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THREE-PARAMETER OPTIMIZATION OF AN AXIALLY LOADED BEAM ON A FOUNDATION

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A beam of circular cross-section, made of viscoelastic material of Kelvin–Voigt type, is considered. The beam is symmetric with respect to its center, the length and volume of the beam are fixed and its ends are simply supported. The radius of the cross-section is a cubic function of co-ordinate. The beam interacts with a foundation of Winkler, Pasternak or Hetényi type and is axially loaded by a non-conservative force $P(t) = P_0 + P_1 \cos \vartheta t$. Only the first instability region is taken into account. The shape of the beam is optimal if the critical value of P_1 is maximal. A few numerical examples are presented on graphs.

1. Introduction

Problem of stability and parametric optimization of an axially loaded beam interacting with a foundation is considered. The radius of the circular cross-section of the beam is assumed to be a cubic polynomial of co-ordinate and an additional strength condition is added. Foundations of Winkler, Pasternak and Hetényi type are taken into account. Only the first instability region is considered. The optimal shape of the beam is characterized by variation of its cross-section radius. This paper is a continuation of the papers [1, 2] in which one-parameter and two-parameter optimization of the problem were considered.

Optimization of viscoelastic cantilever beam with respect to its dynamic stability was presented by A. Gajewski and A. S. Foryś during Euromech Colloquium 190 [3], cf. [4]. Optimization of structures is the subject of monograph by A. Gajewski and M. Życzkowski [5]. A study concerning to optimization of mechanical systems in conditions of parametric resonances was written by A. Foryś [6]. Some new approach to the solution of optimization problem for a compressed column is given by A. Gajewski [7, 8].

The Lagrange problem on an optimal column is analysed in the paper by A. P. Seyranian, O. G. Privalova [9]. Parametrically excited beam and its optimal shape is considered in the paper by A. A. Mailybaev, H. Yabuno and H. Kaneko [10].

2. Formulation of the problem

A straight beam of circular cross-section (see Fig. 1) is made of viscoelastic material of Kelvin–Voigt type. The undeformed beam axis coincides with the x-axis. In view of symmetry of the problem we assume that the beam is symmetric with respect to its centre $x = l/2$. The length l of the beam and its volume V are fixed. We assume that the cross-section of the beam is non-zero i.e. the radius $r(\xi)$ (where $\xi = x/l$) of the cross-section satisfies the assumption

$$r(\xi) > 0, \xi \in [0, 1]. \tag{2.1}$$

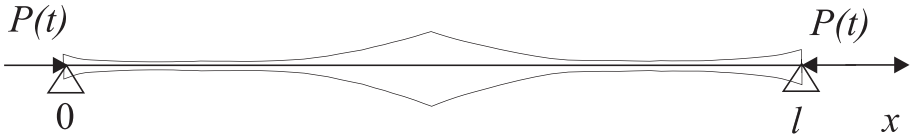

Fig. 1. The shape of the beam.

We assume that the radius is given by the following formulae:

$$r(\xi) \equiv r_0(\varepsilon_1, \varepsilon_2, \varepsilon_3)\varphi(\xi; \varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{2.2}$$

$$= \begin{cases} r_0 \left(1 - \dfrac{\varepsilon_1}{2} - \dfrac{\varepsilon_2}{4} - \dfrac{\varepsilon_3}{8} + \varepsilon_1\xi + \varepsilon_2\xi^2 + \varepsilon_3\xi^3\right), & \xi \in \left[0, \dfrac{1}{2}\right] \\ r_0 \left[1 + \dfrac{\varepsilon_1}{2} + \dfrac{3\varepsilon_2}{4} + \dfrac{7\varepsilon_3}{8} - (\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3)\xi + (\varepsilon_2 + 3\varepsilon_3)\xi^2 - \varepsilon_3\xi^3\right], & \xi \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

where $r_0 > 0$. Values of the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ determine the shape of the beam. For a prismatic beam $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$. In this paper one assumes that $\varepsilon_3 \neq 0$. On the basis of the assumption (2.1) employed for $\xi = 0$ i.e. $r(0) > 0$, one has the inequality

$$1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{8} > 0. \tag{2.3}$$

The volume of the beam is

$$V = 2\pi l \int_0^{1/2} r^2(\xi) d\xi = \pi r_0^2 l / f(\varepsilon_1, \varepsilon_2, \varepsilon_3),$$

where

$$f(\varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{2.4}$$

$$= \left[1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{3} - \frac{3\varepsilon_3}{16} + \frac{\varepsilon_1^2}{12} + \frac{\varepsilon_2^2}{30} + \frac{9\varepsilon_3^2}{896} + \frac{5\varepsilon_1\varepsilon_2}{48} + \frac{9\varepsilon_1\varepsilon_3}{160} + \frac{7\varepsilon_2\varepsilon_3}{192}\right]^{-1}.$$

Therefore one obtains the following formula:

$$r_0 = \sqrt{\frac{Vf(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{\pi l}}. \tag{2.5}$$

The three independent parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the optimization parameters. Their admissible values must belong to a set in R^3 in which the inequality (2.1) is satisfied. One boundary (a plane) of this set is given by (2.3).

So we look for such values of parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for which the value of the cubic polynomial $\varphi(\xi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ given by the first formula (2.2) is positive for $\xi \in \left[0, \frac{1}{2}\right]$.

To this end the following reasoning is applied. Beacause of the assumption that $\varepsilon_3 \neq 0$, the possible graphs of this polynomial can be of the forms shown in Figs. 2–5, where $\varphi = \varphi(\xi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$.

For the graphs shown in Figs. 2 and 3 the polynomial has no extremes, so the following condition is fulfilled:

$$\Delta \equiv 3\varepsilon_1\varepsilon_3 - \varepsilon_2^2 \geq 0. \tag{2.6}$$

For the graphs shown in Figs. 4 and 5 the polynomial has two extremes, so the following condition is fulfilled:

$$\Delta < 0. \tag{2.7}$$

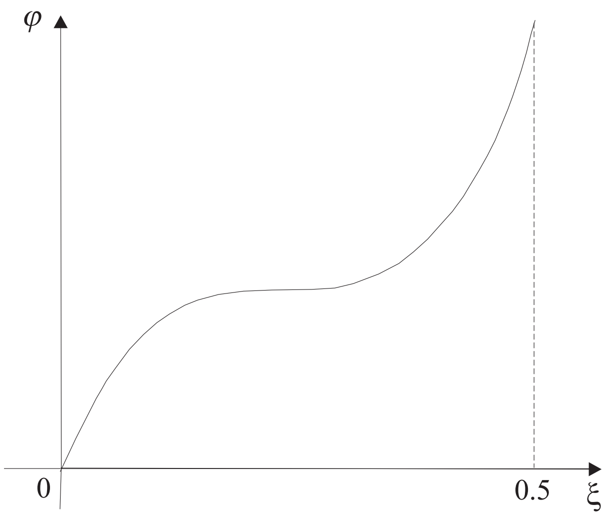

FIG. 2. The cubic polynomial.

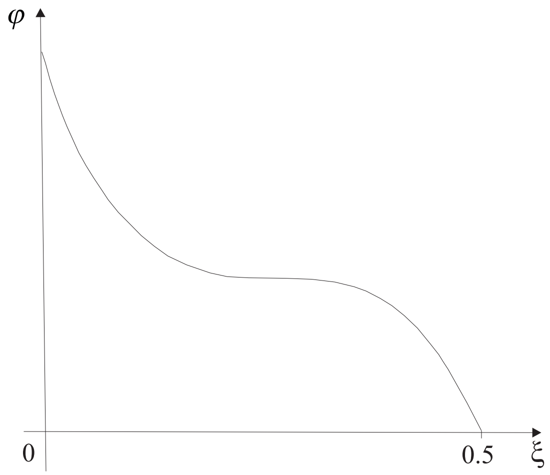

FIG. 3. The cubic polynomial.

For the cases shown in Figs. 2 and 3, from the inequalities $r(0) > 0, r\left(\frac{1}{2}\right) = r_0 > 0$ the condition (2.1) results. Therefore the conditions (2.3) and (2.6) define the set of admissible values of the optimization parameters for these cases.

If $\Delta < 0$ the situation is more complicated. In this case the polynomial φ given by the first formula (2.2) has two extremes at the points $\xi_{\min} = (-\varepsilon_2 + \sqrt{-\Delta})/3\varepsilon_3$ and $\xi_{\max} = (-\varepsilon_2 - \sqrt{-\Delta})/3\varepsilon_3$.

First we cosider the case $\varepsilon_3 > 0$, shown in Fig. 4. The points $\xi = 0$ and $\xi = \frac{1}{2}$ i.e. the ends of the interval $\left[0, \frac{1}{2}\right]$, can be situated in the following six positions: (11), (33), (22), (21), (31), (32), where each digit denotes the interval indicated in Fig. 4. To satisfy the condition (2.1) the following inequalities must be satisfied for succesive positions:

(2.8) $$(11): \ \xi_{\min} \le 0,$$

(2.9) $$(33): \ \xi_{\max} \ge \frac{1}{2},$$

(2.10) $$(22): \ \xi_{\max} \le 0 \wedge \xi_{\min} \ge \frac{1}{2},$$

(2.11) $$(21): \ \xi_{\max} \le 0 \wedge \xi_{\min} \in \left(0, \frac{1}{2}\right] \wedge \varphi(\xi_{\min}) > 0,$$

(2.12) $$(31): \ \xi_{\max} > 0 \wedge \xi_{\min} \le \frac{1}{2} \wedge \varphi(\xi_{\min}) > 0,$$

(2.13) $$(32): \ \xi_{\min} \ge \frac{1}{2} \wedge \xi_{\max} \in \left(0, \frac{1}{2}\right].$$

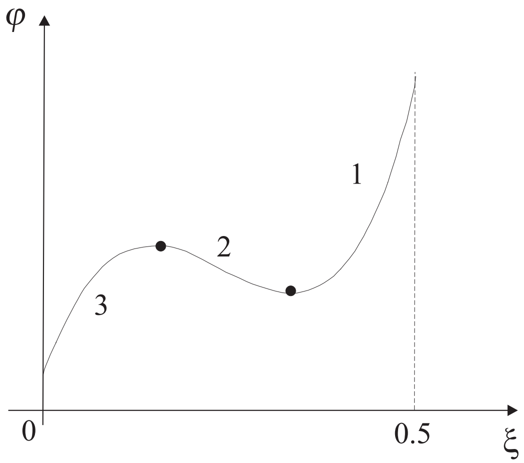

FIG. 4. The cubic polynomial.

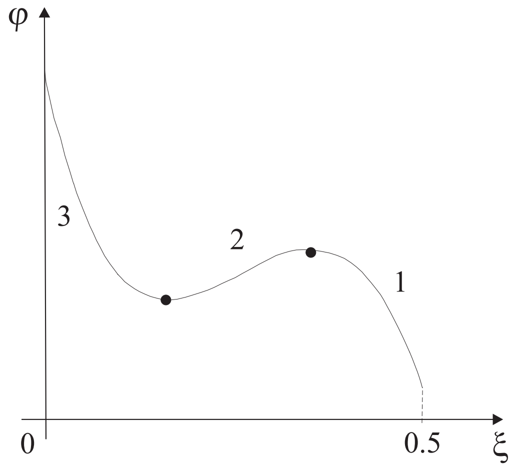

FIG. 5. The cubic polynomial.

Next we consider the case $\varepsilon_3 < 0$, illustrated in Fig. 5. To satisfy the condition (2.1), the following inequalities must be satisfied for succesive positions:

$$(11): \ \xi_{\max} \leq 0, \tag{2.14}$$

$$(33): \ \xi_{\min} \geq \frac{1}{2}, \tag{2.15}$$

$$(22): \ \xi_{\min} \leq 0 \land \xi_{\max} \geq \frac{1}{2}, \tag{2.16}$$

$$(21): \ \xi_{\min} \leq 0 \land \xi_{\max} > 0 \land \xi_{\max} \leq \frac{1}{2}, \tag{2.17}$$

$$(31): \ \xi_{\min} > 0 \wedge \xi_{\max} < \frac{1}{2} \wedge \varphi(\xi_{\min}) > 0, \tag{2.18}$$

$$(32): \ \xi_{\min} > 0 \wedge \xi_{\min} < \frac{1}{2} \wedge \xi_{\max} \geq \frac{1}{2} \wedge \varphi(\xi_{\min}) > 0. \tag{2.19}$$

For $\Delta < 0$ the above conditions define the admissible set in $\varepsilon_1\varepsilon_2\varepsilon_3$ space, i.e. the set in which the radius of cross-section of the beam is positive.

The beam under consideration is axially loaded by a non-conservative force of the form

$$P(t) = P_0 + P_1 \cos\vartheta t, \tag{2.20}$$

where t is time and P_0, P_1, ϑ are positive constants. The beam interacts with a foundation of Winkler, Pasternak or Hetényi type with damping. A study of different foundation models has been presented by Kerr [11].

The following dimensionless quantities are introduced [1, 12]:

$$\begin{array}{lll}
\upsilon = \dfrac{w}{l}, & \tau = (\pi/2l^2)\sqrt{\pi EV/\rho l}t, & \alpha = 4l^4P_0/\pi EV^2,\\
\beta = 4l^4P_1/\pi EV^2, & \varLambda = (\pi\lambda/2l^2)\sqrt{\pi V/\rho lE}, & \kappa = 4kl^6/\pi^3EV^2,\\
\mu = 4Gl^4/\pi EV^2, & \theta = (2l^2/\pi)\sqrt{\rho l/\pi EV}\vartheta, & \gamma = (2cl^4/\pi^2EV^2)\sqrt{\pi EV/\rho l},\\
\delta = 4\pi Dl^2/EV^2, & f \equiv f(\varepsilon_1, \varepsilon_2, \varepsilon_3), & \varphi \equiv \varphi(\xi; \varepsilon_1, \varepsilon_2, \varepsilon_3),
\end{array}$$

where $w(x,t)$ is the transverse displacement of the cross-section in the space coordinate x at the time t, E is Young's modulus, λ and c are the coefficients of internal and external damping respectively, ρ is the mass density of the beam, k is the foundation stiffness per unit length, G is the foundation modulus and D is the foundation flexural stiffness.

The equation of the transverse vibrations of the beam on its foundation has the form [1, 12]

$$\begin{aligned}
\frac{1}{\pi^2}f^2\frac{\partial^2}{\partial\xi^2}\left(\varphi^4\frac{\partial^2\upsilon}{\partial\xi^2} + \varLambda\varphi^4\frac{\partial^3\upsilon}{\partial\xi^2\partial\tau}\right) + \alpha\frac{\partial^2\upsilon}{\partial\xi^2} + \beta\frac{\partial^2\upsilon}{\partial\xi^2}\cos\theta\tau \\
+ \pi^2 f\varphi^2\frac{\partial^2\upsilon}{\partial\tau^2} + \pi^2\kappa\upsilon + \pi^2\gamma\frac{\partial\upsilon}{\partial\tau} + \frac{\delta}{\pi^2}\frac{\partial^4\upsilon}{\partial\xi^4} - \mu\frac{\partial^2\upsilon}{\partial\xi^2} = 0,
\end{aligned} \tag{2.21}$$

where $\mu \equiv \delta \equiv 0$ for a Winkler foundation, $\delta \equiv 0$ for a Pasternak foundation and $\mu \equiv 0$ for a Hetényi foundation respectively.

It is assumed that the two ends of the beam are simply supported:

$$\begin{aligned}
&\upsilon(0,\tau) = 0, \quad [\varphi^4(\partial^2\upsilon/\partial\xi^2 + \Lambda\partial^3\upsilon/\partial^2\xi\partial\tau)](0,\tau) = 0,\\
&\upsilon(1,\tau) = 0, \quad [\varphi^4(\partial^2\upsilon/\partial\xi^2 + \Lambda\partial^3\upsilon/\partial^2\xi\partial\tau)](1,\tau) = 0.
\end{aligned} \tag{2.22}$$

From the Eq. (2.21) with the boundary conditions (2.22) one can determine the first instability region for the beam under consideration. Parametric optimization of the shape of the beam consists in finding those admissible values of the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for which the value of P_1 i.e. the oscillatory component of the loading force, causing beam's instability, is maximal.

3. Solution of the problem

The problem is approximately solved by the Galerkin method; cf. [1, 2]. Therefore one looks for the solution of Eq. (2.21) in the form

$$v(\xi,\tau) = \sum_{n=1}^{N} q_n(\tau) \sin n\pi\xi \tag{3.1}$$

and obtains the set of ordinary differerential equations for the unknown functions $q_n(\tau)$

$$\sum_{k=1}^{N} (A_{nk}\ddot{q}_k + B_{nk}\dot{q}_k + C_{nk}q_k + D_{nk}q_k \cos\theta\tau) = 0, \qquad n = 1, ..., N, \tag{3.2}$$

where

$$\begin{aligned}
A_{nk} &= f(\varepsilon_1,\varepsilon_2,\varepsilon_3) \int_0^1 \varphi^2(\xi;\varepsilon_1,\varepsilon_2,\varepsilon_3) \sin n\pi\xi \sin k\pi\xi d\xi,\\
B_{nk} &= \frac{1}{2}\gamma\delta_{nk} + \Lambda n^2 k^2 f^2(\varepsilon_1,\varepsilon_2,\varepsilon_3) \int_0^1 \varphi^4(\xi;\varepsilon_1,\varepsilon_2,\varepsilon_3) \sin n\pi\xi \sin k\pi\xi d\xi,\\
C_{nk} &= \frac{1}{2}(\kappa + \mu n^2 + \delta n^4 - \alpha n^2)\delta_{nk}\\
&\quad + n^2 k^2 f^2(\varepsilon_1,\varepsilon_2,\varepsilon_3) \int_0^1 \varphi^4(\xi;\varepsilon_1,\varepsilon_2,\varepsilon_3) \sin n\pi\xi \sin k\pi\xi d\xi,\\
D_{nk} &= -\frac{1}{2}\beta n^2 \delta_{nk}.
\end{aligned} \tag{3.3}$$

Here δ_{nk} is the Kronecker delta.

In further considerations only the first two Eqs. (3.2) are retained; these are equations for the functions $q_1(\tau), q_2(\tau)$. From these two equations the boundaries of the first instability region of the beam are determined. The instability region occurs in the neighbourhood of twice the value of the first natural frequency of the beam [3, 4].

To determine the boundaries of the first instability region one assumes the solution of Eqs. (3.2) in the following form [13, 14]:

$$q_k(\tau) = A_k \sin\frac{\theta\tau}{2} + B_k \cos\frac{\theta\tau}{2}, \qquad k = 1, 2, \tag{3.4}$$

where A_k, B_k are constants. After inserting (3.4) into the system of Eq. (3.2), a system of four algebraic linear homogeneous equations for A_k, B_k is obtained. The non-zero solution of these equations exists if the determinant of the system equals zero. This leads to the biquadratic equation for dimensionless amplitude β of the oscillating component of loading, in the form ([1])

$$\frac{1}{16}\beta^4 - \left[h_{11}^2 + \frac{1}{16}h_{22}^2 + (\theta^2/4)B_{11}^2 + (\theta^2/64)B_{22}^2\right]\beta^2 + h_{11}^2 h_{22}^2 + (\theta^2/4)\left[h_{11}^2 B_{22}^2 + h_{22}^2 B_{11}^2 + (\theta^2/4)B_{11}^2 B_{22}^2\right] = 0, \tag{3.5}$$

where

$$h_{11} = -(\theta^2/4)A_{11} + C_{11}, h_{22} = -(\theta^2/4)A_{22} + C_{22}. \tag{3.6}$$

From Eq. (3.5) one determines the boundaries of the first instability region i.e. the critical value of β as a function of θ. Inside the instability region the critical value attains its minimal value given by the formula

$$\beta_{\min} = 4\left|B_{11}\right|\sqrt{\frac{C_{11}}{A_{11}} - \frac{{B_{11}}^2}{4A_{11}^2}}. \tag{3.7}$$

The critical value of β depends on the values of optimization parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$. The shape of the beam is optimal if the value of $\beta_{\min}$ is maximal.

The results of paper [2] show one difficulty: in many cases the radius of the cross-section of the beam attains very small values. This fact undoubtedly questions the obtained results. Thus in the present paper the following strength condition is added – one assumes that the maximal stress, i.e. the stress, at the smallest cross-section of the beam, does not exceed the limity stress σ_0 of the material:

$$\frac{P_0 + P_1}{\pi r_{\min}^2} \le \sigma_0. \tag{3.8}$$

4. Parametric optimization of the shape of the beam

To realize a few numerical calculations one assumes: $E = 2.1 \cdot 10^{11}$ Pa, $\sigma_0 = 2.2 \cdot 10^8$ Pa, $V/l^3 = 10^{-4}$. Therefore the strength condition (3.8) takes the form

$$\frac{\alpha+\beta}{f(\varepsilon_1,\varepsilon_2,\varepsilon_3)\varphi^2_{\min}} \le 13.37. \tag{4.1}$$

The numerical calculations have been performed for $\varepsilon_k \in [-10, 10]$, $k = 1, 2, 3$ with step 0.1 and the maximum of $\beta_{\min}$ has been found out.

The following denotations are adopted:

Winkler model ($\mu = \delta = 0$)

case 1: $\gamma = 0.03$, $\kappa = 0.1$, $\alpha = 0.5$, $\Lambda = 0.01$,

case 2: $\gamma = 0.1$, $\kappa = 0.1$, $\alpha = 0.5$, $\Lambda = 0.01$,

case 3: $\gamma = 0.1$, $\kappa = 0.1$, $\alpha = 0.8$, $\Lambda = 0.01$.

Pasternak model ($\delta = 0$)

case 4: $\gamma = 0.005$, $\kappa = 0.1$, $\mu = 0.2$, $\alpha = 0.5$, $\Lambda = 0.002$,

case 5: $\gamma = 0.1$, $\kappa = 0.1$, $\mu = 0.2$, $\alpha = 0.5$, $\Lambda = 0.01$.

Hetényi model ($\mu = 0$)

case 6: $\gamma = 0.005$, $\kappa = 0.1$, $\delta = 0.2$, $\alpha = 0.5$, $\Lambda = 0.002$,

case 7: $\gamma = 0.1$, $\kappa = 0.1$, $\delta = 0.2$, $\alpha = 0.5$, $\Lambda = 0.01$.

The results are tabulated and illustrated on the graphs which are shown in Figs. 6–10, where

$$R(\xi) = r(\xi)/l. \tag{4.2}$$

For comparison in Figs. 8–10 the graphs obtained in virtue of reference [2] are shown by dashed lines.

Case	ε_1	ε_2	ε_3	β_{opt}
1	−1.0	0.2	9.9	0.382
2	−1.1	0.5	9.5	0.623
3	−1.2	0.3	10.0	0.584
4, 6	−3.4	4.2	8.8	0.316
5, 7	−3.2	4.4	7.4	1.404

The optimal shape of the beam depends on the values of parameters describing materials of the beam and foundation. The optimal shape of the beam is not a universal one.

Sensitivity of the shape of the beam i.e. $R(\xi)$ to the values of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is presented by Fig. 11 where the graphs for optimal case and for $\varepsilon_k - 0.1$, $k = 1, 2, 3$ are shown.

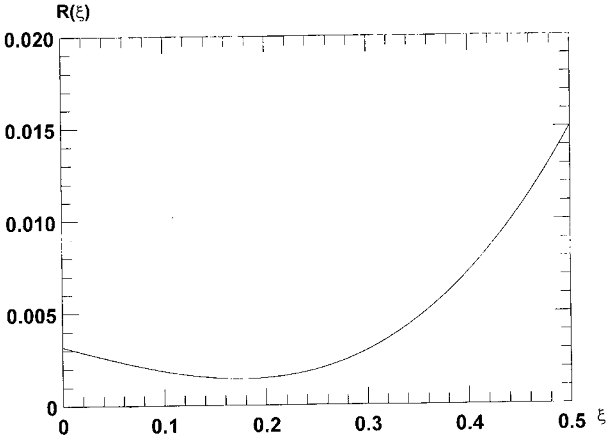

FIG. 6. The graph $R(\xi)$, case 1.

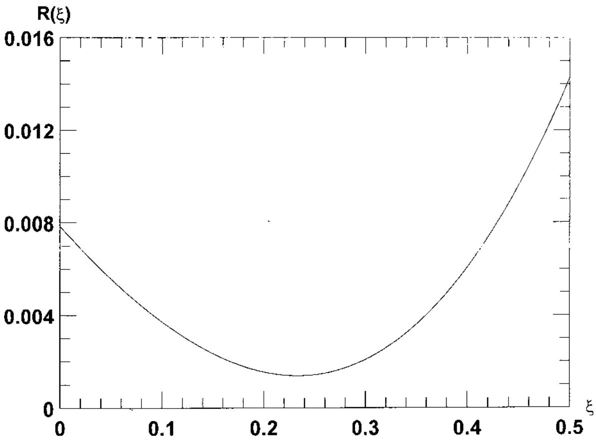

FIG. 7. The graph $R(\xi)$, case 4, 6.

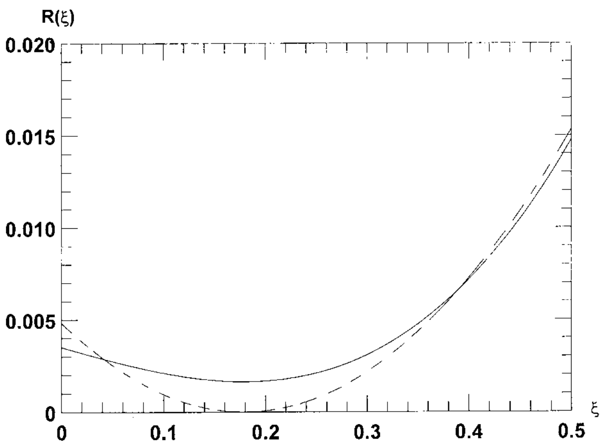

FIG. 8. The graph $R(\xi)$, case 2.

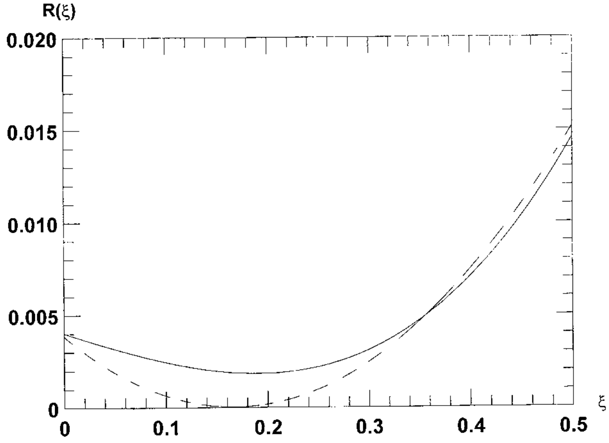

FIG. 9. The graph $R(\xi)$, case 3.

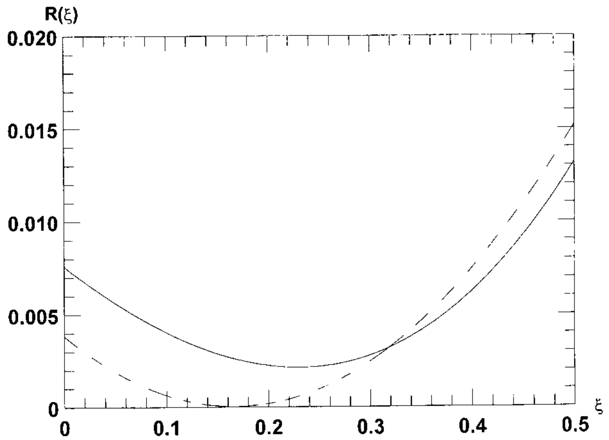

FIG. 10. The graph $R(\xi)$, case 5, 7.

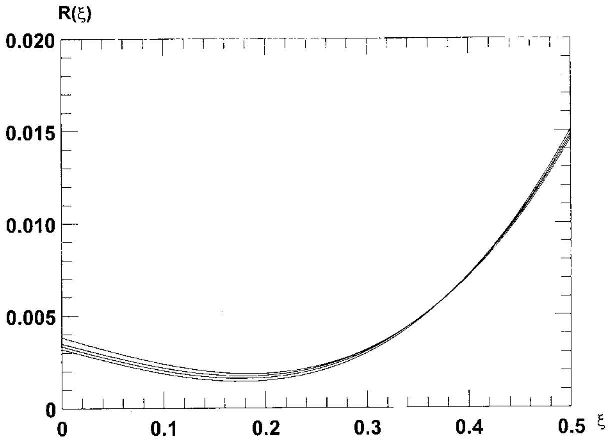

FIG. 11. The graph $R(\xi)$, case 1, for different ε_k.

5. Final remarks

The parametrical optimization of an axially loaded viscoelastic beam on a foundation has been discussed. The radius of the cross-section of the beam is a cubic function of co-ordinate. The beam performs transverse vibration and interacts with a foundation of Winkler, Pasternak or Hetényi type. The values of three optimization parameters defining optimal shape of the beam are calculated for a number cases. Results are shown on the graphs.

The results of the paper are the extension and confirmation of the results obtained in the previous papers [1, 2].

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