical terms $\overset{*}{\lambda}$, $\overset{*}{v}$. Note that this difficulty has nothing to do with small deformations and arises even if the deformation becomes zero (initial state). It may formally be avoided if a fixed reference configuration [61, 67, 68, 98, 110] were considered, for which $\hat{n}$ remains constant. Then, however, $\hat{T}$ represents no longer the true stresses but those won by an appropriate spatial transformation which would involve $\hat{n}$ itself, so that the inconvenient term appears again.

There are not yet very many practical, or numerical applications of the basic extremum theorems (3.11), (3.12), but those shown in Fig. 9 fortunately refer to (civil engineering, small strain) situations in which the surface does not rotate much under the action of external (including reactive) forces, so that the additional term in the Eq. (3.13) either disappears or remains small.

The second extremum theorem (3.12) remains in any case rigorously valid [66]. An approach using functional analysis to construct admissible states is given in [62].

3.3. Generalizations

Attempts have been made to generalize the elastic principles (3.4), and especially the theorem on the minimum of complementary energy $(3.4)_2$, to elastic-plastic bodies without passing to the incremental state. Hodge [70] therefore imposed further restrictions on the statically admissible fields, so that the true solution could no longer be reached, but obtained the following result: The complementary energy calculated as if the body were purely elastic forms an upper bound to the suitably defined elastic-plastic complementary energy (cf. also Save [107]). This definition alone is tricky because energy integrals such as (3.5), and accordingly for the complementary energy V, become path-dependent functionals rather than functions of the state only. So Martin [71] as well as (without a mutual reference) Soechting and Lance [72] introduced "maximum" paths along which V reaches a maximum value, and thus proved bounding theorems which again need not converge on the true solution. Different constitutive laws have been examined by Trifan [104, 105].

Another problem if displacements and strains x, q, are to be considered rather than rates v, λ arises from the fact that compatibility (1.11), though transferrable to x, q in a generalized form, may even then become wrong if the elastic, and plastic parts q_e, q_p of strain are separately examined. This means that q_e splits off into compatible elastic strains q_e and incompatible ones q_r due to residual stresses or dislocations, so that q_r and q_p form together a compatible state. The first analysis(5), given by CERADINI [92], was continued by DE DONATO and MAIER [68, 69, 97, 98].

4. RATE-DEPENDENT AND DYNAMIC PLASTICITY

A "standard" rigid-plastic material (Sec. 2.1), the local yield surface of which is "rate-dependent" (i.e. this, or the distribution of uniaxial yield stress Y, say, depends on λ), obeys without any change the basic upper and lower bound theorems (2.1) in which, however, the material parameters (yield surfaces, yield stress etc.) must not be varied with the admissible states. Instead, iterative procedures might be tried ([25], see Sec. 2.2).

A different approach considers rate-dependent plastic bodies as viscous fluids (MARTIN 1966), a sufficiently large class of which fulfill dual extremum theorems which are immediately derived from those for elastic materials (3.4) if q, x were replaced, also in (3.5), by the rates λ, v, respectively. The resulting theorems (cf. [1] also for further references) become, in contrast to those (3.4), even rigorously correct because the virtual work theorem (1.8) could be applied without modification. A "limit"-material showing a yield limit but behaving otherwise than a Newtonian fluid, is the so-called "BINGHAM"-solid the extremum principles for which were first given by PRAGER [74], and extended to elastic-viscoplastic continua by PERZYNA (see [101] also for further references). HADDOW and LUMIG [75, 76] give examples (torsion), while LECKIE and PONTER [77, 78] superimpose elasticity as well as rate-independent plasticity, and apply their principles to beams, plates, and cylindrical shells. MRÓZ and RANIECKI [111] deal with thermo-viscoplasticity, while GAJEWSKI [93] develops an approach based on functional analysis. DE BOER [114] generalizes PERZYNA's constitutive law, and PONTER [117] considers creep again.

Let us now turn to dynamic plasticity in its proper meaning, namely plastokinetics. Then, the kinetic work balance yields

$$T+W_{\text{in}}=T_0+W_{\text{ex}}$$

where W_{in}, W_{ex} are the current values of internal and external work, while T, T_0 are the instantaneous and initial kinetic energies, respectively. Thus, for an initial pulse load $T_0>0$, $W_{\text{ex}}\equiv 0$ one obtains at the very end of the deformation, $T=0$, that $W_{\text{in}}=T_0$, from which equation simple bounds of maximum displacements may be derived [79]. The method was extended by MARTIN [71] using the maximum-path concept referred to in Sec. 3.3, as well as by DE DONATO and MAIER [95] to structures taking geometric changes into account. See also Ponter [116].

From now on we shall examine extremum theorems in which the state of acceleration $\dot{v}$ will be varied and /or bounded, and denote by $-\rho\dot{v}$ the state of related body forces (ρ: mass, or mass density) to be added to the state of "statical" external

(5) Obviously containing computational errors.

loads F^s, so that symbolically $F = F^s - \rho v^{\bullet}$ becomes the complete external load. Then TAMUZH [80] established the minimum principle

$$(4.1) \qquad J(\overset{*}{v}^{\bullet}, \overset{*}{\lambda}^{\bullet}) = A_{ex}\left(\frac{1}{2}\rho v^{\bullet} - \hat{F}, \overset{*}{v}^{\bullet}\right) + A_{in}(\hat{Q}^*, \overset{*}{\lambda}^{\bullet}) \Rightarrow \text{Min},$$

where the modified "kinematically admissible" state $\overset{*}{v}^{\bullet}$, $\overset{*}{\lambda}^{\bullet}$, $\hat{Q}^*$ has to be varied, while the instantaneous state of rate $\hat{\lambda}$, and therefore of force $\hat{Q} = H\hat{\lambda}$, $\hat{F} = L\hat{Q}$ is assumed to be known. H denotes the rigid plastic constitutive operator, cf. (1.12). According to a more precise interpretation given by MARTIN [81], the admissible state of strain-acceleration $\overset{*}{\lambda}^{\bullet}$ should not be won by a differentiation of $\overset{*}{\lambda}$ but formally by means of the compatibility conditions (1.11), so that $\overset{*}{\lambda}^{\bullet} = C\overset{*}{v}^{\bullet}$, and belong to the domain of H (apart from its physical dimension) at least in all points where the known strain-rate $\hat{\lambda}$ vanishes (rigid region of the body). There, $\hat{Q}^* = H\overset{*}{\lambda}^{\bullet}$ by definition, while $\hat{Q}^* = \hat{Q}$ (true stresses) in the deforming region $\hat{\lambda} \neq 0$.

It is seen that for an everywhere deforming body there is no internal constraint imposed by the constitutive law at all, and the Tamuzh principle is as valid as, say, the work-and-energy theorem. The constitutive operator H was introduced, very strangely, only with respect to rigid zones (where it should properly be of minor importance), so that the meaning of the theorem is somewhat obscure. Of course it holds, like the constitutive law (1.12) itself, for viscous fluids as well [94].

Much more straightforward, and related to rigid plastic bodies are two dual extremum theorems which follow immediately from the basic upper and lower-bound theorems (2.1) by introducing an "admissible state of acceleration" $\overset{0}{v}^{\bullet}$, so that $\overset{0}{F} = \hat{F}^s - \rho\overset{0}{v}^{\bullet}$ is statically admissible in the sense of (1.17), and $\hat{F}^s$ denotes the statical part of the true external load which is assumed to be predetermined. Then, $\hat{F} = \hat{F}^s - \rho\hat{v}^{\bullet}$ yields the true external load, and we obtain

$$(4.2) \qquad \begin{aligned} A_{ex}(\rho\overset{0}{v}^{\bullet}, \hat{v}) &\geq A_{ex}(\rho\hat{v}^{\bullet}, \hat{v}), \\ A_{ex}(\rho\hat{v}^{\bullet}, \overset{*}{v}) &\geq A_{ex}(\hat{F}^s, \overset{*}{v}) - A_{in}(\overset{*}{Q}, \overset{*}{\lambda}). \end{aligned}$$

The Eq. $(4.2)_1$ was without any reference to (2.1) proved by MARTIN [81], and in a more specialized shape by CAPURSO [94], while MARTIN gave a much narrower form of $(4.2)_2$ as well [81].

As an example, let us consider again our standard truss in Fig. 1 the rods of which are assumed to be equal with respect to cross sectional area as well as yield limit,

$$A_1 = A_2 = A_3 = A, \qquad Y_1 = Y_2 = Y_3 = Y,$$

and where the masses m are concentrated in the nodal points. Only the (true) force $\hat{F}^s_2 \neq 0$ is to act, while $F^s_1 = F^s_3 = 0$ and the initial state $\hat{v}^1 = \hat{v}^2 = \hat{v}^3 = 0$ refers to the structure at rest. Using then the kinematically admissible state given in the Eq. (2.2), we obtain from $(4.2)_2$ the estimate

$$m\left(-\frac{1}{2}\hat{v}^{\cdot 1}+\frac{1}{2\sqrt{3}}\hat{v}^{\cdot 2}-\hat{v}^{\cdot 3}\right)>\frac{1}{2\sqrt{3}}\hat{F}^s_2-YA \tag{4.3}$$

(in which intuitively $\hat{v}^{\cdot 1}<0$, $\hat{v}^{\cdot 2}>0$, $\hat{v}^{\cdot 3}<0$ if $\hat{F}^s_2>0$). The inequality $(4.2)_1$ is, because of $\hat{v}=0$, identically satisfied. If we, however, substitute the subsequent state of velocity $\hat{v}=\hat{v}^{\cdot}dt$, $dt>0$ (time increment), then we obtain, using the statically admissible state (2.2) in order to construct the admissible state of acceleration[6],

$$\overset{0}{v}{}^{\cdot 1}=\overset{0}{v}{}^{\cdot 3}=0\,,\qquad \overset{0}{v}{}^{\cdot 2}=\frac{1}{m}(\hat{F}^s_2-\sqrt{3}\,YA)$$

and

$$(\hat{F}^s_2-\sqrt{3}\,YA)\hat{v}^{\cdot 2}\geqslant m\,[(\hat{v}^{\cdot 1})^2+(\hat{v}^{\cdot 2})^2+(\hat{v}^{\cdot 3})^2]\,. \tag{4.4}$$

It is seen that in the case of $\hat{F}^s_2\to\sqrt{3}\,Ya$, only statical deformation ($m=0$) may occur, so that $\hat{F}^s_2>\sqrt{3}\,YA$ should be expected. Therefore (4.3), (4.4) generalize, as could be expected before, the former inequality (2.3).

Another approach is examined by SYMONDS [118].

5. FURTHER GENERALIZATIONS AND APPLICATIONS

Non-standard plastic behaviour cannot be illustrated by truss-like structures but may play a role, e.g. in soil mechanics. Denote by Q, as before, the global state of generalized stress in discrete systems but the local one in the points of a continuum, so that Q may always be considered to have a finite number of n components Q_j only, which define a vector ("stress vector") in the n-dimensional Euclidean "space of state". Accordingly, the state of rate λ may be represented in the same space of state, by the "rate vector" having n components λ^j. The inner product

$$Q\lambda=Q_j\,\lambda^j \tag{5.1}$$

gives us, according to the Eq. (1.5), the total internal rate of work $A_{\text{in}}(Q,\lambda)$ in discrete systems, while in a continuum a body integration needs to be carried out before, so that $Q\lambda$ is then the rate-of-work "density" only.

As in standard plasticity we assume the existence of a convex "yield surface" in the space of state, containing the origin 0 in its interior,

$$f(Q)=0\,,\quad f(0)\leqslant 0\,, \tag{5.2}$$

so that the statically admissible states $\overset{0}{Q}$ shall always be defined by

$$f(\overset{0}{Q})\leqslant 0\,. \tag{5.3}$$

---

[6] Obviously useful only if $\hat{F}^s_2>0$

In the event of the tip of the vector Q fixed at the origin lying on the yield surface so that $f(Q)=0$, then plastic yielding might occur in standard plasticity, so that λ forms an outward normal to the yield surface in the tip of Q, i.e.

$$\lambda^j=\bar{\lambda}\frac{\partial f}{\partial Q_j}, \quad \bar{\lambda}\geqslant 0, \tag{5.4}$$

where $\bar{\lambda}$ denotes an additional variable, and f is assumed sufficiently smooth or regular(7). Now, an often discussed class of non-standard rigid-plastic constitutive operators H uses, instead, a different function g, so that (smoothness and regularity provided)

$$\lambda^j=\bar{\lambda}\frac{\partial g}{\partial Q_j}, \quad \bar{\lambda}\geqslant 0 \tag{5.5}$$

is to hold. Here, $g(Q)$ shall, like f, define a convex "flow surface" in the space of state containing the origin 0 in its interior, i.e.

$$g(Q)=0, \quad g(0)\leqslant 0. \tag{5.6}$$

A kinematically admissible state is characterized by a compatible rate-field $\overset{*}{\lambda}$, so that (5.5) is invertible, to give any $\overset{*}{Q}$ obeying $f(\overset{*}{Q})=0$.

Because of

$$\frac{\partial g(\alpha Q)}{\partial Q_j}=\alpha\left(\frac{\partial g(\bar{Q})}{\partial \bar{Q}}\right)_{\bar{Q}=\alpha Q}$$

it follows from (5.5) using $\alpha>0$ that an elongated (or shortened) vector $\bar{Q}$ may be considered instead of Q, for which $g(\bar{Q})=0$ is to hold if in the positive direction of Q there is no point of infinity of the flow surface. *Vice versa*, each compatible field $\overset{*}{\lambda}$ for which at least one solution $\overset{*}{\bar{Q}}$ of (5.5) exists is admissible if in the positive direction of $\overset{*}{\bar{Q}}$ there is no point of infinity of the yield surface.

Regarding this, the non-standard law of plasticity described above may be illustrated by Fig. 10 where (outward) normality holds with respect to the flow surface, in the positive elongation $\bar{Q}$ of the vector Q which ends on the yield surface. Standard plasticity arises as the special case $g=af$, $a=\text{const}>0$.

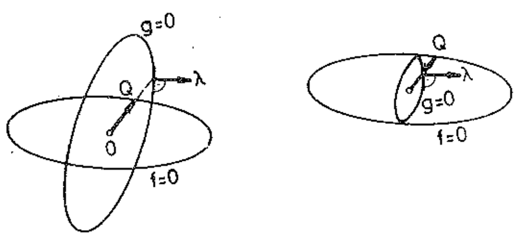


Fig. 10. Non-standard rigid-plastic constitutive law.

Rather complicated extremum theorems for such non-standard elastic-plastic materials have been derived by MAIER [69, 98] though his condition of admissibility $f\leqslant 0$ seems artificial and might exclude the true solution. Taking this consciously into account one can much more easily set up rigid-plastic bounding theorems first

(7) For generalizations, see KOITER [6]

stated by RADENKOVIĆ 1961 [82], which are again an immediate consequence of the basic rigid-plastic upper and lower-bound theorems (2.1).

First we assume that the flow surface $g=0$ either possesses no points of infinity, or possesses those only in common with the yield surface, so that both surfaces form cylinders in the related directions of infinity. Then by a simple transformation

$$g(Q) \to g(AQ)+C; \quad A=\text{const}>0\,, \quad C=\text{const}, \tag{5.7}$$

it will always be found that the new right-hand expression becomes non-negative if f vanishes, so that it generates a new flow surface to be denoted again by $g(Q)=0$, which is completely enclosed by the yield surface $f=0$ as shown in the right half of Fig. 10. For practical reasons it is desirable that this internal surface should not become too tiny, so that for the sake of a better adaption, two constants A and C have been introduced in the Eq. (5.7) rather than only one C as would have been sufficient in principle.

Relating now the upper and lower bound theorem (2.1) to the inner flow surface only ($\hat{\bar{F}}$ instead of $\hat{F}$, $\overset{0}{\bar{F}}$ instead of $\overset{0}{F}$, $\overset{*}{\bar{Q}}$ instead of $\overset{*}{Q}$), and observing $\hat{F}=\alpha\hat{\bar{F}}$ where $\alpha\gtrless 1$, it is seen that the lower bound theorem $(2.1)_1$ may immediately be extended to give

$$A_{\text{ex}}(\overset{0}{\bar{F}}, \hat{v}) \leqslant A_{\text{ex}}(\hat{\bar{F}}, \hat{v}) \leqslant A_{\text{ex}}(\hat{F}, \hat{v})\,, \tag{5.8}$$

where only the outer terms are the interesting ones, while the statical admissibility of $\overset{0}{\bar{F}}=L\overset{0}{\bar{Q}}$ means $g(\overset{0}{\bar{Q}})\leqslant 0$.

Assume now inversely that there are no points of infinity of the yield surface, or only common ones with the flow surface, so that both surfaces form cylinders in the related directions of infinity. Then by a transformation (5.7) it can always be enforced that the new flow surface being again denoted by $g(Q)=0$, lies completely outside the yield surface f (no Figure). Then we get from the upper bound theorem applied to the flow surface, using $\hat{\bar{F}}=\alpha\hat{F}$ ($\alpha\geqslant 1$) that, in the case of a positive left-hand side,

$$A_{\text{ex}}(\hat{F}, \overset{*}{v}) \leqslant A_{\text{ex}}(\hat{\bar{F}}, \overset{*}{v}) \leqslant A_{\text{in}}(\overset{*}{\bar{Q}}, \overset{*}{\lambda}) \tag{5.9}$$

holds, where again only the outer terms are relevant, and fulfil the same inequality in a trivial manner if the left-hand term should be negative, or vanish. Kinematical admissibility in the generalization of (1.17) means here that $\overset{*}{\lambda}=C\overset{*}{v}$ forms on outward normal to the modified flow surface in the point $\overset{*}{\bar{Q}}$, so that (5.5) would have to be inverted mutually with $g(\overset{*}{\bar{Q}})=0$.

All the variational or extremum theorems considered in this paper may be modified according to the method of Euler-Lagrangian multipliers by introducing side-conditions and redundant variables, or by adding, or subtracting them, but eventually then lose their extremum property [1]. Those procedures have been repeatedly proposed in elasticity (Reissner-Hellinger principles, etc., cf. [1], resulting from (3.4)), and were applied to rigid-plastic materials by SCHROEDER and SHERBOURNE

[83], to elastic-plastic ones by KLEIBER [112], while LUNG [25], MAHRENHOLTZ and KLIE [113] used them in numerical applications.

Also for numerical purposes it becomes unnecessary to ask for any physical meaning of the extremum theorems, so that quite artificial ones, combined with the method of weighted residuals [85], may be installed which may additionally involve arbitrary side and boundary conditions. After a first step done by HILL [90], applied in [99], KOLAROV [86] proposed a method which was extended and widely applied to metal forming processes by ADLER, DALHEIMER, and STECK [87, 88, 89, 91]. BHANDARI and ODEN [84] tried to elaborate a very general scheme of how such extremum theorems should be set up. It turned out (according to this author's feeling) that the proper idea would be that h^2 becomes a minimum if $h=0$ represents any constitutive equation. To deal with such purely mathematical or numerical procedures in greater detail would break the frame of the present review.

REFERENCES

1. H. LIPPMANN, *Extremum and Variational Principles in Mechanics*, Udine: CISM Courses and Lectures No. 54 (1970), Wien-New York, Springer, 1972.
2. D. RADENKOVIC and Q. S. NGUYEN, *Duality of Limit Theorems for Structures with Standard Rigid-Plastic Behaviour*, Lab. Méc. Solides, Paris, École Polytechnique, 1972.
3. M. SAYIR und H. ZIEGLER, *Der Verträglichkeitssatz der Plastizitätstheorie und seine Anwendung auf räumlich unstetige Felder*, Z. Angew. Math. Phys., **20** (1969), 78 – 93.
4. H. LIPPMANN, K. ANDRESEN, A. BEHRENS und D. BESDO, *Grundlagen der Umformtechnik*, Stahl und Eisen, **87** (1967), 389 – 393, 464 – 466 und 672 – 678.
5. I. F. COLLINS, *The Upper Bound Theorem for Rigid/Plastic Solids Generalized to Include Coulomb Friction*, J. Mech. Phys. Solids, **17** (1969), 323 – 338.
6. W. T. KOITER, *General Theorems for Elastic-Plastic Solids*, Progress in Solid Mech. (I. N. Sneddon, R. Hill eds.), **1** (1960), 167 – 221.
7. H. LIPPMANN, *Plastomechanik der Umformung metallischer Werkstoffe*, Vol. 1 — Berlin-Heidelberg-New York, Springer, 1967. Vol. 2 — under preparation.
8. H. SAMSON, *Warmscheren von Stahl*, Dissertation, Techn. Universität Clausthal, 1973.
9. H. ERNST and M. E. MERCHANT, *Chips Formation, Friction and High Quality Machined Surfaces* — in: Surface Treatment of Metals, Trans. Am. Soc. Metals, **29** (1941), 299–328.
10. E. H. LEE and B. W. SHAFFER, *The Theory of Plasticity Applied to a Problem of Machining*, Trans. ASME J. Appl. Mech., **18** (1951), 405 – 413.
11. G. W. ROWE and P. T. SPICK, *A New Approach to Determination of the Shear Plane Angle in Machining*, Trans. ASME J. Engng. Ind., **89** (1967), 530–538.
12. W. H. FEILBACH and B. AVITZUR, *Analysis of Rod Shaving and Orthogonal Cutting*, Trans. ASME J. Engng. Ind., **90** (1968), 393–403.
13. B. ZÜNKLER, *Herleitung der spezifischen Schnittkraft k_s nach dem Prinzip des geringsten Zwanges, ausgehend von der Fließkurve des Werkstoffes*, TZ f. prakt. Metallbearbeitung, **65** (1971), 76-79.
14. W. JOHNSON and H. KUDO, *The Mechanics of Metal Extrusion*, Manchester, University Press, 1962.
15. JU. A. ALJUŠIN, *Analysis of Deformation Processes of Metals by Tools with Curvilinear Contours*, Mašinovedeije no. 5 (1965), 83–88 [in Russian].
16. H. LIPPMANN, *Abschätzen oberer und unterer Schranken für Umformleistungen und Kräfte, besonders beim Strangpressen. Grundlagen der bildsamen Formgebung*, Düsseldorf, Stahleisen, 1966, 83–98.

17. B. AVITZUR, *Strain-Hardening and Strain-Rate Effects in Plastic Flow Through Conical Converging Dies*, Trans. ASME J. Engng. Ind., **89** (1967), 556–562.
18. H. C. SORTAIS and S. KOBAYASHI, *An Optimum Die Profile for Axisymmetric Extrusion*, Int. J. Machine Tool Design and Research, **8** (1968), 61–72.
19. C. T. CHEN and F. F. LING, *Upper Bound Solutions to Axisymmetric Extrusion Problems*, Int. J. Mech. Sci., **10** (1968), 863–880.
20. E. R. LAMBERT and S. KOBAYASHI, *A Theory on the Mechanics of Axisymmetric Extrusion Through Conical Dies*, J. Mech. Engng. Sci., **10** (1968), 367–380.
21. A. H. SHABAIK and E. G. THOMSEN, *Comparison of Two Complete Solutions in Axisymmetric Extrusion with Experiment*, Trans. ASME J. Engng. Ind., **91** (1969), 543–548.
22. Z. ZIMERMAN and B. AVITZUR, *Metal Flow through Conical Converging Dies — A Lower Upper Bound Approach Using Generalized Boundariesof the Plastic Zone*, Trans. ASME J. Engng. Ind., **92** (1970), 119–129.
23. H. S. MEHTA, A. H. SHABAIK and S. KOBAYASHI, *Analysis of Tube Extrusion*, Trans. ASME J. Engng. Ind., **92** (1970), 403–411.
24. T. MUROTA, T. JIMMA and K. KATO, *Analysis of Axisymmetric Extrusion*, Bull. JSME 13, **65**, (1970), 1366–1374.
25. M. LUNG, *Ein Verfahren zur Berechnung des Geschwindigkeits- und Spannungsfeldes bei stationären starr-plastischen Formänderungen mit finiten Elementen*, Dissertation Hannover, Techn. Universität, 1971.
26. N. AHMED, *Conical Flows in Metal Forming*, Trans. ASME J. Basic Engng., **94** (1972), 213–222.
27. A. TROOST, *Zur elementaren Theorie des axialsymmetrischen Vorwärtsstrangpressens; Ermittlung der Umformgeometrie und des Kraftbedarfs durch Variationsrechnung*, Arch. Eisenhüttenwes, **49** (1973), 315–320.
28. B. AVITZUR, E. D. BISHOP and W. C. HAHN jr., *Impact Extrusion — Upper Bound Analysis of the Early Stage*, Trans. ASME J. Engng. Ind., **94** (1972), 1079–1086.
29. S. KOBAYASHI and E. G. THOMSEN, *Approximate Solutions to a Problem of Press Forging*, Trans. ASME J. Engng. Ind., **81** (1958), 217–227.
30. S. KOBAYASHI, R. HERZOG, J. T. LAPSLEY jr. and E. G. THOMSEN, *Theory and Experiment of Press Forging Asymmetric Parts of Aluminium and Lead*, Trans. ASME J. Engng. Ind., **81** (1959), 228–236.
31. S. KOBAYASHI and E. G. THOMSEN, *Upper- and Lower-Bound Solutions to Axisymmetric Compression and Extrusion Problems*, Int. J. Mech. Sci., **7** (1965), 127–143.
32. H. KUDO, *An Upper-Bound Approach to Plane Strain Forging and Extrusion*, Int. J. Mech. Sci., **1** (1960), (I) 57–83, (II) 229–252, (III) 366–368.
33. J. B. HADDOW and W. JOHNSON, *Bounds for the Load to Compress Plastically Square Discs Between Rough Dies*, Appl. Sci. Res., Sect. A, **10** (1961), 471–477.
34. B. AVITZUR, *Forging of Hollow Disks*, Israel J. Technol., **2** (1964), 295–304.
35. B. ZÜNKLER, *Ermittlung der beim Gesenkschmieden stabförmiger Teile auftretenden Spannungen und Kräfte*, Ind.-Anzeiger: Werkzeugmaschinen und Fert.-Techn., **87** (1965), 569–576.
36. K. KWASZCZYŃSKA and Z. MRÓZ, *A Theoretical Analysis of Plastic Compression of Short Circular Cylinders*, Arch. Mech. Stos., **19** (1967), 787–797.
37. S. K. SAMANTA, *The Application of the Upper Bound Theorem to the Prediction of Indenting and Compressing Loads for Circular and Rectangular Disks*, Acta Polytechn. Scandin., Mech. Engng. Ser., **38** (1968), 5–36.
38. J. Y. LIU, *Upper Bound Solutions of Some Axisymmetric Cold Forging Problems*, Trans. ASME J. Engng. Ind., **93** (1971), 1134–1144.
39. K. ANDRESEN, *Blockstauchen zwischen ebenen parallelen Bahnen*, Arch. Eisenhüttenwes., **44** (1973), 595–598.
40. K. ANDRESEN, *Die numerische Behandlung eines Schrankenverfahrens mit Anwendung auf axialsymmetrisches Stauchen*, Acta Mech., **17** (1973), 291–303.

41. I. Ja. TARNOVSKIJ, A. A. POZDEEV, V. L. KOLMOGOROV, *A Variational Principle in the Theory of Metal Processing under Pressure. Theory of Rolling.* Moscow, Sci. Techn. Gov. Publ. Lit. Iron and Non-Iron Metals, 1962, 57–67 [in Russian].
42. A. TROOST, *Zur elementaren Plastizitätstheorie räumlicher Umformvorgänge,* Arch. Eisenhüttenwesen, **35** (1964), 847–853.
43. J. W. GREEN, L. G. M. SPARLING and J. F. WALLACE, *Shear Plane Theories of Hot and Cold Flat Rolling,* J. Mech. Engng. Sci., **6** (1964), 219–235.
44. V. MANNL und H. LIPPMANN, *Elementare Theorie und Schrankenverfahren der Plastomechanik,* Rev. Roum. Sci. Techn. — Méc. Appl., **15** (1970), 539–553+**16** (1971), 739.
45. H. LIPPMANN und R. KÜMMERLING, *On Spread in Rollin,* Mech. Res. Comm., **2** (1975), 113—118.
46. D. C. DRUCKER, W. PRAGER, *Soil Mechanics and Plastic Analysis of Limit Design,* Qu. Appl. Math., **10** (1952), 157–165.
47. R. T. SHIELD, *On Coulomb's Law of Failure in Solids,* J. Mech. Phys. Solids, **4** (1955), 10–16.
48. J. B. CHEATHAM, P. R. PASLAY, C. W. G. FULCHER, *Analysis of the Plastic Flow of Rock under a Lubricated Punch,* Trans. ASME J. Appl. Mech., **35** (1968), 87–94.
49. P. R. PASLAY, J. B. CHEATHAM, C. W. G. FULCHER, *Plastic Flow of Rock Under a Pointed Punch in Plane Strain,* Trans. ASME J. Appl. Mech., **35** (1968), 95–101.
50. H. LIPPMANN, *Plasticity in Rock Mechanics,* Int. J. Mech. Sci., **13** (1971), 291–297.
51. M. CAPURSO, *Minimum Principle for the Incremental Solution of Elastoplastic Problems,* Atti Accad. Naz. Lincei, Rend. Sci. Fis., Mat. Nat., **46** (1969), 417–425 and 552–560 [in Italian].
52. I. F. COLLINS, *A Note on the Interpretation of Coulomb's Analysis of the Thrust on a Rough Retaining Wall in Terms of the Limit Theorems of Plasticity Theory,* Géotechnique, 1973, 442–447.
53. J. F. ADIE and J. M. ALEXANDER, *A Graphical Method of Obtaining Hodographs for Upper--Bound Solutions to Axisymmetric Problems,* Int. J. Mech. Sci., **9** (1967), 349–357.
54. H. GAJEWSKI, *Ein Verfahren zur Konstruktion statisch zulässiger Spannungsfelder,* Z. Angew. Math. Mech., **47** (1967), 19–30.
55. A. NAGAMATSU, T. MUROTA and T. JIMMA, *On the Nonuniform Deformation of Material in Axially Symmetric Compression Caused by Friction.* Part. 1. *Comparison of Results Using Calculus of Variations with Experiments Concerning Compression of Cylinder.* Part 2. *Theoretical Analysis of Compression of Circular Cylinders,* Bull. JSME, 14, **70** (1971), 331–338 and 339–347.
56. R. H. LANCE, *On Automatic Construction of Velocity Fields in Plastic Limit Analysis,* Acta Mech., **3** (1967), 22–33.
57. H. GAJEWSKI, *Zur Lösung einer Klasse konvexer Minimumprobleme der Plastizitätstheorie,* Z. Angew. Math. Mech., **49** (1969), 83–90.
58. W. OLSZAK and W. URBANOWSKI, *Plastic Non-Homogeneity. A Survey on Theoretical and Experimental Research.* Non-Homogeneity in Elasticity and Plasticity. Sympos. Warsaw Sept. 2–9 (1958), 259–300 (1959).
59. P. G. HODGE jr., *Numerical Applications of Minimum Principles in Plasticity,* Engineering Plasticity (J. Heyman, F. Leckie eds.), Oxford Univ. Press, 1968, 237–256.
60. W. OLSZAK and P. PERZYNA, *Extremum Theorems in the Theory of Plasticity of Non-Homogeneous and Anisotropic Bodies,* Arch. Mech. Stos., **9** (1958), 659–712.
61. K. W. NEALE, *A General Variational Theorem for the Rate Problem in Elasto-Plasticity,* Int. J. Solids Structures, **8** (1972), 865–876.
62. A. LANGENBACH, *Variationsmethoden in der nichtlinearen Elastizitäts- und Plastizitätstheorie,* Wiss. Z. Humboldt-Univ. Berlin math. -naturwiss., R. **9** (1959/60), 145–164.
63. R. B. STOUT and Ph. G. HODGE, *Elastic/Plastic Torsion of Hollow Cylinders,* Int. J. Mech. Sci., **12** (1970), 91–108.

64. C. H. LEE and S. KOBAYASHI, *Elastoplastic Analysis of Plane-Strain and Axisymmetric Flat Punch Indentation by the Finite-Element Method*, Int. J. Mech. Sci., **12** (1970), 349–370.

65. H. EGGERS, *Die Berechnung von Kontinua aus elastisch-plastischem Material mit Verfestigung nach einer finiten Elementenmethode*, Dissertation Braunschweig, Techn. Univ., 1972.

66. H. LIPPMANN, *On the Incremental Extremum Theorems for Elastic-Plastic Media*, Mech. Res. Comm., **1** (1974), 33–36.

67. R. H. LANCE and J. F. SOECHTING, *A Displacement Bounding Principle in Finite Plasticity*, Int. J. Solids Structures, **6** (1970), 1103–1118.

68. O. De DONATO, *Some Extremum Theorems for the Elastoplastic Boundary Value Problem in the Presence of Large Displacements*, Meccanica, **5** (1970), 210–218.

69. G. MAIER, *A Minimum Principle for Incremental Elastoplasticity with Non-Associated Flow Laws*, J. Mech. Phys. Solids, **18** (1970), 319–330.

70. P. G. HODGE jr., *A Deformation Bounding Theorem for Flow Law Plasticity*, Quart. Appl. Math., **24** (1966), 171–174.

71. J. B. MARTIN, *Extended Displacement Bound Theorems for Work-Hardening Continua Subject to Impulsive Loading*, Int. J. Solids Structures, **2** (1966), 9–26.

72. J. F. SOECHTING and R. H. LANCE, *A Bounding Principle in the Theory of Work-Hardening Plasticity*, Trans. ASME J. Appl. Mech., **36** (1969), 228–232.

73. W. JOHNSON, M. C. de MALHERBE and R. VENTER, *Upper Bounds to the Load for the Plane Strain Working of Anisotropic Metals*, J. Mech. Engng. Sci., **14** (1972), 297–306.

74. W. PRAGER, *On Slow Visco-Plastic Flow*, Studies in Math. and Mech.; presented to R. v. Mises. New York, Academic Press, 1954, 208–216.

75. J. B. HADDOW and H. LUMING, *An Extension of One of the Extremum Principles for a Bingham Solid*, Appl. Sci. Res., (A) **15** (1965), 81–86.

76. J. B. HADDOW and H. LUMING, *The Application of the Extremum Principles for a Bingham Solid*, Appl. Sci. Res., **16** (1966), 469–477.

77. F. A. LECKIE and A. R. S. PONTER, *Deformation Bounds for Bodies which Creep in the Plastic Range*, Trans. ASME J. Appl. Mech., **37** (1970), 426–430.

78. A. R. S. PONTER and F. A. LECKIE, *The Application of Energy Theorems to Bodies which Creep in the Plastic Range*, Trans. ASME J. Appl. Mech., **37** (1970), 753–758.

79. J. B. MARTIN, *A Displacement Bound Principle for Inelastic Continua Subjected to Certain Classes of Dynamic Loading*, Trans. ASME J. App. Mech., **32** (1965), 1–6.

80. V. D. TAMUZH, *On a Minimum Principle in Dynamics of Rigid-Plastic Bodies*, J. Appl. Math. Mech., **26** (1963), 1067–1077.

81. J. B. MARTIN, *Extremum Principles for a Class of Dynamic Rigid-Plastic Problems*, Int. J. Solids Structures, **8** (1972), 1185–1204.

82. D. RADENKOVIC, *Théorie des charges limites: extension à la mécanique des sols*, Séminaire de plasticité (Ecole Polytechnique). Paris, Publ. Scientifiques et Techniques du Ministère de l'Air, no. N. T. 116 (1961), 129–142.

83. J. SCHROEDER and A. N. SHERBOURNE, *A General Variational Theorem in Classical Plasticity*, J. Math. Phys., **46** (1967), 175–187.

84. D. R. BHANDARI and J. T. ODEN, *A Variational Principle for Non-Linear Thermoviscoplasticity*, Int. J. Non-Linear Mech., **7** (1972), 489–501.

85. B. A. FINLAYSON and L. E. SCRIVEN, *The Method of Weighted Residuals—A Review*, Appl. Mech. Rev., **19** (1966), 735–748.

86. D. KOLAROV, *Methods of Solving the Equations of the Mechanics of Deformable Media*, Archiv. Mech. Stos., **16** (1964), 989–1007.

87. R. DALHEIMER, *Beiträge zur Frage der Spannungen, Formänderungen und Temperaturen beim axialsymmetrischen Strangpressen*, Ber. Inst. Umformtechnik, Univ. Stuttgart (TH) Nr 20. Essen. Girardet , 1970.

88. E. STECK, *Numerische Behandlung von Verfahren der Umformtechnik*, Ber. Inst. Umformtechnik, Univ. Stuttgart (TH) Nr . 22. Essen. Girardet, 1971.

89. G. ADLER und R. DALHEIMER, *Ein numerisches Verfahren zum Lösen von Problemen der Umformtechnik*, Werkstattstechnik, **62** (1972), 194–198.
90. R. HILL, *A General Method of Analysis for Metal-Working Processes*, J. Mech. Phys. Solids, **11** (1963), 305—326.
91. E. STECK, *Ein Verfahren zur näherungsweisen Berechnung des Spannungs- und Verformungszustandes bei Umformvorgängen*, Annals C.I.R.P., **17** (1969), 251–258.
92. G. CERADINI, *A Maximum Principle for the Analysis of Elastic-Plastic Systems*, Meccanica, **1** (1966), 77–82.
93. H. GAJEWSKI, *Iterations- , Projektions- und Projektions-Iterationsverfahren zur Berechnug viscoplastischer Strömungen*, Z. Angew. Math. Mech., **50** (1970), 485–490.
94. M. CAPURSO, *Minimum Principles in the Dynamics of Isotropic Rigid-Plastic and Rigid-Viscoplastic Continuous Media*, Meccanica, **7**, 92–97 (1972).
95. O. De DONATO, and G. MAIER, *Bounds on Total Dynamic Deflections of Elastoplastic Structures Allowing for Second-Order Geometric Effects*, Mach. Res. Comm., **1** (1974), 37–41.
96. G. MAIER, *Extremum Theorems for the Analysis of Elastic Plastic Structures Containing Unstable Elements*, Meccanica, **2** (1967), 235–242.
97. M. CAPURSO and G. MAIER, *Incremental Elastoplastic Analysis and Quadratic Optimization*, Meccanica, **5** (1970), 1–10.
98. G. MAIER, *Incremental Plastic Analysis in the Presence of Large Displacements and Physical Instabilizing Effects*, Int. J. Solids Struct., **7** (1971), 345–372.
99. G. D. LAHOTI and S. KOBAYASHI, *On Hill's General Method of Analysis for Metal-Working Processes*, Int. J. Mech. Sci., **16** (1974), 521–540.
100. C. S. HARTLEY, *Upper Bound Analysis of Extrusion of Axisymmetric, Piecewise Homogeneous Tubes*, Int. J. Mech. Sci., **15** (1973), 651–663.
101. P. PERZYNA, *Fundamental Problems in Viscoplasticity*, Adv. Appl. Mech., no. 9, New York, Academic Press, 1966.
102. Z. MRÓZ, *Limit Analysis of Plastic Structures Subject to Boundary Variations*, Arch. Mech. Stos., **15** (1963), 63–75.
103. G. GRANDORI, *Formulatione variazionale della condizione di plasticità per continui idealmente elastoplastici*, Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat., **90** (1956), 541–546.
104. D. TRIFAN, *A Minimum Principle of Plasticity*, Quart. Appl. Math., **13** (1955), 337–339.
105. D. TRIFAN, *On a Complementary Energy Principle*, J. Math. Mech., **6** (1957), 87–90.
106. L. STABILINI, *Contributo allo studio dei sistemi elasto-plastici*, Österr. Ing.- Archiv, **15** (1961), 214–221.
107. M. SAVE, *Une interpretation du théorème de Colonetti*, Z. angew. Math. Phys., **13** (1962), 479–499.
108. D. C. DRUCKER, *On Uniqueness in the Theory of Plasticity*, Quart. Appl. Math., **14** (1956), 35–42.
109. V. K. GARG, S. C. ANAND and P. G. HODGE jr., *Elastic-Plastic Analysis of a Wheel Rolling on a Rigid Track*, Int. J. Solids Structures, **10** (1974), 945–956.
110. M. DUSZEK, *On Minimum Principles in Finite Plasticity*, Bull. Acad. Polon. Sci., Série Sci. Techn., **21** (1973), 89–96.
111. Z. MRÓZ and B. RANIECKI, *Variational Principles in Uncoupled Thermoplasticity*, Int. J. Engng. Sci., **11** (1973), 1133–1141.
112. M. KLEIBER, *A Three-Fields Variational Theorem for the Rate Problem of Finite Strain Elasto--Plasticity with Discontinuous Fields*, Paper presented to EUROMECH 54 Colloqiuum on Finite Deformations in Plasticity, Warsaw, Sept. 30 — Oct. 3, 1974.
113. O. MAHRENHOLTZ and W. KLIE, *Axisymmetric Plastic Deformation Using Finite Element Method*, Paper presented to EUROMECH 54 Colloquium on Finite Deformations in Plasticity, Warsaw, Sept. 30 — Oct. 3, 1974.

114. R. de Boer, *Zur Theorie viskoplastischer Stoffe*, J. Appl. Math. Phys. (ZAMP), **25** (1974), 195–208.
115. S. I. Oh and S. Kobayashi, *An Approximate Method for a Three-Dimensional Analysis of Rolling*, Int. J. Mech. Sci., **17** (1975), 293-305.
116. A. R. S. Ponter, *General Displacement and Work Bounds For Dynamically Loaded Bodies*, J. Mech. Phys. Solids, **23** (1975), 151-163.
117. A. R. S. Ponter, *Deformation Bounds For the Bailey-Orowan Theory of Creep*, Trans. ASME J. Appl. Mech., **42** (1975), 619-624.
118. P. S, Symonds, *On an Extremum Principle For Mode From Solutions in Plastic Structural Dynamics*, Trans. ASME J. Appl., **42** (1975), 630-640.

S t r e s z c z e n i e

ZASADY EKSTREMALNE I WARIACYJNE W PLASTYCZNOŚCI

Ekstremum funkcji (funkcjonałów) W jest często z przyczyn numerycznych stosowane w słabszej formie wariacyjnej $\delta W=0$. W pracy przedyskutowano współzależność pomiędzy tymi dwoma sformułowaniami. W większości teorii W oznacza pracę lub wielkość analogiczną, natomiast $\delta W=0$ zasadę prac przygotowanych (wirtualnych).

W punkcie 2 rozważono zasady wariacyjne i ekstremalne dla materiałów sztywno plastycznych. Klasyczne zasady Haara-Kármána-Sadowsky'ego-Phillipsa-Hilla o dolnej i górnej ocenie przedstawiono w postaci ogólnej nie ograniczając się do szczególnych warunków brzegowych, nieściśliwości, niezależności od prędkości, jednorodności czy izotropii. Zasady te stały się najsilniejszym narzędziem dla stosowania teorii do problemów praktycznych. Zilustrowano to kilkoma przykładami z mechaniki budowli, obróbki, obróbki metali i mechaniki gruntów pokazując również pewne ostatnio badane właściwości dotyczące tarcia powierzchniowego, działania sił masowych i ściśliwości materiału.

Punkt 3 dotyczy zagadnień statycznych dla materiału sprężysto-plastycznego. Wychodząc z zasady Cotterill'a-Castigliano dla sprężystości, Prager, Hodge, Greenberg i Bauer wyprowadzili analogiczne zasady dla materiałów idealnie plastycznych lub ze wzmocnieniem. Jednakże zasady te znalazły mało zastosowań numerycznych. W ostatnim czasie wykazano, że trzeba być przy tym bardzo ostrożnym, aby uniknąć błędów systematycznych.

W punkcie 4 przedyskutowano zagadnienia zależne od prędkości deformacji lub dynamicznej plastyczności. Oprócz ogólnych zasad mechaniki, jak zasada Hamiltona, istnieje kilka twierdzeń specjalnych, związanych z pracą i energią, pozwalających szacować wielkość całkowitych deformacji lub inne wielkości.

Punkt 5 poświęcono pewnym uogólnieniom i zastosowaniom przedyskutowanych zasad. Podobnie jak w sprężystości czyniono próby bezpośredniego zastosowania zasady prac przygotowanych (wirtualnych) $\delta W=0$ do otrzymania informacji o nieznanym, poszukiwanym rozwiązaniu. Mogą być również rozpatrywane materiały bardziej ogólne, jak np. o niestowarzyszonym prawie płynięcia. Najbardziej ogólne zasady są jednak ściśle związane z metodą ważonych reziduów.

Р е з ю м е

ЭКСТРЕМАЛЬНЫЕ И ВАРИАЦИОННЫЕ ПРИНЦИПЫ В ПЛАСТИЧНОСТИ

Экстремум функций (функционалов) W часто, по численных причинам, применяется в более слабой вариационной форме $\delta W=0$. В работе обсуждена взаимозависимость между этими двумя формулировками. В большинстве теорий W обозначает работу или аналогичную величину, а $\delta W=0$ -- пинцип возможных (виртуальных) работ.

В пункте 2 рассмотрены вариационные и экстремальные принципы для жестко пластических материалов. Классические принципы Хаара-Кармана-Садовского-Филипса-Хилла о нижней и верхней оценках представлены в общем виде, не ограничиваясь частными граничными условиями, несжимаемостью, независимостью от скорости, однородностью или изотропией. Эти принципы стали самым сильным аппаратом для применения теории к практическим задачам. Это иллюстрируется несколькими примерами из строительной механики, обработки, обработки металлов и механики грунтов, показывая тоже некоторые исследуемые в последнее время свойства, касающиеся поверхностного трения, действия массовых сил и сжимаемости материала.

Пункт 3 касается статических вопросов для упруго-пластического материала. Исходя из принципа Котерилла-Кастильяно для упругости, Прагер, Ходж, Гринберг и Бауэр вывели аналитические принципы для идеально пластических материалов или материалов с упрочнением. Однако эти принципы нашли мало численных применений. В последнее время показано, что надо быть при этом очень осторожным, чтобы избежать систематических ошибок.

В пункте 4 обсуждены задачи зависящие от скорости деформации или динамической пластичности. Кроме общих принципов механики, как принцип Гамильтона, существует несколько специальных теорем, связанных с работой и энергией, которые позволяют оценить величину полных деформаций или другие величины.

Пункт 5 посвящен некоторым обобщениям и применениям обсуждаемых принципов. Аналогично как в упругости предпринимались попытки непосредственного применения принципа возможных (виртуальных) работ $\delta W=0$ для получения информаций о неизвестном, искомом решении. Могут тоже рассматриваться более общие материалы чем о неассоциированном законе течения. Наиболее общие принципы однако близко связаны с методом вычетов с весом.