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ON THE DETERMINATION AND USE OF PRINCIPAL LINES IN PLASTIC FLOW

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Assuming plane strain for Huber-Mises yield condition (for a Coulomb-Tresca one, both plane strain and stress are admitted) of incompressible homogeneous isotropic ideal plastic medium, basic relation of principal lines (of curvatures and their derivatives), consisting with known relations is derived. It leads, in the case of family of curves translated to one another, to an ordinary differential equation of the second order in y_1' where $y_1(x_1)$ denotes principal line. The solution of this equation is known (the respective integrals are tabulated in the paper) and thus the result — fully consisting with the well-known Prandtl solution is obtained by the direct method. Similarly in the case of family of homothetic principal lines analogical (although more complicated) equation is obtained in $t=dq/d\varphi$, where $q=\ln r$ and r, φ are polar coordinates. In this case the principal lines are directly proved to be logarithmic spirals with an arbitrary slope with respect to radii vectors (the latter case is degenerated one) or other spirals — that is presumably a "new" solution — which, for some values of parameters, may be fairly good approximated by Galileo spirals. Calculation and displaying have been performed by means of ODRA 1204 computer. A brief note on possible degenerations is added.

Introduction

The principal lines in axially symmetric three dimensional problems of the ideal plastic medium have been considered by H. Lippmann [1, 2]. In the present paper it will be shown that at least in two simple cases principal lines may be used for direct solution of the plane problem and, moreover, in the second case, this leads, apart from the known solutions (as in the first case), to a "new" one.

2. Assumptions

Homogeneous isotropic ideal plastic medium (body forces neglected), plane strain (or some cases of plane stress) with Coulomb-Tresca yield condition, alternatively, plane strain of incompressible medium with C.-T. or Huber-Mises yield condition. Moreover, all functions specified are assumed to be differentiable as shown in the text and, on the basis of some analysis, principal lines are assumed to be in translation (Sect. 4) or homothety (Sect. 5) with respect to each other.

3. General relation

Consider the two Lamé-Maxwell equations of the form:

$$(3.1)\qquad \frac{\partial\sigma_i}{\partial s_i}=\frac{\sigma_i-\sigma_j}{\rho_j},\qquad i,j=1,2,\qquad i\neq j,$$

where σ_i, σ_j denote principal stresses, s_i and ρ_j — length of the arc and curvature radius of trajectory of principal stress marked by the respective index. (Sign convention: cf [3, 4] and note: $ds_i = -d\rho_j$). Let us take into account $\sigma_1 - \sigma_2 = k$ (cf. Sect. 2.), observing that for the Huber-Mises yield condition $k = 2\sigma_0/\sqrt{3}$, whereas for the Coulomb-Tresca yield condition $k = \sigma_0$, where σ_0 is yield strength in the tensile test. We obtain:

$$\frac{\partial \sigma_i}{\partial s_i} = -(-1)^i k\kappa_j, \tag{3.2}$$

where κ_j denotes curvature of the trajectory of σ_j. Differentiation of the Eq. (3.2) gives

$$\frac{\partial^2 \sigma_i}{\partial s_i\, \partial s_j} = -(-1)^i k \frac{\partial \kappa_j}{\partial s_j}. \tag{3.3}$$

Observe that in general

$$\frac{\partial^2 \sigma_i}{\partial s_i\, \partial s_j} \neq \frac{\partial^2 \sigma_i}{\partial s_i\, \partial s_j}.$$

On introducing Lamé's coefficients H_1 and H_2, in view of

$$\frac{\partial^2 \sigma_1}{\partial \alpha_1\, \partial \alpha_2} = \frac{\partial^2 \sigma_1}{\partial \alpha_2\, \partial \alpha_1},$$

we have:

$$H_{1,2} \frac{\partial \sigma_1}{\partial s_1} + H_1 H_2 \frac{\partial^2 \sigma_1}{\partial s_1\, \partial s_2} = H_{2,1} \frac{\partial \sigma_1}{\partial s_2} + H_2 H_1 \frac{\partial^2 \sigma_1}{\partial s_2\, \partial s_1}, \tag{3.4}$$

where $H_{1,2}$ and $H_{2,1}$ denote derivatives of H_1 and H_2 with respect to the curvilinear coordinates α_2 and α_1. Substituting the known relation $\kappa_i = -\frac{H_{i,j}}{H_i H_j}$ (cf. [3, 4]) in the Eq. (3.2) and equating k found from the two equations thus obtained, we have

$$\frac{\partial \sigma_1}{\partial s_1} \frac{H_{1,2}}{H_1 H_2} = -\frac{\partial \sigma_1}{\partial s_2} \frac{H_{2,1}}{H_1 H_2}.$$

Substitution in the Eq. (3.4) after taking into account the Eq. (3.3), gives

$$\frac{\partial \kappa_1}{\partial s_1} + \frac{\partial \kappa_2}{\partial s_2} = 2\kappa_1 \kappa_2. \tag{3.5}$$

This relation follows directly — in the case of homogeneity — from more general considerations of an inhomogeneous medium ([5] 2.42, p. 47).

It may be seen that, if we adopt rectangular coordinates, the relation represented by the Eq. (3.5) does not involve ordinates y, y_1 of the two principal lines orthogonal to each other, but only their derivatives. However, since two orthogonal families of curves intervene, in general, integral constants are also involved.

4. THE SIMPLEST CASE — A FAMILY OF CURVES TRANSLATED TO EACH OTHER

In two cases at least — closely related with each other — the integral constants vanish in the process of differentiation. The first is when one of the families considered is of the form $y=f(x)+c$, since y', y'', y''' do not depend on c. Thus, simple calculations give the direct relations of y', y'', y''' with the respective ones y_1', y_1'', y_1''' of the orthogonal family. Consequently, bearing in mind the well-known relations $s' \equiv \frac{ds}{dx} = \sqrt{1+y'^2}$ and $\kappa = \frac{y''}{(1+y'^2)^{3/2}}$, we deduce: $s'' = \frac{y''y'}{s'}$ and

$$\frac{\partial \kappa}{\partial s} = \frac{s'^2 y''' - 3y''^2 y'}{s'^6} \tag{4.1}$$

(with omission of the index $i=1, 2$ (not written) which is the same for all symbols). Further, we obtain:

$$y'' = \frac{y_1''}{y_1'^2}, \qquad y''' = \frac{y_1' y_1''' - 2y_1''^2}{y_1'^3}, \qquad s' = \frac{s_1'}{y_1'},$$

$$\kappa = \frac{y_1'' y_1'}{s_1'^3}, \qquad \frac{\partial \kappa}{\partial s} = \frac{(1+y_1'^2)(y_1' y_1''' - 2y_1''^2)y_1' + 3y_1''^2 y_1'}{s_1'^6}.$$

As regards $\partial\kappa/\partial s$, note that there are two ways of calculation, both leading to the same result. In the sign convention of the Eq. (3.5) (cf. Eq. (3.1)) when replacing in the Eq. (4.1) and in the last formula for κ, quantities denoted with indices by corresponding quantities denoted without indices and vice versa, the sign of one member of the equation should be changed. Consequently, the Eq. (3.5) in the rectangular Cartesian frame takes a form as if only the sign of the term $\partial\kappa/\partial s$ were changed. The functions of y_1', y_1'', and y_1''' obtained determine all quantities of the Eq. (3.5). Thus substitution in that equation eliminates y', y'', y''' and leads, by means of simple calculation, to the relation determining the principal lines in the case considered:

$$(1-z^4)z'' = 2(3-z^2)zz'^2, \quad \text{where} \quad z = y_1'. \tag{4.2}$$

The correctness of the Eq. (4.2) is seen by its fulfilling two conditions. The first — that is the condition of orthogonality — is proved by the fact that two forms of the Eq. (4.2), namely after two substitutions: 1) $z=w$, 2) $z=-\frac{1}{w}$ are identical. (In case 2) we have: $z' = \frac{w'}{w^2}$, $z'' = \frac{ww''-2w'^2}{w^3}$.

The second necessary condition is that the Eq. (4.2) should be verified by the principal trajectories of the system, where slip lines in each family are cycloids translated parallelly to each other along the coordinate line y (Prandtl solution).

Observe that the slope of the cycloid is:

$$y_s' = \pm\sqrt{\frac{h \pm x}{h \mp x}} \tag{4.3}$$

and that of principal lines is:

$$z \equiv y' = \frac{y'_s \mp 1}{1 \pm y'_s}. \tag{4.4}$$

Thus, assuming in the Eq. (4.3) "the upper signs", we find:

$$z = \frac{h - \sqrt{h^2 - x^2}}{x}, \qquad z' = \frac{h(h - \sqrt{h^2 - x^2})}{x^2 \sqrt{h^2 - x^2}},$$

$$z'' = \frac{2h(h^2 - x^2)^{3/2} - 2h^2(h^2 - x^2) + h^2 x^2}{x^3 (h^2 - x^2)^{3/2}}.$$

Substitution of these quantities into the Eq. (4.2) turns all its terms into zeros, which means that the Eq. (4.2) is verified by the principal line of the Prandtl solution. The calculations are facilitated by the relation: $1 + z^2 = 2z' \sqrt{h^2 - x^2}$. Assumption of „other signs" in the Eq. (4.3) leads to the same result, since they refer to curves with the same symmetry as that inherent in the Eq. (4.2) (cf. signs of z and z''). The symmetries of curves $y = f(x)$ with respect to the axes y and x are geometrically obvious, and tacitly admitted by the very formulation of the problem.

The Eq. (4.2) may be solved (cf. [6], p. 651) and gives

$$|z'| = c\, \frac{(1+z^2)^2}{1 - z^2}, \tag{4.5}$$

$$x = \frac{1}{c} \int \frac{1 - z^2}{(1+z^2)^2}\, dz = \frac{1}{c}\, \frac{z}{1+z^2} + c_1. \tag{4.6}$$

For a fixed value of c, sgn $(z'x) = \text{const}$ (cf. Eq. (4.5)), whence also it may be seen that 1) $y(x) = y(-x)$, 2) $y_a(x) = -y_b(x)$, where $y = y_a, y_b$, 3) $z(-x) = -z(x)$. The two forms with various signs of z' are symmetric with respect to the axis x, but in the given region only one of these forms may appear. It suffices to assume one of these—for instance, $z' < 0$. We have $z = 0$ for $x = 0$, whence (cf. Eq. (4.6)) $c_1 = 0$. Then for $x_1 \to 0$, $z_1 \to \infty$ and also $c_1 = 0$. Thus the Eq. (4.6) may be rewritten in the form:

$$z^2 + \frac{z}{cx} + 1 = 0, \tag{4.7}$$

with the discriminant $\Delta = \frac{1}{c^2 x^2} - 4$, whence, in view of $-1 \leqslant x \leqslant 1$: $\Delta \geqslant 0$ for $|c| \leqslant \frac{1}{2}$. Denoting the roots of the Eq. (4.7) $z_{\mathrm{I}}, z_{\mathrm{II}}$, we have $z_{\mathrm{I}} z_{\mathrm{II}} = 1$. It follows that any pair of corresponding roots refers to the curves of different families; not, however, at the same point, but at two different symmetric ones (x, y and $-x, y$). On the other hand, the change of the sign of c alone (or x alone) leads to the change of the sign of both roots of the Eq. (4.7) (Cf. $z_{\mathrm{I}} + z_{\mathrm{II}}$ and $z_{\mathrm{I}} z_{\mathrm{II}}$). Thus we may replace the Eq. (4.7), where c proved to be ambivalent, by

$$z^2 \pm \frac{z}{cx} + 1 = 0, \tag{4.8}$$

where c has a fixed value. The Eq. (4.8) represents explicitely two coupled equations A and B, with roots z_{AI}, z_{AII}, z_{BI}, z_{BII}, where $z_{AI}=-z_{BI}$ and $z_{AII}=-z_{BII}$. It follows $z_{IA}\, z_{IIB}=-1$ (and $z_{IB}\, z_{IIA}=-1$) — that is, these are pairs of roots referring to the orthogonal lines at a given point. This agrees with the concrete value of c as determined from boundary conditions. From the condition of antisymmetry (or symmetry), for one family of curves for $x=0$, that is in the central plane of compressed and sheared layer we have $z=0$, and on its boundary for $x=\pm 1$ we have $z=\mp 1$ (bearing in mind $z'<0$) and $z=\max$ as in the Prandtl solution. Thus, from the Eq. (4.7) for $x=\pm 1$ it follows that $cx=\pm\frac{1}{2}$, and in both cases $c=\frac{1}{2}$. Substituting this in the Eq. (4.7), we have

$$xz^2+2z+x=0, \quad z=\frac{-1\pm\sqrt{1-x^2}}{x}=y'. \tag{4.9}$$

Further integration gives

$$y_1=\sqrt{1-x^2}-\ln\left(1+\sqrt{1-x^2}\right)+C, \tag{4.10}$$

$$y_2=-2\ln|x|-\sqrt{1-x^2}+\ln\left(1+\sqrt{1-x^2}\right)+C. \tag{4.11}$$

The curve orthogonal with respect to $y_1=f(x)$ is obtained from the coupled equation (cf. Eqs. (4.7) and —(4.8)) or by changing the sign: of y_2 Eq. (4.11) (+ into − and vice versa). Since the functions $y_1(x)$ and $y_2(x)$ seem not to be tabulated, the respective values as found by the computer are listed in Table 1. The respective graphs are presented in Fig. 1. Note from the Eq. (4.6) that all curves from the family $x=F_1(z)$ may be obtained from each other by simple compression or extension (change of c) and translation (change of c_1). From the Eq. (4.6) and $y=\int z\,dx+c_2$, it may be seen that for a given direction of the plate (layer) c, c_1, c_2 have no effect on the form of the principal lines (except their prolongation) but only on their translation in directions x and y (change of c_1 and c_2 respectively), and on the scale of the graph (homothety). In fact, any change of c causes change of coordinates x and y in the same ratio, $z=dy/dx$ being unchanged.

It should be stated that the solution obtained in the present section is, in principle, apart from minor additions, the same as that by Prandtl; the method adopted is, however, new one — the direct method, based on the use of principal lines.

The three integration constants determining $y=f(x)$ are to be found from three conditions (cf. Eq. (4.2)). In particular cases, when the coordinate of the plane of symmetry or antisymmetry (plate in compression and in shear respectively) is known, the two first conditions are as follows: on the symmetry plane 1) for $x=0$ $y=0$ 2) for $x=0$ $y'=0$. In general, conditions are as follows: 1) on the surface of the plate (layer) for $x=0$, $y=0$; 2) on the same boundary $(x=0)$ $y'=y'_0$; 3) on the other boundary for $x=2a$ (or plane of antisymmetry $x=a$) $y'=y'_a$, where y'_0 and y'_a are calcualted from respective Mohr circles, or, alternatively, determined by photoelastic observation on a strip of finite width.

Certain generalizations are possible which, although they affect to some extent the general assumptions, may be taken into account by respective modifications

Table 1. Functions $y_1 = \sqrt{1-x^2} - \ln(1+\sqrt{1-x^2})$ and $y_2 = -2\ln|x| - \sqrt{1-x^2} + \ln(1+\sqrt{1-x^2})$ (see Eqs. (4.10), (4.11), and Fig. 1).

$\pm x$	y_1	y_2	$\pm x$	y_1	y_2
1.00	0.000000	0.0000000	0.49	−1.1818	−0.24486
0.99	−0.010997	−0.0091033	0.48	−1.2205	−0.24745
0.98	−0.022894	−0.017512	0.47	−1.2601	−0.24998
0.97	−0.035426	−0.025493	0.46	−1.3006	−0.25244
0.96	−0.048504	−0.033140	0.45	−1.3422	−0.25485
0.95	−0.062080	−0.040507	0.44	−1.3848	−0.25720
0.94	−0.076122	−0.047629	0.43	−1.4285	−0.25949
0.93	−0.090610	−0.054532	0.42	−1.4733	−0.26172
0.92	−0.10553	−0.061235	0.41	−1.5193	−0.26389
0.91	−0.12087	−0.067756	0.40	−1.5666	−0.26601
0.90	−0.13662	−0.074105	0.39	−1.6152	−0.26807
0.89	−0.15277	−0.080295	0.38	−1.6651	−0.27007
0.88	−0.16933	−0.086334	0.37	−1.7165	−0.27201
0.87	−0.18629	−0.092230	0.36	−1.7694	−0.27390
0.86	−0.20366	−0.097990	0.35	−1.8239	−0.27574
0.85	−0.22142	−0.10362	0.34	−1.8801	−0.27752
0.84	−0.23958	−0.10913	0.33	−1.9381	−0.27924
0.83	−0.25815	−0.11451	0.32	−1.9980	−0.28091
0.82	−0.27712	−0.11978	0.31	−2.0598	−0.28253
0.81	−0.29650	−0.12494	0.30	−2.1239	−0.28409
0.80	−0.31629	−0.13000	0.29	−2.1901	−0.28560
0.79	−0.33650	−0.13494	0.28	−2.2589	−0.28706
0.78	−0.35713	−0.13979	0.27	−2.3302	−0.28846
0.77	−0.37819	−0.14454	0.26	−2.4043	−0.28981
0.76	−0.39968	−0.14919	0.25	−2.4815	−0.29110
0.75	−0.42161	−0.15375	0.24	−2.5619	−0.29235
0.74	−0.44399	−0.15822	0.23	−2.6458	−0.29354
0.73	−0.46682	−0.16260	0.22	−2.7336	−0.29468
0.72	−0.49011	−0.16690	0.21	−2.8255	−0.29577
0.71	−0.51388	−0.17110	0.20	−2.9221	−0.29680
0.70	−0.53812	−0.17523	0.19	−3.0237	−0.29779
0.69	−0.56285	−0.17927	0.18	−3.1309	−0.29872
0.68	−0.58809	−0.18324	0.17	−3.2443	−0.29960
0.67	−0.61384	−0.18712	0.16	−3.3647	−0.30043
0.66	−0.64010	−0.19093	0.15	−3.4930	−0.30121
0.65	−0.66691	−0.19466	0.14	−3.6303	−0.30194
0.64	−0.69426	−0.19831	0.13	−3.7778	−0.30262
0.63	−0.72217	−0.20190	0.12	−3.9373	−0.30325
0.62	−0.75067	−0.20541	0.11	−4.1107	−0.30382
0.61	−0.77975	−0.20885	0.10	−4.3008	−0.30435
0.60	−0.80944	−0.21221	0.09	−4.5111	−0.30483
0.59	−0.83975	−0.21551	0.08	−4.7462	−0.30525
0.58	−0.87071	−0.21874	0.07	−5.0129	−0.30563
0.57	−0.90233	−0.22190	0.06	−5.3209	−0.30595
0.56	−0.93464	−0.22500	0.05	−5.6852	−0.30623
0.55	−0.96764	−0.22803	0.04	−6.1313	−0.30645
0.54	−1.0014	−0.23099	0.03	−6.7065	−0.30663
0.53	−1.0359	−0.23389	0.02	−7.5173	−0.30675
0.52	−1.0711	−0.23673	0.01	−8.9035	−0.30683
0.51	−1.1072	−0.23950	0.00	$-\infty$	−0.30685
0.50	−1.1441	−0.24221			

of boundary conditions. These generalizations are: 1) residual stresses in thin layers of the plate parallel to its surface; 2) consideration of the weight of the vertical plate (or layer).

Numerical example. From the Eq. (4.6) for $c=1$ and $c_1=0$, we have $x=F(z)\equiv x_r$. For $y_0'=0.165$ and $y_a'=0.25$ (cf. Table 1 and Fig. 1), we have $x_{r0}=0.310702$, $y_0=$

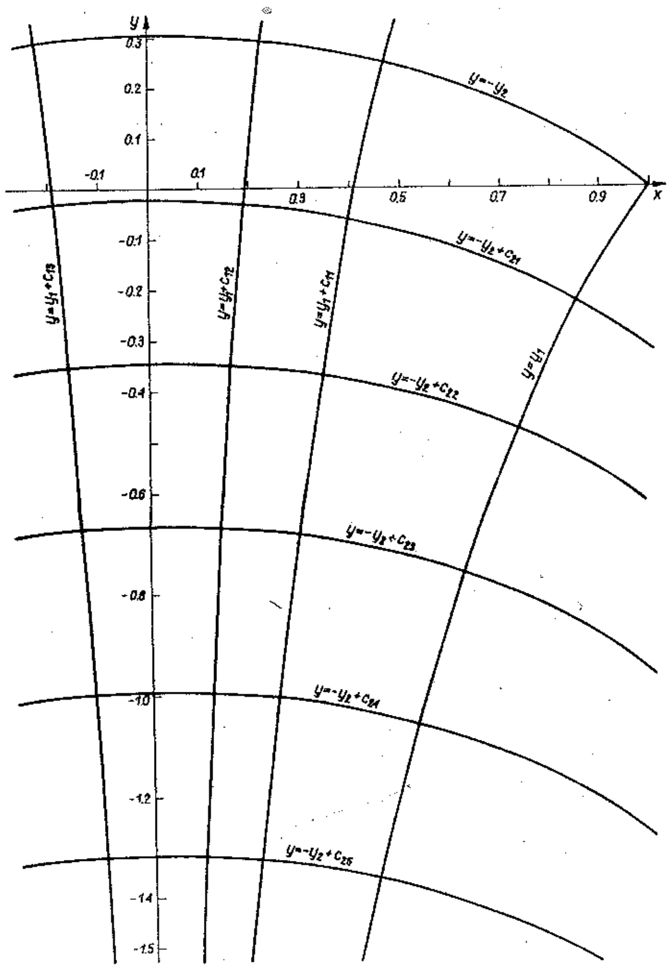

FIG. 1.

$=0.049756$; $x_{ra}=0.441542$, $y_{ra}=0.103715$. Taking into account $x=cx_r+c_1$ and the Eq. (4.6), we have $\pm 0.310702\,c+c_1=0$ and $\pm 0.441542\,c+c_1=a$, whence c and c_1 are determined. Since $y'=y_r'$, we may write $y_0=cy_{r0}=0.381a$; $y_a=cy_{ra}=$ $=0.785\,a$.

In simple problem of plastic layer, the stress is fully determined by the principal direction. In fact, from the equilibrium equation, we have $\sigma_{m\,\mathrm{surf}}+k\cos 2\alpha_{\mathrm{surf}}=$ $=\sigma_{m\,\mathrm{sect}}+k\cos 2\alpha_{\mathrm{sect}}$ where σ_m denotes mean normal stress, $k\cos\alpha$ presents a normal stress due to the deviator, $\tan\alpha$ is the slope of principal line, and the subscript „surf" denotes quantities on the surface, while "sect" denotes respective quantities in an arbitrary section parallel to the surface.

5. Homothetic family of principal lines

Translation of the curves (Sect. 4) may be regarded as homothety with the centre in infinity. Replacement of the rectangular Cartesian frame by a polar one, associated with such change of principal lines that they from homothety with the centre in the origin of polar reference frame, gives generalization of the problem. Thus we have

$$cr = f(\varphi), \tag{5.1}$$

where r, φ are polar coordinates. It may be seen that the slope, being in general

$$\frac{dr}{r\,d\varphi} = \frac{d(\ln r)}{d\varphi}, \tag{5.2}$$

in this case (cf. Eq. (5.1)) is expressed by:

$$\frac{d(\ln r)}{d\varphi} = \frac{d[\ln f(\varphi)]}{d\varphi}, \tag{5.3}$$

that is, depends only on φ while c vanishes as constant summand similarly, as in the case of translation (cf. Sect. 4). This means that in this respect the transformation introduced is equivalent, apart from the transformation of the coordinates, to the replacement of variables by their natural logarithms. It eliminates, in particular, coefficient $1/r$ (cf. Eq. (5.2)) Confronting the result with that of Sect. 4., we note that $(\ln r)'$ replaces y' and $[\ln f(\varphi)]'$ replaces $f'(x)$. Consequently, independence of the results of the integral constants of other family, as stated in Sect. 4., holds also in this case, with variables $x, y, f(x)$ replaced by φ, $\ln r$, $\ln f(\varphi)$, respectively.

However, in view of the change of coordinates, and thus of geometry, calculation analogical to that of Sect. 4. should be carried out.

On using the orthogonality condition $r_1'/r_1 = -r/r'$, where subscript 1 and its omission have meaning as in Sect. 4., and on denoting $\ln r = q$, $\ln r_1 = q_1$, we obtain:

$$q' = -\frac{1}{q_1'}, \quad q'' = \frac{q_1''}{q_1'^2}, \quad q''' = \frac{q_1' q_1''' - 2q_1''^2}{q_1'^3}, \quad \kappa = \frac{1+q'^2-q''}{e^q(1+q'^2)^{3/2}}, \tag{5.4}$$

$$\kappa_1 = q'\kappa$$

and

$$\frac{\partial\kappa}{\partial s} = \frac{-(1+q'^2)(q'''+q'+q'^3)+3q'q''^2}{e^{2q}(1+q'^2)^3}. \tag{5.5}$$

On calculating, analogically to Sect. 4, in two ways, we obtain an identical result:

$$\frac{\partial\kappa_1}{\partial s_1} = q'\frac{(1+q'^2)(-q'q'''+2q''^2+q'^2+1)-3q''^2}{e^{2q}(q'^2+1)^3}. \tag{5.6}$$

(Difficulties arising in correct determination of signs of $\partial\kappa/\partial s$ and $\partial\kappa_1/\partial s_1$, for substitutions into the Eq. (3.5), are avoided by means of the condition of orthogonality). This leads to the Eq.:

$$(1-t^2)\,t''+\left(1-\frac{4}{1+t^2}\right)2tt'^2+4tt'=0, \tag{5.7}$$

which by substitution $p(t)=t'$, and division by p (cf. [6]) may be reduced to:

$$p'+2\,\frac{t^2-3}{1-t^4}\,tp+\frac{4t}{1-t^2}=0. \tag{5.8}$$

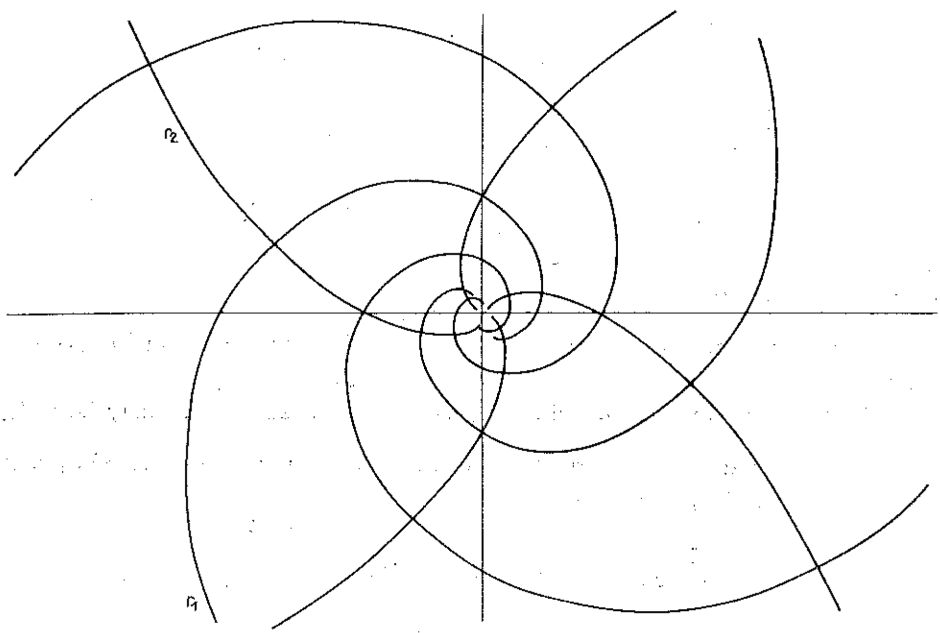

FIG. 2. $r=ae^{k\varphi}$, $r_1=ae^{2\varphi}$, $r_2=ae^{-0.5\varphi}$.

The solution $p=0$ (excluded in the above treatment) gives two orthogonal families of logarithmic spirals with an arbitrary slope, with respect to the radii--vectors, of one family (Fig. 2). In particular, in the case where the slope equals $\pi/4$, it is a trivial, well-known solution. Let us, however, confront the well-known relations for curvature radius R, length of the arc L, $dR_i/d\varphi$, $dL/d\varphi$, where index i refers to the involute being also a logarithimic spiral (see Fig. 3)). It may be proved, that $dL/d\varphi=dR_i/d\varphi$. This suffices to show that spirals of the Eq.: $\rho=ae^{m\varphi}$ form Hencky-Prandtl trajectories for any m (cf. [3]).

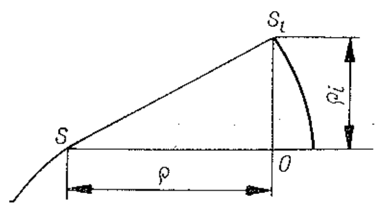

FIG. 3.

In the case $p\neq 0$, the integral curve passing the point has the form:

$$p(t)=e^{-F}\left(\eta+\int_{\xi}^{t} g(t)\, e^{F}\, dt\right)=\frac{1-\xi^2}{(1+\xi^2)^2}\,\frac{(1+t^2)^2}{(1-t^2)}\left[\eta+2\frac{1+\xi^2}{1-\xi^2}\left(\frac{(1+\xi^2)}{1+t^2}-1\right)\right], \tag{5.9}$$

where

$$F(t)=\int_{\xi}^{t} 2\frac{t^2-3}{1-t^4}\, t dt.$$

Since $p\,(t)=t'\,[\varphi\,(t)]$, from the preceding relation we obtain the expression for the independent variable of the Eq. (5.7):

$$\varphi=\frac{1}{2}\int\frac{(1-t^2)\,dt}{(1+t^2)+N(1+t^2)^2}+C, \tag{5.10}$$

where $N=\dfrac{1-\xi^2}{(1+\xi^2)^2}-\dfrac{2}{1+\xi^2}$. On integrating, we have:

$$\varphi=\text{arc tan}\, t-\frac{1}{2}\left(P+\frac{1}{P}\right)\text{arc tan}\frac{t}{P}+C, \tag{5.11}$$

where $P=\sqrt{1+\dfrac{1}{N}}$.

Further integration with respect to φ leads to q' (note: $t=q$) and to r, which equals e^q.

To this end, the well known relation between $\int\limits_{t_1(\varphi_1)}^{t_2(\varphi_2)}\varphi dt$ and $\int\limits_{\varphi_1}^{\varphi_2} t d\varphi$ may be used, and thus the integration of the Eq. (5.11) with respect to φ may be replaced by that with respect to t. This gives:

$$\int_{t(0)}^{t(\varphi_1)}\varphi dt=t_1\,\varphi_1+\ln\left[\frac{1}{\sqrt{1+t_1^2}}\left(\frac{P^2+t_1^2}{P^2}\right)^{\frac{P^2+1}{4}}\right], \tag{5.12}$$

whence

$$q=\int_{0}^{t(\varphi_1)} td\varphi=\ln\left[\sqrt{1+t_1^2}\left(\frac{P^2}{P^2+t_1^2}\right)^{\frac{P^2+1}{4}}\right]$$

and

$$r=n\sqrt{1+t_1^2}\left(\frac{P^2}{P^2+t_1^2}\right)^{\frac{P^2+1}{4}}, \tag{5.13}$$

where the parameter n determines particular homothetic curves of one family. For numerical reversing of the function $\varphi\,(t)$ from the Eq. (5.11), a computer is used.

The family of orthogonal curves satisfying also the Eq. (5.11) is found by replacing t by $u=-\dfrac{1}{t}$, while φ remains unchanged—that is, P is replaced by $P_1=\dfrac{1}{P}$, as follows from the Eq. (5.11). Thus denoting $\psi=\text{arc tan}\, u-\dfrac{1}{2}\left(P_1+\dfrac{1}{P_1}\right)\times$

$\times$ arc tan $\frac{u}{P_1}+C_1$ we arrive at (cf. Eq. (5.11)) $\psi=\varphi+\frac{\pi}{2}\left[\frac{1}{2}\left(P+\frac{1}{P}\right)-1\right]$. Example: for $P=10$, $\psi=\varphi+2\pi+0.0785$. Graphs are obtained by means of a computer and plotter. Some principal lines cover fairly wide angles (up to 2π). These curves (cf. dotted lines in Fig. 4) are similar to logarithmic spirals, but may be found with much

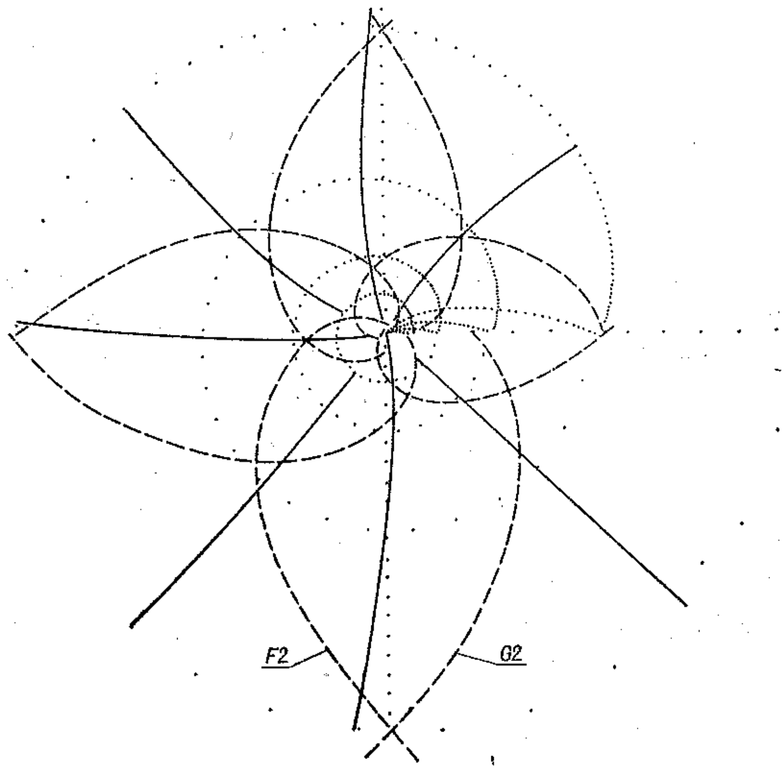

Fig. 4. ..... orthogonal Cartesian reference frame and principal lines as plotted by computer and plotter, — — — additionally calculated principal lines, —·—·— additionally calculated slip lines.

better approximation than replacement by these spirals, and in a simpler way than a strict calculation performed by the computer. In fact, neglect in the Eq. (5.11) of the term arctan t gives for small values of P (of the order of 0.1) slight errors. Then, from the Eq. (5.11), we arrive at

$$t=0.1\ \tan\left(-\frac{\varphi}{5.05}\right), \tag{5.14}$$

and from $r_2=r_1\ \exp\int t d\varphi$ it follows that $r_2^2\cos\frac{\varphi_1}{5.05}=r_1^2$.

Three consecutive approximations by series, with preservations in each of two terms only yield:

$$r_1=c\left(1-\frac{\varphi_1}{102}\right), \tag{5.15}$$

where $c = r_2$, and the cumulative error of all three approximations for $P=0.1$ and $\varphi = \pi/2$ amounts to less than 0.01%. The Eq. (5.15) presents a Galileo spiral (cf. [7], vol. II, p. 47) — that is, the trajectory of a freely falling material point with respect to a vertical line rotating with the earth. In an analogical manner it is possible to calculate directly φ, r for respective points of the slip lines, with the slopes s with respect to radii-vectors. In fact, from the reversed Eq. (4.4) we have for P=0.1:

$$\int s d\varphi = -5.05 \int \frac{\tan \chi}{10 \pm \tan \chi} d\chi \pm 50.5 \int \frac{d\chi}{10 \pm \tan \chi} \tag{5.16}$$

(cf. Eq. (5.14)), where $\chi = \varphi/5.05$. On integrating, we obtain:

$$r = r_0 \,|10 \cos \chi \pm \sin \chi \exp(-4.95\chi)|, \tag{5.17}$$

where r_0 is the radius at a given point on the slip line.

Moreover, the Eq. (5.14) enables us to find directly in the whole region the second family of principal lines, while a computer, unless with sophisticated programs, determines the selines in a narrow wedge region only (for $P=0.1$). In this case, we have:

$$\frac{r_2}{r_1} = \left(\sin \frac{\varphi_2}{5.05} \bigg/ \sin \frac{\varphi_1}{5.05}\right),$$

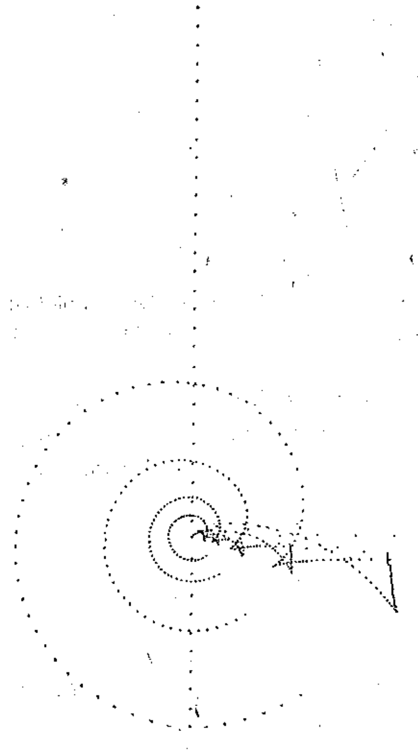

FIG. 5.

where r_2 and r_1 are radii at arbitrary points of the line. However, the use of antiderivatives of the function from the Eq. (5.16) is more convenient, since to obtain the value of the definite integral for any limits it suffices to subtract the respective values F2 or G2 (Fig. 4).

The symmetry of all curves with respect to the polar axis is evident from both the physical sense and the Eq. (4.11). One only of these symmetrical forms is considered. The functions $r(\varphi)$, as presented in Fig. 5, are not single-valued; actual principal lines are represented, however, by one branch only, the second being of minor interest due to the limited included angle of the wedge region, and rather slight deviation from the straight line.

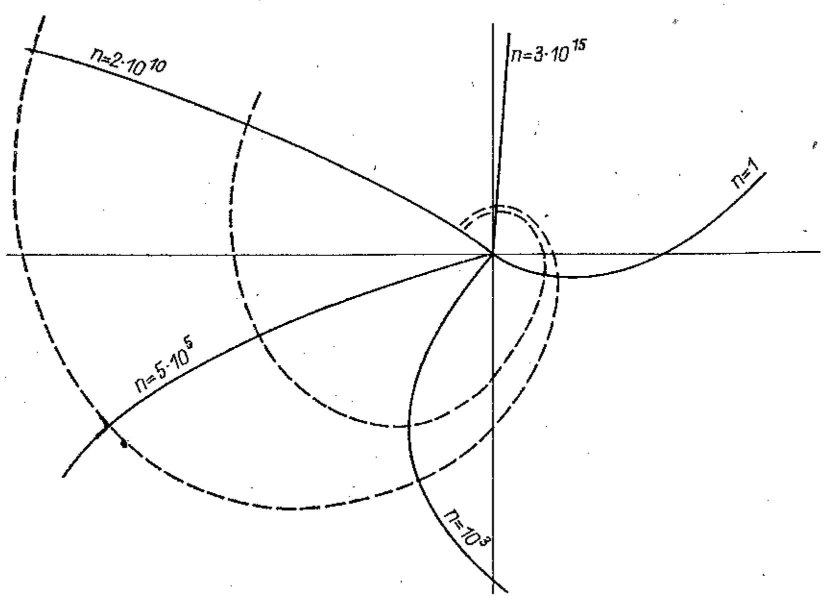

FIG. 6. – – – first family of principal lines (as calculated), ——— second family of principal lines (obtained by graphical approximation).

This refers to the case $P=0.1$ (cf. Eq. (5.15)), while regarding other value of P it may be stated that:

1) On expanding arctanα and arctanα/P into power series, the stability of these functions may be shown, and also of $r(\varphi)$ with respect to dP.

2) From values P=0.1, 0.8, 0.97, 1, 1.2, (10), 11, 100, for 0.1 (10) and 100 (0.01) only are there possible principal lines in a large included angle (up to 2π) of the wedge region; for the remaining values listed this angle is very small. Consequently, the respective cases have been neglected, especially since they may be assumed to be replaced, with sufficient approximation, by suitable logarithmic spirals (cf. Eq. (5.7) and Fig. 2).

3. In the case of $P=100$ (Fig. 6), principal lines may cover the angles up to 2π, but any particular principal line in practice covers a very small angle only, since

otherwise the angle covered is associated with the change of the radius-vector by several orders of magnitude.

The apparent "novelty" of the solution found may be ascribed, in the author's opinion, to the triviality of the well-known solution with the logarithmic spirals, while, in the case of translation, the Prandtl solution, being not trivial, may be only generalized in some respects by matching various regions.

6. On the degeneration of the general relation (Eq. (3.5))

It may be shown (note:

$$(1+z^2)\,z-3z'^2\,z=0 \tag{6.1}$$

after integration gives $\kappa=c$, where $c=0$, full proofs are omitted) that for families of curves in translation (Sect. 4), cases in which in the Eq. (3.5) one, two or three quantities vanish are not possible. Thus, possible is only the case in which all four quantities disappear, which means that pricnipal lines form a rectangular net of straight lines. In the case of homothety of principal lines, degeneration may be of two kinds — that is, with principal lines forming: 1) concentric circular arcs and their radii, and 2) a rectangular net of strainght lines.

As regards the degenerated solution containing straight principal lines, it may be stated that, in contrast with the curvilinear solution, it is not stable with respect to convexity or concavity of the surface of the plate. This statement disregards the highly improbable possibility of dead zones on the surface, and is based only on the assumption that the high degeneration of the Eqs. (3.5) and (4.2) has greater influence than a slight modification of the region.

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STRESZCZENIE

O WYZNACZANIU I STOSOWANIU TRAJEKTORII NAPRĘŻEŃ GŁÓWNYCH PRZY PŁYNIĘCIU PLASTYCZNYM

W założeniu płaskiego stanu odkształcenia dla warunku plastyczności Hubera-Misesa (dla warunku Coulomba-Treski dopuszczalny jest ponadto płaski stan naprężenia) idealnego ośrodka plastycznego, nieściśliwego, jednorodnego i izotropowego wyprowadza się podstawową zależność trajektorii naprężeń głównych (krzywizn i ich pochodnych w układzie naturalnym), zgodną z zależnościami znanymi. Prowadzi to, w przypadku układu trajektorii równolegle względem siebie przesuniętych, do równania różniczkowego zwyczajnego drugiego rzędu względem y_1', gdzie $y_1(x_1)$ jest równaniem trajektorii głównej. Rozwiązanie tego równania jest znane (odpowiednie wartości całek podane są w pracy), a zatem otrzymuje się metodą "prostą" rozwiązanie zgodne ze znanym rozwiązaniem Prandtla. Podobnie, w przypadku gdy trajektorie główne tworzą rodzinę krzywych jednokładnych, otrzymuje się analogiczne, chociaż bardziej złożone równanie ze względu na $t=dq/d\varphi$, gdzie $q=\ln r$, r, φ zaś są współrzędnymi biegunowymi. W tym przypadku uzyskuje się bezpośredni dowód, że trajektorie naprężeń głównych są spiralami logarytmicznymi o dowolnym kącie pochylenia względem promieni wodzących (jest to zresztą przypadek zwyrodnienia), albo też są innymi spiralami — a to jest przypuszczalnie „nowe" rozwiązanie, które dla pewnych wartości parametrów można z bardzo niewielkim błędem zastąpić spiralami Galileusza. Obliczenia i wykresy uzyskano za pomocą komputera ODRA 1204. Dodano krótkie uwagi na temat możliwych zwyrodnień.

Резюме

ОБ ОПРЕДЕЛЕНИИ И ПРИМЕНЕНИИ ТРАЕКТОРИЙ ГЛАВНЫХ НАПРЯЖЕНИЙ ПРИ ПЛАСТИЧЕСКОМ ТЕЧЕНИИ

В предположении плоского деформационного состояния, для условия пластичности Губера-Мизеса (для условия Кулона-Треска кроме этого допустимо плоское напрячженное состояние) идеально пластической, несжимаемой, однородной и изотропной среды, выводится основная зависимость траекторий главных напряжений (кривизн и их производнысх в натуральной системе) совпадающая с известными зависимостями. Это приводит, в случае системы траекторий сдвинутых параллельно друг относительно друга, к обыкновенному дифференциальному уравнению второго порядка по отнишению к y_1', где $y_1(x_1)$ является уравнением главной траектории. Решение этого уравнения известно (соответствующие значения интегралов приведены в работе), следовательно получается „простым" методом решение совпадающее с известным решением Прандтля. Аналогично, в случае, когда главные траектории образуют семейство подобных кривых, получается аналогичное, хотя более сложное, уравнение по отношению к $t=dq/d\varphi$, где $q=\ln r$; r, φ — полярные координаты. В этом случае получается непосредственное доказательство, что траектории главных напряжений это логарифмические спирали с произвольным углом наклона по отношению к радиусу — вектору (это впрочем случай вырождения), или же это другие српиали — это вероятно новое решение, которое для некоторых значений параметров можно с очень небольшой ошибкой заменить спиралями Галилея. Расчеты и графики получены при помощи ЭВМ Одра 1204. Даются краткие замечания на тему возможных вырождений.

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