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CIRCULAR ARC CRACK AND CONCENTRIC INHOMOGENEITY IN AN INFINITE ISOTROPIC ELASTIC PLATE UNDER TENSION

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A circular inhomogeneity is embedded in an infinite elastic material which also contains a circular arc crack. The inhomogeneity and the crack are concentric but the radius of the crack is greater than that of inhomogeneity. The infinite plate is subject to a traction at infinity. The above elasticity problem is solved in this paper using complex variable method, in the circular region bounded by the radius of the crack including inhomogeneity. Some numerical calculations have been done. It is seen that a more flexible inhomogeneity than outside material decreases the stress intensity factor at the tips of the crack. Also, as expected, the stress intensity factor increases as the crack moves away from the inhomogeneity in the case when the inhomogeneity is more flexible than outside material, while it decreases in the case of more rigid inhomogeneity.

1. INTRODUCTION

The study of crack problems was first initiated by INGLIS [1], GRIFFITH [2] and NEUBER [3]. A number of problems [4–10] have been solved in elasticity theory on the stress distribution around a crack in an infinite flat plate subject to either extensional or flexural loading at infinity. A simple method for some types of two-dimensional crack problems has been given by MUSKHELISHVILI [11]. The method is based upon complex variable formulation of the problem. ENGLAND [12] extended Muskhelishvili's technique and gave a solution to an arc crack around the circular elastic inhomogeneity in an infinite plate under traction at infinity. In this case (considered by England) the crack exists at the boundary of the circular inhomogeneity.

As is well known, inclusion problems have been considered by various authors, i.e., by FRENKEL [13], MOTT and NABARRO [14], ESHELBY [15], JASWON and BHARGAVA [16] and others. The problem, when there is a circular inhomogeneity and another circular hole in an infinite medium under traction at infinity, has been considered by BHARGAVA and KAPOOR [17]. The problem of a straight line crack and inhomogeneity in an infinite medium has been considered by TAMATE [18] and BHARGAVA R. D. and BHARGAVA R. R. [19].

In the present paper, the problem of two-dimensional elastic circular inhomogeneity and a circular arc crack in an infinite isotropic elastic material is considered. The circular crack has the same centre (taken as origin) as the inhomogeneity but is of different radius. The radii of the crack and the inhomogeneity are a and C, respectively, and $a > C$. The crack subtends an angle 2α at the centre. The edges of the cut are free from external stresses and the plate is subject to a tension

N at infinity in X, Y plane making an angle β with the X-axis. The X-axis passes through the mid-point of the crack (Fig. 1). The problem is considered to be the plane problem, and thus the Y-axis is in the plane of the section. The inhomogeneity is welded to outside material to avoid slipping.

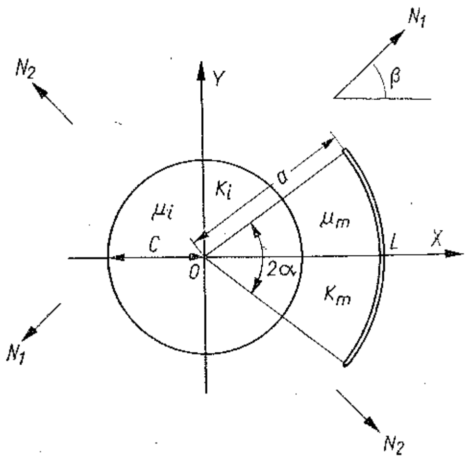

Fig. 1. Configuration and coordinate system.

As is well known, the stresses P_{ij} ($i, j = r, \theta$ in polar coordinates) are known if two complex potential functions $\Phi(z)$ and $\Psi(z)$ are known. These are related to stresses P_{ij} by the relations:

$$P_{rr}+P_{\theta\theta}=2\,[\Phi(z)+\overline{\Phi(z)}],\tag{1.1}$$
$$P_{rr}+iP_{r\theta}=[\Phi(z)+\overline{\Phi(z)}-z\overline{\Phi'(z)}-(\bar{z}/z)\,\overline{\Psi(z)}].$$

Similarly the displacement components U_r, U_θ are related to $\Phi(z)$, $\Psi(z)$ by the relation

$$2\mu\frac{\partial}{\partial\theta}\{e^{i\theta}(U_r+iU_\theta)\}=iz\,[K\Phi(z)-\overline{\Phi(z)}+z\overline{\Phi'(z)}+(\bar{z}/z)\,\overline{\Psi(z)}],\tag{1.2}$$

where $z=re^{i\theta}$, μ is the shear modulus, $K=3$–$4\,\sigma$ for the plane strain, and $K=(3-\sigma)/(1+\sigma)$ for generalized plane stress, σ being Poisson's ratio.

If the resultant traction at outer boundary is zero, the functions $\Phi(z)$ and $\Psi(z)$ at large distances from the origin are of the form:

$$\Phi(z)=\Gamma+O(z^{-2}),\qquad \Psi(z)=\Gamma'+O(z^{-2}),\tag{1.3}$$

where

$$\Gamma=\frac{1}{4}(N_1+N_2)+i\frac{2\mu\varepsilon_\infty}{1+K},$$

and

$$\Gamma'=-\frac{1}{2}(N_1-N_2)\,e^{-2i\beta}.$$

Here N_1, N_2 are the values of the principal stresses at infinity, β is the angle made by the direction of N_1 with X-axis and ε_∞ denotes the value of the rotation at infinity. It will be assumed that ε_∞ is zero.

It is convenient to define a new function $\Omega(z)$ by the relation

$$\Omega(z)=\bar{\Phi}\left(\frac{a^2}{z}\right)-\frac{a^2}{z}\,\bar{\Phi}'\left(\frac{a^2}{z}\right)-\frac{a^2}{z^2}\,\bar{\Psi}\left(\frac{a^2}{z}\right),\tag{1.4}$$

whence

$$\Psi(z)=\frac{a^2}{z^2}\Phi(z)-\frac{a^2}{z^2}\bar{\Omega}\left(\frac{a^2}{z}\right)-\frac{a^2}{z}\Phi'(z). \tag{1.5}$$

The behaviour of $\Omega(z)$ near the origin is given by

$$\Omega(z)=-\bar{\Gamma}'\frac{a^2}{z^2}+O(1). \tag{1.6}$$

From $(1.1)_2$, (1.2), (1.4) it may be seen that

$$\begin{aligned} P_{rr}+iP_{r\theta}&=\Phi(z)+\Omega\left(\frac{a^2}{\bar{z}}\right)+\bar{z}\left(\frac{\bar{z}}{a^2}-\frac{1}{z}\right)\overline{\Psi(z)},\\ 2\mu\frac{\partial}{\partial\theta}\{e^{i\theta}(U_r+iU_\theta)\}&=iz\left[K\Phi(z)-\Omega\left(\frac{a^2}{\bar{z}}\right)-\bar{z}\left(\frac{\bar{z}}{a^2}-\frac{1}{z}\right)\overline{\Psi(z)}\right]. \end{aligned} \tag{1.7}$$

For solving the problem, two sets of functions $\{\Phi(z), \Psi(z), \Omega(z)\}$ are to be evaluated. One will relate to circular inhomogeneity. This will be distinguished by writing the subscripts i to $\Phi(z)$, $\Psi(z)$ and $\Omega(z)$. The other will relate to the rest of the region extending up to infinity, including the arc crack. The subscripts m affixed to the functions will cater for this region.

2. Boundary conditions and solution of the problem

Thus the boundary condition for the outer region may be written as follows: At infinity,

$$\Phi_m(z)=\Gamma+O(z^{-2}),\qquad \Psi_m(z)=\Gamma'+O(z^{-2}), \tag{2.1}$$

whence, at the origin,

$$\Omega_m(z)=-\bar{\Gamma}'\frac{a^2}{z^2}+O(1). \tag{2.2}$$

Also at the rims of the crack, stresses are not applied. Thus $(P_{rr}^{\pm}+iP_{r\theta}^{\pm})_{\substack{|\theta|<\alpha\\ r=a}}=0.$

Note that we have used the superscripts + and − for stresses on the left and right regions of the crack (Fig. 1). In terms of the functions $\Phi(z)$ and $\Omega(z)$, this condition may be written as

$$\Phi_m^+(t)+\Omega_m^-(t)=0,\qquad \Phi_m^+(t)+\Omega_m^-(t)=0, \tag{2.3}$$

where the point z on the crack is distinguished by writing t for z. The Eqs. (2.3) may readily be transformed into

$$\begin{aligned} [\Phi_m(t)+\Omega_m(t)]^+ + [\Phi_m(t)+\Omega_m(t)]^- &=0,\\ [\Phi_m(t)-\Omega_m(t)]^+ - [\Phi_m(t)-\Omega_m(t)]^- &=0, \end{aligned} \tag{2.4}$$

where the symbols have obvious meanings.

Also on the common interface between the inhomogeneity and the outer material the stresses P_{rr}, $P_{r\theta}$ and displacements U_r, U_θ are continuous. Noting that at the boundary of the interface $z=Ce^{i\theta}$, the above conditions may be written in terms of Φ_j, Ψ_j $(j=i, m)$ with the help of $(1.1)_2$ and (1.2) as follows:

$$\begin{aligned}(2.5)\quad &\Phi_m(ce^{i\theta})+\bar{\Phi}_m(Ce^{-i\theta})-Ce^{-i\theta}\bar{\Phi}'_m(Ce^{-i\theta})-e^{-2i\theta}\bar{\Psi}_m(Ce^{-i\theta})=\Phi_i(Ce^{i\theta})+\\ &\qquad+\bar{\Phi}_i(Ce^{-i\theta})-Ce^{-i\theta}\bar{\Phi}'_i(Ce^{-i\theta})-e^{-2i\theta}\bar{\Psi}_i(Ce^{-i\theta}),\\ &\frac{1}{\mu_m}[K_m\Phi_m(Ce^{i\theta})-\bar{\Phi}_m(Ce^{-i\theta})+Ce^{-i\theta}\bar{\Phi}'_m(Ce^{-i\theta})+e^{-2i\theta}\bar{\Psi}_m(Ce^{-i\theta})]=\\ &\qquad=\frac{1}{\mu_i}[K_i\Phi_i(Ce^{i\theta})-\bar{\Phi}_i(Ce^{-i\theta})+Ce^{-i\theta}\bar{\Phi}'_i(Ce^{-i\theta})+e^{-2i\theta}\bar{\Psi}_i(Ce^{-i\theta})].\end{aligned}$$

We shall construct the complex potentials $\Phi_m(z)$, $\Psi_m(z)$ for the infinite plate which satisfy the conditions given at infinity as well as along the rims of the crack L and at the interface. For this purpose, we divide the complex potentials into two parts

$$(2.6)\qquad \Phi_m(z)=\Phi_{1m}(z)+\Phi_{2m}(z),\qquad \Psi_m(z)=\Psi_{1m}(z)+\Psi_{2m}(z),$$

and hence

$$(2.7)\qquad \Omega_m(z)=\Omega_{1m}(z)+\Omega_{2m}(z).$$

The problem of the circular arc crack in an infinite plate under tension T at infinity is already solved. The solution of the problem is given in [11] and [20]. This solution denoted by $\Phi_{1m}(z)$, $\Omega_{1m}(z)$; $\Phi_{2m}(z)$, $\Omega_{2m}(z)$ may be taken as the perturbation due to the presence of inhomogeneity. The functions $\Phi_{1m}(z)$ and $\Omega_{1m}(z)$ are given by

$$\begin{aligned}(2.8)\quad &\Phi_{1m}(z)=\frac{1}{2X(z)}\left[C_1\frac{z}{a}+C_0+C_{-1}\frac{a}{z}+C_{-2}\frac{a^2}{z^2}\right]+\frac{D_0}{2}+\frac{D_{-2}}{2}\frac{a^2}{z^2},\\ &\Omega_{1m}(z)=\frac{1}{2X(z)}\left[C_1\frac{z}{a}+C_0+C_{-1}\frac{a}{z}+C_{-2}\frac{a^2}{z^2}\right]-\frac{D_0}{2}-\frac{D_{-2}}{2}\frac{a^2}{z^2},\end{aligned}$$

where $X(z)$ denotes that branch of

$$(2.9)\qquad X(z)=\left(\frac{z}{a}-e^{-i\alpha}\right)^{1/2}\left(\frac{z}{a}-e^{i\alpha}\right)^{1/2}$$

which is single-valued in the entire plane cut along L such that $X(z)\to\left(\frac{z}{a}\right)$ for $|z|\to\infty$. Note that [11], $\Phi_{1m}(z)$, $\Omega_{1m}(z)$ satisfy the Eqs. (2.4). The constants C_1, C_0, C_{-1}, C_{-2}, D_0 and D_{-2} are given by

$$\begin{aligned}(2.10)\quad &D_0+p_{-1}C_1=2\Gamma,\quad p_{-1}C_0+p_{-2}C_1=0,\quad D_{-2}-p_0C_{-2}=2\bar{\Gamma}',\\ &p_0C_{-1}+p_1C_{-2}=0,\quad D_{-2}+p_0C_{-2}=0,\\ &-\bar{D}_0+p_{-1}\bar{C}_1=D_0+p_0C_0+p_1C_{-1}+p_2C_{-2},\end{aligned}$$

where the constants p_{-n} are the coefficients of the following power series expansions:

$$(2.11)\qquad (1-z_1 e^{-i\alpha})^{-1/2}(1-z_1 e^{i\alpha})^{-1/2}=\sum_{n=0}^{\infty} p_{-(n+1)} z_1^n, \quad z<1,$$

$$p_n=-p_{-(n+1)}, \quad n=0,1,2,\ldots.$$

From (2.8), for $|z|<a$, Φ_{1m} may be written as

$$(2.12)\qquad \Phi_{1m}(z)=\frac{1}{2}\sum_{n=0}^{\infty} C_n^* \frac{z^n}{a^n},$$

where

$$(2.13)\qquad \begin{aligned} C_0^* &= D_0+p_0 C_0+p_1 C_{-1}+p_2 C_{-2}, \\ C_n^* &= p_{(n-1)} C_1+p_n C_0+p_{n+1} C_{-1}+p_{n+2} C_{-2}, \quad n\geqslant 1. \end{aligned}$$

The corresponding function $\Psi_{1m}(z)$ which is derived from (1.5) may be expanded in the region $|z|<a$ as

$$(2.14)\qquad \Psi_{1m}(z)=-\frac{1}{2}\sum_{n=0}^{\infty} D_n^*\left(\frac{z}{a}\right)^n,$$

where

$$(2.15)\qquad \begin{aligned} D_n^* &= (n+1) C_{n+2}^*+(-\bar{D}_{-2}+p_{-1}\bar{C}_{-1}+p_{-2}\bar{C}_0+p_{-3}\bar{C}_1)\delta_{0,n}+ \\ &+(p_{-n}\bar{C}_{-2}+p_{-(n+1)}\bar{C}_{-1}+p_{-(n+2)}\bar{C}_0+p_{-(n+3)}\bar{C}_1)(1-\delta_{0,n}), \\ &\qquad n=0,1,2,\ldots, \end{aligned}$$

where $\delta_{0,n}$ denotes the usual Kronecker delta.

Next, we construct the auxiliary potentials $\Phi_{2m}(z)$ and $\Omega_{2m}(z)$. Note that Φ_{1m}, Ψ_{1m} already satisfy the conditions at infinity. Therefore Φ_{2m}, Ψ_{2m} are taken to be such functions which give zero tractions at infinity. This implies that

$$(2.16)\qquad \Phi_{2m}(z)=O(z^{-2}), \quad \Psi_{2m}(z)=O(z^{-2}) \quad \text{if} \quad |z|\to\infty,$$

whence

$$(2.17)\qquad \Omega_{2m}(z)=O(1) \quad \text{if} \quad |z|\to 0.$$

Since the Eqs. (2.4) are dual homogeneous Hilbert problems for two functions $\Phi_m(z)$ and $\Omega_m(z)$ and $\Phi_{1m}(z)$, $\Omega_{1m}(z)$ already satisfy these equations, hence Φ_{2m}, Ω_{2m} will also satisfy these Eqs. (2.4). Moreover, $\Phi_{2m}(z)$ and $\Omega_{2m}(z)$ are analytic in the entire plane cut along L. Hence these functions may be constructed by the use of Muskhelishvili's technique. Considering the conditions (2.16), (2.17) and the fact that $\Phi_{2m}(z)$ and $\Psi_{2m}(z)$ could have poles of various orders at the origin, we get

$$(2.18)\qquad \Phi_{2m}(z)=\frac{1}{2}\left[\sum_{n=2}^{\infty} A_{-n}\left(\frac{a}{z}\right)^n+\sum_{n=0}^{\infty} A_n\left(\frac{z}{a}\right)^n+ \right.$$

$$\left. +\frac{1}{X(z)}\left\{\sum_{n=1}^{\infty} B_{-n}\left(\frac{a}{z}\right)^n+\sum_{n=0}^{\infty} B_n\left(\frac{z}{a}\right)^n\right\}\right],$$

$$(2.19)\qquad \Omega_{2m}(z)=\frac{1}{2}\left[-\sum_{n=2}^{\infty}A_{-n}\left(\frac{a}{z}\right)^n-\sum_{n=0}^{\infty}A_n\left(\frac{z}{a}\right)^n+\right.$$
$$\left.+\frac{1}{X(z)}\left\{\sum_{n=1}^{\infty}B_{-n}\left(\frac{a}{z}\right)^n+\sum_{n=0}^{\infty}B_n\left(\frac{z}{a}\right)^n\right]\right\},$$

where

$$(2.20)\qquad \begin{aligned} A_n&=-\sum_{k=1}^{\infty}p_{-k}B_{n+k}, \quad n=0,1,2,\ldots, \qquad \sum_{k=1}^{\infty}p_{-k}B_{k-1}=0,\\ A_{-n}&=\sum_{k=0}^{\infty}p_kB_{-(n+k)}, \quad n=2,3,\ldots, \qquad \sum_{k=0}^{\infty}p_kB_{-(k+1)}=0. \end{aligned}$$

The complex potential $\Phi_{2m}(z)$ in the Eq. (2.19) and the corresponding function $\Psi_{2m}(z)$, obtained from the Eq. (1.5) for $C<|z|<a$, may be expanded in Laurent series as follows:

$$(2.21)\qquad \Phi_{2m}(z)=\frac{1}{2}\sum_{n=-\infty}^{\infty}E_n\left(\frac{z}{a}\right)^n, \qquad \Psi_{2m}(z)=\frac{1}{2}\sum_{n=-\infty}^{\infty}F_n\left(\frac{z}{a}\right)^n,$$

where

$$(2.22)\qquad \begin{aligned} &E_{-1}=0,\\ &E_{-n}=2\sum_{k=0}^{\infty}p_kB_{-(k+n)}, \quad n=2,3,\ldots,\\ &E_n=\sum_{k=0}^{\infty}p_kB_{n-k}-\sum_{k=1}^{\infty}p_{-k}B_{n+k}, \quad n=0,1,2,\ldots. \end{aligned}$$

$$(2.23)\qquad \begin{aligned} &F_{-1}=0,\\ &F_{-n}=(n-1)E_{-(n-2)}-2\sum_{k=1}^{\infty}p_{-k}\bar{B}_{n+k-2}, \quad n=2,3,\ldots,\\ &F_n=(n+1)\left(\sum_{k=1}^{\infty}(p_{-k}B_{n+k+2}-\sum_{k=0}^{\infty}p_kB_{n+2-k}\right)+\sum_{k=0}^{\infty}p_k\bar{B}_{-(n+k+2)}-\\ &\qquad -\sum_{k=1}^{\infty}p_{-k}\bar{B}_{-(n-k+2)}, \quad n=0,1,2,\ldots. \end{aligned}$$

Substituting the Eqs. (2.12), (2.14) and (2.21) into the Eq. (2.6), we get the expression of the complex potentials valid in the region $C<|z|<a$ for infinite plate which satisfy the conditions at infinity and along the rims of the crack L. Thus it may be seen that

$$(2.24)\qquad \begin{aligned} \Phi_m(z)&=\frac{1}{2}\left[\sum_{n=1}^{\infty}E_{-n}\left(\frac{a}{z}\right)^n+\sum_{n=0}^{\infty}(C_n^*+E_n)\left(\frac{z}{a}\right)^n\right],\\ \Psi_m(z)&=\frac{1}{2}\left[\sum_{n=1}^{\infty}F_{-n}\left(\frac{a}{z}\right)^n+\sum_{n=0}^{\infty}(-D_n^*+F_n)\left(\frac{z}{a}\right)^n\right], \end{aligned}$$

where C_n^* and D_n^* are known constants, while E_n, E_{-n} and F_n, F_{-n} are linear functions of unknown constants B_n and B_{-n}.

On the other hand, the complex potentials Φ_i and Ψ_i for the circular inhomogeneity are holomorphic in the region $|z|\leqslant C$. They can directly be expanded into Taylor series as

$$\Phi_1(z)=\frac{1}{2}\sum_{n=0}^{\infty}G_n\left(\frac{z}{C}\right)^n,\qquad \Psi_i(z)=\frac{1}{2}\sum_{n=0}^{\infty}H_n\left(\frac{z}{C}\right)^n, \tag{2.25}$$

where G_n and H_n are unknown constants which are determined by the conditions of continuity on $|z|=C$. Substituting the Eqs. (2.24) and (2.25) into (2.5) to satisfy the requirements of continuity of $(P_{rr}+iP_{r\theta})$ and (U_r+iU_θ), and comparing the coefficients of the same powers of $e^{i\theta}$ on both sides, we obtain

$$\begin{aligned}
&(C_0^*+\bar{C}_0^*)+(E_0+\bar{E}_0)-\left(\frac{a}{C}\right)^2\bar{F}_{-2}=G_0+\bar{G}_0\,,\\
&\frac{\mu_i}{\mu_m}(K_m C_0^*-\bar{C}_0^*)+\frac{\mu_i}{\mu_m}(K_m E_0-\bar{E}_0)+\frac{\mu_i}{\mu_m}\bar{F}_{-2}\left(\frac{a}{C}\right)^2=K_i G_0-\bar{G}_0\,,\\
&E_{-n}\left(\frac{a}{C}\right)^n-(n-1)(\bar{C}_n^*+\bar{E}_n)\left(\frac{C}{a}\right)^n+(\bar{D}_{n-2}^*-\bar{F}_{n-2})\left(\frac{C}{a}\right)^{n-2}=\\
&\qquad\qquad=-\{(n-1)\bar{G}_n+\bar{H}_{n-2}\}\,,\\
&\frac{\mu_i}{\mu_m}K_m E_{-n}\left(\frac{a}{C}\right)^n+\frac{\mu_i}{\mu_m}(n-1)(\bar{C}_n^*+\bar{E}_n)\left(\frac{C}{a}\right)^n-\frac{\mu_i}{\mu_m}(\bar{D}_{n-2}^*-\bar{F}_{n-2})\left(\frac{C}{a}\right)^{n-2}=\\
&\qquad\qquad=(n-1)\bar{G}_n+\bar{H}_{n-2}\,,\qquad n=2,3,\ldots,\\
&(C_n^*+E_n)\left(\frac{C}{a}\right)^n+(n+1)\bar{E}_{-n}\left(\frac{a}{C}\right)^n-\bar{F}_{-(n+2)}\left(\frac{a}{C}\right)^{n+2}=G_n,\\
&\frac{\mu_i}{\mu_m}K_m(C_n^*+E_n)\left(\frac{C}{a}\right)^n-\frac{\mu_i}{\mu_m}(n+1)\bar{E}_{-n}\left(\frac{a}{C}\right)^n+\\
&\qquad\qquad+\frac{\mu_i}{\mu_m}\bar{F}_{-(n+2)}\left(\frac{a}{C}\right)^{n+2}=K_i G_n,\qquad n=1,2,\ldots.
\end{aligned}\tag{2.26}$$

By eliminating G_n and H_{n-2} from the Eqs. $(2.26)_{2,3}$, we get

$$\begin{aligned}
E_{-n}=&-\frac{\left(\dfrac{\mu_i}{\mu_m}\right)-1}{\left(\dfrac{\mu_i K_m}{\mu_m}\right)+1}(n-1)\left(\frac{C}{a}\right)^{2n}(\bar{C}_n^*+\bar{E}_n)+\\
&+\frac{\left(\dfrac{\mu_i}{\mu_m}\right)-1}{\left(\dfrac{K_m\mu_i}{\mu_m}\right)+1}\left(\frac{C}{a}\right)^{2n-2}(\bar{D}_{n-2}^*-\bar{F}_{n-2}),\qquad n=2,3,\ldots.
\end{aligned}\tag{2.27}$$

The Eqs. $(2.26)_{1,2}$ and $(2.26)_6$ give

(2.28)
$$F_{-2}=-\frac{\left(\frac{\mu_i K_m}{\mu_m}\right)-K_i-\left(\frac{\mu_i}{\mu_m}\right)+1}{2\left(\frac{\mu_i}{\mu_m}\right)+K_i-1}\left(\frac{C}{a}\right)^2(C_0^*+\bar{C}_0^*+E_0+\bar{E}_0),$$
$$F_{-(n+2)}=(n+1)\left(\frac{C}{a}\right)^2 E_{-n}-\frac{\left(\frac{\mu_i K_m}{\mu_m}\right)-K_i}{\left(\frac{\mu_i}{\mu_m}\right)+K_i}\left(\frac{C}{a}\right)^{2n+2}(\bar{C}_n^*+\bar{E}_n),\quad n=1,2,\dots .$$

Substituting the Eqs. (2.22) into (2.27) and (2.28), the following infinite sets of linear equations are obtained for B_n, B_{-n}:

(2.29)
$$\sum_{k=0}^{\infty} p_k B_{-(k+1)}=0,$$
$$\sum_{k=0}^{\infty} p_k B_{-(k+n)}=\frac{1}{2}\left[-\frac{\left(\frac{\mu_i}{\mu_m}\right)-1}{\left(\frac{\mu_i K_m}{\mu_m}\right)+1}\left\{(n-1)\left(\frac{C}{a}\right)^{2n}\bar{C}_n^*-\left(\frac{C}{a}\right)^{2n-2}\bar{D}_{n-2}^*\right\}+\right.$$
$$+\frac{\left(\frac{\mu_i}{\mu_m}\right)-1}{\left(\frac{\mu_i K_m}{\mu_m}\right)+1}\left(\frac{C}{a}\right)^{2n-2}\left\{(n-1)\left(1-\frac{C^2}{a^2}\right)\left(\sum_{k=0}^{\infty}p_k\bar{B}_{n-k}-\sum_{k=1}^{\infty}p_{-k}\bar{B}_{k+n}\right)+\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}p_{-k}B_{k-n}-\sum_{k=0}^{\infty}p_k B_{-(k+n)}\right\}\right],\qquad n=2,3,\dots,$$

(2.30)
$$\sum_{k=1}^{\infty} p_{-k}\bar{B}_{k-1}=0,$$
$$\sum_{k=1}^{\infty}p_{-k}\bar{B}_k=\frac{1}{2}\left(\sum_{k=0}^{\infty}p_k B_{-k}-\sum_{k=1}^{\infty}p_{-k}B_k\right)+\frac{1}{2}\,\frac{\left(\frac{\mu_i K_m}{\mu_m}\right)-K_i-\left(\frac{\mu_i}{\mu_m}\right)+1}{2\left(\frac{\mu_i}{\mu_m}\right)+K_i-1}\times$$
$$\times\left(\frac{C}{a}\right)^2(C_0^*+\bar{C}_0^*)+\frac{1}{2}\,\frac{\left(\frac{\mu_i K_m}{\mu_m}\right)-K_i-\left(\frac{\mu_i}{\mu_m}\right)+1}{2\left(\frac{\mu_i}{\mu_m}\right)+K_i-1}\left(\frac{C}{a}\right)^2\times$$
$$\times\left(\sum_{k=0}^{\infty}p_k B_{-k}-\sum_{k=1}^{\infty}p_{-k}B_k+\sum_{k=0}^{\infty}p_k\bar{B}_{-k}-\sum_{k=1}^{\infty}p_{-k}\bar{B}_k\right),$$

$$(2.31)\quad \sum_{k=1}^{\infty} p_{-k}\bar{B}_{k+n}=(n+1)\left(1-\frac{C^2}{a^2}\right)\sum_{k=0}^{\infty} p_k B_{-(k+n)}+ \\ +\frac{1}{2}\frac{\left(\frac{\mu_i K_m}{\mu_m}\right)-K_i}{\left(\frac{\mu_i}{\mu_m}\right)+K_i}\left(\frac{C}{a}\right)^{2n+2}\bar{C}_n^*+\frac{1}{2}\frac{\left(\frac{\mu_i K_m}{\mu_m}\right)-K_i}{\left(\frac{\mu_i}{\mu_m}\right)+K_i}\left(\frac{C}{a}\right)^{2n+2}\times \\ \times\left(\sum_{k=0}^{\infty} p_k\bar{B}_{n-k}-\sum_{k=1}^{\infty} p_{-k}\bar{B}_{n+k}\right),\qquad n=1,2,\dots .$$

After the determination of B_n and B_{-n}, the coefficients A_n and A_{-n} are calculated by the Eqs. (2.20) and the coefficients G_n and H_n by the Eqs. (2.26). Thus the problem is reduced to the solution of infinite set of linear equations (2.29) and (2.30) in the unknowns B_n, B_{-n} in which the values of Γ, Γ', K_i, μ_i/μ_m, K_m, C/a are given and p_n, p_{-n} can be found out by (2.11).

Once the constants E_n, F_n, E_{-n}, F_{-n} are known, these values are substituted into (2.24) to obtain the explicit values of Φ_m and Ψ_m; note that C_n^*, D_n^* are already known from (2.13), (2.15). Also the values of G_n, H_n are substituted into (2.25) to find explicit expressions for $\Phi_i(z)$, $\Psi_i(z)$.

The stresses and displacements can directly be found out by substituting these values of (Φ_i, Ψ_i) and (Φ_m, Ψ_m) into (1.1) and (1.2).

A few particular cases of interest can be considered. Thus, for example, let the elastic moduli of the inhomogeneity be the same as those of the outside material, i.e., let $\mu_i=\mu_m$ and $K_i=K_m$. It may be shown from (2.18) that Φ_{2m} and Ψ_{2m} come out to be identically zero. Similarly the case of cavity and arc crack can be obtained by putting $\mu_i=0$ in the Eqs. (2.29) and (2.30). In this case K_m and K_i are cancelled from (2.29) and (2.30), and therefore the stressed state in the sheet is independent of Poisson's ratio. For rigid inhomogeneity, one may take the limit $\mu_i\to\infty$. In this case K_m remains in the Eqs. (2.29) and (2.30) and hence the distribution of stresses in the infinite plate changes with Poisson's ratio. For rigid outside material, we put $\mu_m\to\infty$.

3. Stress intensity factor at the tips

It is of some interest to find out the effect of inhomogeneity on the stresses at the crack tip. Consider the case when tension at infinity is applied in X-direction. Thus we have $\Gamma=T/4$ and $\Gamma'=-T/2$.

The definition given by Sih, Paris and Erdogan [21] is used to find the stress intensity factor. It is known that, in the case of an infinite plate with only a circular arc crack of radius a subtending an angle 2α at the centre (and no inhomogeneity or cavity), the stress intensity factor at the tips of the crack is given by $(K_{10}-iK_{20})$, where

$$K_{10}=\frac{T}{2}\frac{\sqrt{a\sin\alpha}}{(1+\sin^2\alpha/2)}\left[\left(1-\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\right)\cos\frac{\alpha}{2}+\left(1+\sin^2\frac{\alpha}{2}\right)\cos\frac{3\alpha}{2}\right],$$

$$K_{20}=\frac{T}{2}\frac{\sqrt{a\sin\alpha}}{(1+\sin^2\alpha/2)}\left[\left(1-\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\right)\sin\frac{\alpha}{2}+\left(1+\sin^2\frac{\alpha}{2}\right)\sin\frac{3\alpha}{2}\right].$$

Using the same definition for the stress intensity factor at the crack tips, but noting the presence of inhomogeneity as in this paper, the stress intensity factor comes out to be (K_1-iK_2), where

$$K_1-iK_2=(K_{10}-iK_{20})\left[1+\frac{2(1+\sin^2\alpha/2)}{\sin\alpha}\left(\frac{PQ+RS}{R^2+P^2}\right)-i\frac{2(1+\sin^2\alpha/2)}{\sin\alpha}\left(\frac{PS-RQ}{R^2+P^2}\right)\right],$$

and

$$P=\frac{1}{8}(5+12\cos\alpha-\cos 2\alpha),$$

$$Q=\frac{1}{T}\sum_{n=1}^{\infty}(B_n-B_{-n})\sin n\alpha,$$

$$R=\frac{1}{4}(6\sin\alpha-\sin 2\alpha),$$

$$S=\frac{1}{T}\left\{B_0+\sum_{n=1}^{\infty}(B_n+B_{-n})\cos n\alpha\right\}.$$

4. NUMERICAL EXAMPLE

To know the effect of inhomogeneity on the crack tips, some numerical calculations have been done for the following values: Two cases of cracks were taken, one when it subtends an angle of 10^0 and another when it subtends 20^0 at the centre. Poisson's ratio is taken to be 1/3, therefore $K_i=K_m=1.666$ and the problem is solved as plane strain problem. The radius of the inhomogeneity is taken as plane strain problem. The radius of the inhomogeneity is taken as 1.0 in all cases. The radius of the crack was successively taken as 1.5 (.25) 2.5. Similarly the ratio of shear moduli μ_i/μ_m was successively taken as 0, 1/3, 1/2, 1, and 3.

In each case the set of linear equations in (2.29) and (2.30) have been solved by the method of iteration and the first 15 values of B_{-n} and B_n are determined. This is ensuring good convergence. In Figs. 2, 3 the influence factor, i.e. (the stress intensity factor with inhomogeneity) the stress intensity factor with only the crack

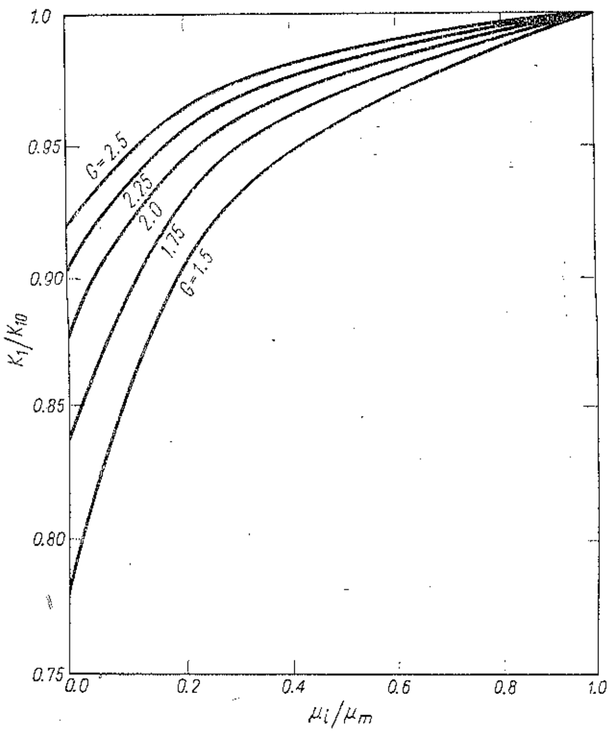

Fig. 2. Crack tip stress intensity factor *vs* shear modulus ratio.

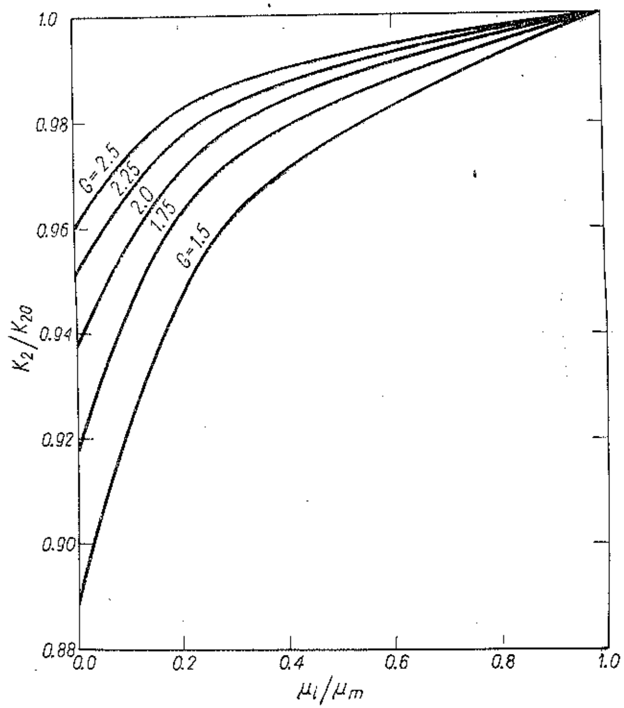

Fig. 3. Crack tip stress intensity factor *vs* shear modulus ratio.

in the infinite matèrial, has been plotted versus the shear modulus ratio μ_i/μ_m for different values of a and for the case when the crack subtends an angle of 10°.

In Fig. 4, the crack subtends an angle of 20°. The influence factor has been plotted versus the radius of the circular arc for $\mu_i/\mu_m=1/3$, 1, 3, respectively. It is observ-

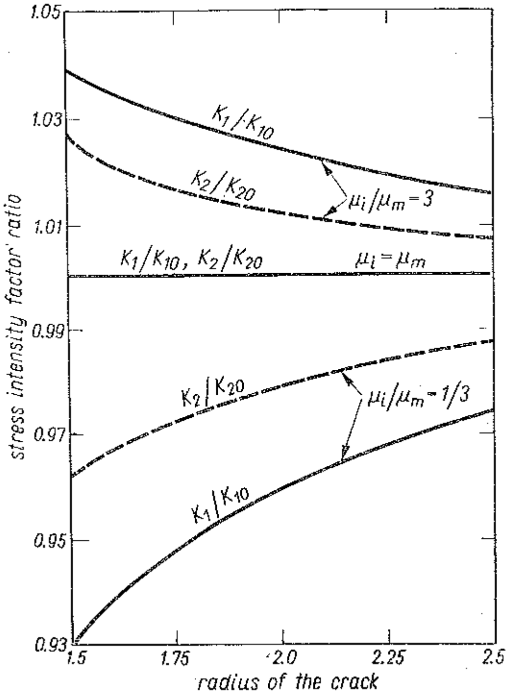

Fig. 4. Crack tip stress intensity factor *vs* radius of the crack.

ed that the presence of more flexible inhomogeneity decreases the stress intensity factor, while the more rigid inhomogeneity increases it; also for $\mu_i<\mu_m$ the stress intensity factor increases as radius of crack increases, while for $\mu_i>\mu_m$ the stress intensity factor decreases as the radius increases.

References

1. C. E. Inglis, *Stress in a plate due to the presence of cracks and sharp corners*, Trans. Inst. Naval Architects, **60**, 219, 1913.
2. A. A. Griffith, *Stresses in a plate bounded by a hyperbolic cylinder*, Aeronautical Research Committee, Reports and Memoranda, No. 1152 (M 55), 1928.
3. H. Neuber, *Kerbspannungslehre*, Springer-Verlag, Berlin 1937.
4. M. L. Williams, *On the stress distribution at the base of a stationary crack*, J. Appl. Mech., **24**, 109, 1957.
5. M. L. Williams, *Bending stress distribution at the base of a stationary crack*, J. Appl. Mech., **28**, 78, 1961.
6. F. Erdogan, *Stress distribution in a nonhomogeneous elastic plane with cracks*, J. Appl. Mech., **30**, 232, 1963.

7. G. G. Sih, and J. R. Rice, *Bending of plates of dissimilar materials with crack*, J. Appl. Mech., **31**, 477, 1964.
8. A. H. England, *A crack between dissimilar media*, J. Appl. Mech., **32**, 400, 1965.
9. W. G. Knauss, *Stress in an infinite strip containing a semi-infinite crack*, J. Appl. Mech., **33**, 356, 1966.
10. O. Tamate, *Flexural problem of a thin plate with a curved crack*, Ing. Archiv., **35**, 323, 1967.
11. N. I. Muskhelishvili, *Some basic problems in the mathematical theory of elasticity*, Groningen, Holland: P. Noordhoff, 1953.
12. A. H. England, *An arc crack around a circular elastic inclusion*, J. Appl. Mech., **33**, 637, 1966.
13. J. Frankel, *Kinetic theory of liquids*, Oxford 1946.
14. N. F. Mott, and F. R. N. Nabarro, *An attempt to estimate the degree of precipitation hardening with a simple model*, Proc. Phys. Soc., **52**, 90, 1940.
15. J. D. Eshelby, *The determination of the elastic field of an ellipsoidal inclusion, and related problems*, Proc. Roy. Soc., **241**, 376, 1957.
16. M. A. Jaswon, and R. D. Bhargava, *Two-dimensional elastic inclusion problems*, Proc. Camb. Phil. Soc., **57**, 669, 1961.
17. R. D. Bhargava, and O. P. Kapoor, *Circular inclusion in an infinite elastic medium with a circular inhomogeneity*, Proc. Camb. Phil. Soc., **62**, 113, 1966.
18. O. Tamate, *The effect of a circular inclusion on the stresses around a line crack in a sheet under tension*, Int. J. Frac. Mech., **4**, 257, 1968.
19. R. D. Bhargava, and R. R. Bhargava, *Elastic circular inclusion in an infinite plane containing two cracks*, Int. J. Engng. Sci., **11**, 437, 1973.
20. L. M. Milne-Thomson, *Plane elastic system*, Springer-Verlag, Berlin 1968.
21. G. C. Sih, P. C. Paris, and F. Erdogan, *Crack-tip, stress-intensity factors for plane extension and plate bending problems*, J. Appl. Mech., **29**, 306, 1962.

Streszczenie

SZCZELINA O ŁUKU KOŁOWYM I KONCENTRYCZNA INKLUZJA W NIESKOŃCZONEJ IZOTROPOWEJ PŁYCIE SPRĘŻYSTEJ PODDANEJ ROZCIĄGANIU

W nieskończonym ośrodku sprężystym umieszczona jest okrągła inkluzja oraz szczelina o łuku kołowym. Inkluzja i szczelina są rozmieszczone koncentrycznie, przy czym promień szczeliny jest większy od promienia inkluzji. Nieskończona płyta poddana jest w nieskończoności rozciąganiu. Powyższy problem sprężysty w niniejszej pracy został rozwiązany przy użyciu metody zmiennej zespolonej w obszarze kołowym ograniczonym promieniem szczeliny i zawierającym wewnątrz inkluzję. Przeprowadzono kilka obliczeń numerycznych. Wykazano, że jeśli inkluzja jest podatniejsza niż materiał otaczający, to współczynnik koncentracji naprężenia w końcach szczeliny maleje. Również, czego należało się spodziewać, współczynnik koncentracji naprężenia wzrasta wraz z oddalaniem się szczeliny od inkluzji w przypadku, gdy inkluzja jest podatniejsza niż materiał otaczający, natomiast maleje w przypadku, gdy inkluzja jest sztywniejsza.

Резюме

ТРЕЩИНА С КРУГОВОЙ ДУГОЙ И КОНЦЕНТРИЧЕСКОЕ ВКЛЮЧЕНИЕ В БЕСКОНЕЧНОЙ ИЗОТРОПНОЙ УПРУГОЙ ПЛИТЕ ПОДВЕРГНУТОЙ РАСТЯЖЕНИЮ

В бесконечной упругой среде помещены круглое включение и трещина с круговой дугой. Включение и трещина расположены концентрически, причем радиус трещины больше, чем радиус включения. Бесконечная плита подвергается в бесконечности растяжению. Вышеприведенная упругая задача решена в настоящей работе при использовании метода комплексной переменной в круговой области ограниченной радиусом трещины и содержавшей внутри включение. Проведено несколько численных расчетов. Показано, что если включение более податливо, чем окружающий материал, тогда коэффициент концентрации напряжений в вершинах трещины убывает. Тоже, что следовало ожидать, коэффициент концентрации напряжений возрастает совместно с удалением трещины от включения в случае, когда включение более податливо, чрм окружающий материал; вместо этого коэффициент убывает в случае, когда включение является более жестким.

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