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OPTIMAL DESIGN OF MULTI-PURPOSE SANDWICH BEAM-COLUMNS

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The paper deals with the minimum-weight design of pin-ended members that have to act as a beam in some situations and as a column in others. The space of design constraints is divided into regions in which the optimal design is governed by either one of the two design requirements or both. The method is illustrated by the example of a member having a linear relationship between stiffness and cross-sectional area (mass). The economy achieved by optimization is discussed.

1. Introduction

We study the problem of minimising the volume (mass) of a pin-ended member of given length that is to serve as a beam for a part of its design life and as a column for the rest, but not both at any given time. Such optimization under multiple design requirements can lead to a rationalised design of mass produced structural/mechanical elements. At the very least it should point towards a rational use of materials.

Optimal elastic design for a single design requirement — maximum fundamental frequency, maximum buckling load, etc. — has been investigated by many authors (see, e.g. [1–4]). Likewise, minimum-weight design of multi-purpose prismatic members was studied in [5 and 6]. Prager and Shield [7] gave the minimum-weight design of a beam-tie of a given length under specified constraints on transverse and longitudinal stiffness. The present work is an extension of this technique to the more frequently met beam-column situations.

The problem under consideration consists in determining the cross-sectional area (mass) distribution of a pin-ended member along its length that has its Euler buckling load under axial compression greater than a specified minimum value maximum deflection under a transverse load less than a specified maximum value. Moreover, the member is to use the minimum possible material. The mathematical problem is formulated in Sect. 2, wherein the necessary optimality is also derived.

The specified values of the minimum Euler buckling load and of the maximum transverse deflection are the design constaints. In the optimization of multi-purpose structures it is not unusual (see, e.g. [5]) that under certain conditions the optimal design is governed by only one of the design constraints. It is important to examine the space of design variables in order to delineate possible regions of single-variable optimal design. This is done in Sect. 3. Finally, in Sect. 4 we present numerical results for the simple case when the member has a linear relationship between stiffness and cross-sectional area (mass) and discuss the economy possible due to optimization.

2. Formulation of the problem and optimality condition

In order to formulae the mathematical optimization problem it is convenient to consider individually the design requirements of the structural member.

2.1. Beam action

Consider a beam of variable cross-section and of a given length, $2L$, pinned at its ends and subjected to a transverse load $2Q$ at its midspan. The transverse deflection $v(x)$ satisfies the following differential equation and boundary conditions in non-dimensional form:

$$\alpha^n(x)\, v_{xx}+Qx=0, \quad 0\leqslant x\leqslant 1; \tag{2.1}$$

$$v(0)=v_x(1)=0, \tag{2.2}$$

where the spatial variable and deflection are non-dimensionalized by L, $Q=Q/EcL^{2n-2}$ $\alpha(x)=A(x)/L^2$ and it is is assumed that the second moment of area $I(x)$ and the cross-sectional area $A(x)$ of the member are related through

$$I(x)=cA^n(x), \tag{2.3}$$

where c and n are constants defined by the shape of the cross-section. In Eqs. (2.1) and (2.2) the subscript x denotes differentiation with respect to the non-dimensional linear variable, and the boundary conditions (2.2) imply symmetry about the midspan. The variation in $\alpha(x)$ has to be such that the following design requirement is met:

$$v(1)=\frac{1}{Q}\int_0^1 \alpha^n(x)\, v_{xx}^2\, dx\leqslant v_0, \tag{2.4}$$

where v_0 is the given maximum allowable deflection.

2.2 Column action

If the member considered in §2.1 is subjected to an axial compressive load P, the deflection $u(x)$ in elastic buckling satisfies the following differential equation and boundary condition in non-dimensional form:

$$\alpha^n(x)\, u_{xx}+Pu=0, \tag{2.5}$$

$$u(0)=u_x(1)=0. \tag{2.6}$$

We are looking for the variation in $\alpha(x)$, that meets the following design requirement:

$$P_{cr}=\int_0^1 \alpha^n\, u_{xx}^2\, dx\Big/\int_0^1 u_x^2\, dx\geqslant P_0, \tag{2.7}$$

where $P_0=P_0/EcL^{2n-2}$ is the minimum permissible Euler buckling load in non-dimensional form.

The optimization problem under consideration reduces to finding the variation $\alpha(x)$ that satisfies Eqs. (2.1), (2,2), (2.5) and (2.6) and the inequalities (2.4) and (2.7) and minimizes the volume (mass) of the member

$$\int_0^1 \alpha(x)\,dx \to \min. \tag{2.8}$$

In writing the functional (2.8) the constant multiplier $2L^3$ has been omitted.

2.3. Optimality condition

To derive the necessary optimality condition for the optimization problem (2.1)–(2.8) we use the standard Lagrange multiplier technique to include the given design constraints (2.4) and (2.7) and write an auxiliary functional

$$\Pi=\int_0^1 \alpha dx+\mu_2\left(Qv_0-\int_0^1 \alpha^n(x)\,v_{xx}^2\,dx\right)+\mu_1\left(P_0\int_0^1 u_x^2\,dx-\int_0^1 \alpha^n(x)\,u_{xx}^2\,dx\right). \tag{2.9}$$

The necessary optimality condition is obtained by setting the variation of π with respect to α equal to zero, whereupon

$$n\alpha^{n-1}(\mu_1\,u_{xx}^2+\mu_2\,v_{xx}^2)=1. \tag{2.10}$$

It is worth noting that the solution $u(x)$ of the homogeneous boundary value problem (2.5)–(2.6) is only determined to within a constant multiplier. Accordingly, the Lagrange multiplier μ_1 could be incorporated into $u(x)$ as a normalisation factor. However, in the numerical solution procedure used here it was found convenient to normalise $u(x)$ in a different manner (see Sect. 4), and hence the multiplier μ_1 is retained in the optimality condition (2.10).

For the sake of definiteness we present below the solution for the case when $n=1$, i.e. when the stiffness and the mass are linearly related. This is true, in particular, of sandwich members and universal rolled sections. In this simple case the necessary optimality condition is independent of the control variable $\alpha(x)$. Before presenting the proper solution it is important to examine the space of the design constraints P_0, v_0, Q to delineate any possible regions of single-variable optimal design.

3. Possible single-variable optimal designs

As mentioned in Introduction, it is quite likely that under certain conditions the optimal design of a multi-purpose structural member is governed by only one of the design constraints involved. Accordingly, we need to examine the maximum deflection of the optimal sandwich column [2] under a transverse load, as well as the buckling load of the optimal sandwich beam [3] under axial compression.

3.1. Maximum deflection of optimal column under transverse load

The minimum-weight design of a pin-ended sandwich column was obtained in [2]. In this case the optimality condition (2.10) reduces to $u_{xx}=-1\sqrt{\mu_1}$, whence it follows that $u(x)=x(2-x)/2\sqrt{\mu_1}$, which satisfies the boundary conditions (2.6).

Finally, the variation of cross-sectional area along the length of the column is given by Eq. (2.5) (with $n=1$) to be $\alpha(x)=Px(2-x)/2$, which satisfies the design constraint (2.7) identically if $P=P_0$. The same result was obtained in [2].

If the optimal sandwich column were to be used as a beam, then the deflection at the point of application of a transverse concentrated load $2Q$ would be $v(1)=$ $=\frac{1}{Q}\int_0^1 \alpha v_{xx}^2\,dx$. Substitution of v_{xx} from Eq. (2.1) with $n=1$ and $\alpha(x)$, gives $v(1)=$ $=0.7726Q/P$. Thus, if the given design constraints v_0, Q, P_0 in the multi-purpose optimization problem satisfy the inequality $P_0 v_0/Q \geqslant 0.7726$, the optimal sandwich column will have sufficient flexuaral stiffness to meet the deflection requirement. In other words, the optimal sandwich column is also the optimal design under constraints on both the Euler buckling load and the maximum deflection under a transverse concentrated load.

3.2. Buckling load of optimal beam under axial compression

The minimum-weight design of a pin-ended sandwich beam under a given constraint on maximum deflection was obtained in [4]. In this particular case the optimality condition reduces to $v_{xx}=-1/\sqrt{\mu_2}$. It follows from Eqs. (2.5), (2.6) and (2.7) that, when $n=1$, $\alpha(x)=Qx/2v_0$. If the optimal sandwich beam were to be used as a column the critical buckling load is given by the solution of Eq. (2.1) and (2.2). The differential equation has a closed form solution in the Bessel function J_1. It transpires that for the optimal beam $p \leqslant 0.7229\ Q/v_0$. Thus, if the given design constraints v_0, P_0, Q in the multi-purpose optimization problem satisfy the inequality $P_0 v_0/Q \leqslant 0.7229$, the optimal sandwich beam will have sufficient stiffness to meet the buckling load requirement. In other words, the optimal sandwich beam is also the optimal design under constraints on both the maximum deflection due to a transverse concentrated load and on the Euler buckling load.

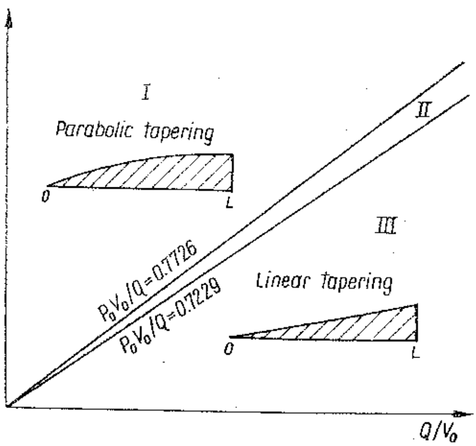

FIG. 1. Plane P_0, Q/v_0 of design constraints.

From the above inequalities it follows that the optimal design is governed by both design requirements only in the narrow region

$$0.7229 \leqslant P_0 v_0/Q \leqslant 0.7726. \tag{3.1}$$

The plane of design variables $(P_0, Q/v_0)$ is thus divided into three regions as shown in Fig. 1. In region I, bounded by the straight line $P_0 v_0/Q=0.7726$, the optimal design of the beam-column is identical to the optimal sandwich column, whereas in region III, bounded by the straight line $P_0 v_0/Q=0.7229$, it is identical to the optimal sandwich beam. Only in the narrow region II bounded by Eq. (3.1) is the optimal design dependent on both the design reguirements. In this region the optimality condition (2.10), with $n=1$, has to be solved in conjunction with the boundary value problems (2.1)–(2.2) and (2.5)–(2.6) and the constraints (2.1) and (2.7).

4. Solution of multi-purpose optimization problem

If the given design constraints are such that they satisfy the inequality (3.1), then the optimal design is governed by both the column and the beam action. In this general case, the optimality condition (2.10), after substitution for v_{xx} and u_{xx} from Eqs. (2.1) and (2.5) respectively, takes the following form:

(4.1)
$$[\alpha/(Q/v_0)]^2=\mu_1\,(Pv_0/Q)^2\,u^2+\mu_2\,v_0^2\,x^2.$$

Moreover, from Eqs. (2.1), (2.4) and (2.10), it follows that

(4.2)
$$v_0\sqrt{\mu_2}=\int_0^1 x\sqrt{1-\mu_1\,u_{xx}^2}\,dx.$$

The differential equation of column buckling (2.5) may be rewritten as

(4.3)
$$u_{xx}=-\frac{Pv_0}{Q}\,\frac{u}{\sqrt{\mu_1\left(\dfrac{Pv_0}{Q}\right)^2 u^2+\mu_2\,v_0^2\,x^2}}$$

and the constraint (2.7) (on buckling load) as

(4.4)
$$\int_0^1[\alpha/(Q/v_0)]\,u_{xx}^2\,dx\Big/\int_0^1 u_x^2\,dx\geqslant Pv_0/Q.$$

Before presenting the numerical procedure used to solve the optimization problem, it is worthwile examining the behaviour of the buckling deflection functon $u(x)$ as $x\to 0$. To do this, let us assume that as $x\to 0$, u varies as $Ax+Bx_p$. Substituting $u=Ax+Bx_p$ into the optimality condition (4.1) and equating the coefficients of like powers in x, we find that the lowest order term corresponds to $p=2$. From the optimality condition (4.1) and the differential equation (2.5) it follows that, near $x=0$,

(4.5)
$$u_{xx}(0)=\lim_{x\to 0}u_{xx}=-\left(\frac{Pv_0}{Q}\right)\frac{u_x(0)}{\sqrt{\mu_1\left(\dfrac{Pv_0}{Q}\right)^2 u_x^2(0)+\mu_2\,v_0^2}},$$

which is a finite quantity. Consequently, to satisfy the condition that the bending moment vanish at $x=0$, the area must vanish, i.e. $\alpha(0)=0$.

The following iteration sequence was followed to solve the multipurpose optimization problem:

(i) Assume $\mu_1 > 0$.

(ii) Estimate $\mu_2 v_0^2$.

(iii) Estimate $K = Pv_0/Q$, and hence find the variation of $\alpha/(Q/v_0)$ from the optimality condition (4.1).

(iv) Integrate u_{xx} (expression (4.3) twice to find u_x and u using the fourth-order Runge-Kutta method from $x=1$ ($u(1)=1$, $u(1)=0$) to $x=h$, where h is the step size. (Note that the deflection function $u(x)$ in buckling is normalised such that $u(1)=1$, which means the other boundary condition, $u(0)=0$, cannot be simultaneously satisfied).

(v) Extrapolate from $x=h$ to $x=0$ by Taylor's series, and check whether $u(0)=0$ is satisfied within permissible limits.

(vi) Calculate a fresh value of $K = Pv_0/Q$ from Eq. (4.4), and repeat steps (iii)–(vi).

(vii) Calculate a fresh value of $\mu_2 v_0^2$ from Eq. (4.2) and repeat steps (ii)–(vii).

The value of $u(0)=0$ in step (v) was used as a check on the accuracy of computations. The error was less than 1.5×10^{-5}. Typical requirements were 5–6 inner loops (steps iii–vi) and 3 outer loops (steps ii–vii).

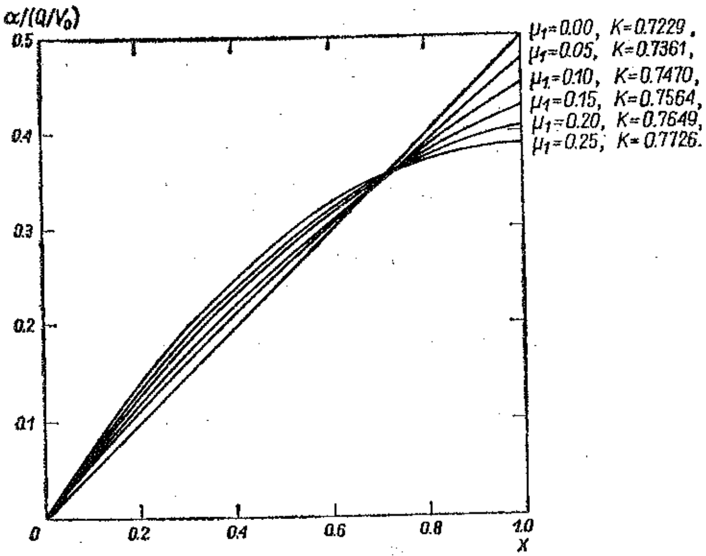

FIG. 2. Variation of α with x.

The variation of the non-dimensional area $\alpha(x)$ is shown in Fig. 2 for various values of $K = P_0 v_0/Q$ within the range $0.7229 \leqslant K \leqslant 0.7726$. The corresponding values of μ_1 are a so given. Actually, the value of μ_1 was assumed and the corresponding value of K calculated in step (vi) from the relation (4.4). Also shown are the extreme cases — the optimal sandwich beam (linear variation in $\alpha(x)$) when $K = 0.7229$ and the optimal sandwich column (parabolic variation in $\alpha(x)$) when $K = 0.7726$.

In order to estimate the economy achieved by optimization (efficiency of optimization) let us compare the volume of the optimal beam-column with that of a prismatic bar of the same length, as well as with that of the optimal sandwich column and optimal sandwich beam.

4.1. Comparison with prismatic bar

The deflection at the midspan of a prismatic (α=constant) pin-ended bar of a length $2L$ under a transverse load $2Q$ is given by $v(1)=Q/3\alpha$. Thus, $\alpha=Q/3v_0$, if the design requirement $v(1)=v_0$ is met. The critical buckling load of the prismatic bar is given by $P=\pi^2$, so that if $P_0\, v_0/Q \leqslant \pi^2/3$ (as in our case), the maximum deflection as a beam is the governing factor. In other words, $\alpha/(Q/v_0)=1/3$.

For a given value of Q/v_0 the volume of the prismatic bar V_{prism} is therefore equal to 1/3 except for the constant multiplier $2L^3$.

The volume of the optimal beam-column V_{opt} to within the constant multiplier $2L^3$ is easily evaluated numerically. The efficiency of optimization as evidenced by the percentage saving in material, $(1-V_{\text{opt}}/V_{\text{prism}})$ 100, may be judged from Fig. 3.

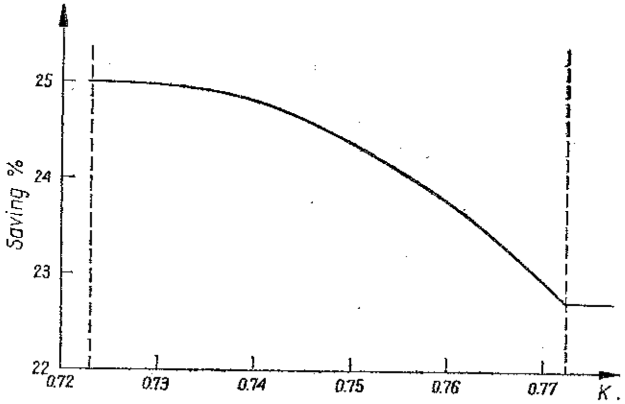

FIG. 3. Saving compared with prismatic member.

As noted above, it is instructive to judge the efficiency of optimal beam-column design against the optimal sandwich beam (linear variation in $\alpha(x)$) and the optimal sandwich column (parabolic variation in $\alpha(x)$).

4.2. Comparison with optimal beam (linear tapering)

For given values of P_0, Q, v_0 the size of the optimal sandwich beam of a length $2L$ is governed by P_0 since the resulting Q/v_0 will then be larger than any specified value of the volume of the optimal sandwich beam V_{beam} to within the constant multiplier $2L^3$ is given by

$$V_{\text{beam}}=0.25\,(Q/v_0)_{\text{beam}}=0.3458\,(P_0\, v_0/Q), \quad \text{as} \quad (P_0\, v_0/Q)_{\text{beam}}=0.7229.$$

The efficiency of optimization in comparison with the optimal sandwich beam, $(1-V_{\text{opt}}/V_{\text{beam}})\cdot 100$, may be judged from Fig. 4.

4.3. *Comparison with optimal column (parabolic tapering)*

For given values of P_0, Q, v_0 the size of the optimal sandwich column of a length $2L$ is controlled by Q/v_0 since the resulting Euler buckling load will then be greater than any specified P_0. The volume of the optimal sandwich column, V_{col}, to within the constant multiplier $2L^3$ is given by $V_{opt}=0.2575\,(Q/v_0)$

The officiency of optimal beam-column design in comparison with the optimal column, $(1-V_{col}/V_{col})\cdot 100$, may be judged from Fig. 4.

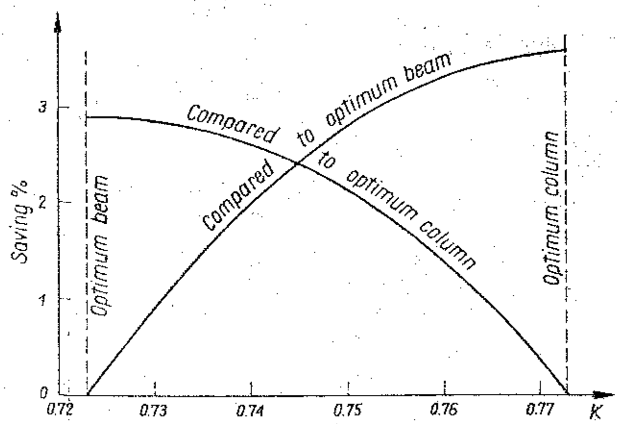

FIG. 4. Savings compared with tapered members.

5. Discussion

The minimum-weight (volume) design of a multi-purpose pin-ended member. that is to serve as a beam for a part of its design life and as a column for the res is illustrated by a member having a linear relationship between stiffness and masst The space of design constraints is closely examined to delineate any regions of single-variable optimal design. It is shown that the regions where both the design constraints are active is rather narrow. The efficiency of optimization is measured by comparing the volume of the optimal multi-purpose member with that of a prismatic bar and a bar with linear (optimal beam) and parabolic (optimal column) tapering. It is shown how much (or rather how little) can be gained by multi-purpose optimization over single-variable optimal designs. However, one can anticipate a substantially bigger gain for other relationships between stiffness and mass ($n=2$ or 3). The present formulation is directly applicable to these situations, although the numerical solution procedure will be further complicated since the control function ($\alpha(x)$) will appear in the optimality condition, too. Moreover, the single--variable optimal designs (optimal column or beam) are not available in a closed--form, although the solutions could easily be obtained within the multi-purpose optimization scheme.

References

1. J. B. Keller, *The shape of the stronest column*, Arch. Rat. Mech. Anal., **5**, 275, 1960.
2. W. Prager and J. E. Taylor, *Problems of optimal structural design*, J. Appl. Mech., **35**, 102, 1968.
3. N. C. Huang, *Optimal design of elastic beams for minimum-maximum deflection*, J. Appl. Mech., **38**, 1078, 1971.
4. B. L. Karihaloo and F. I. Niordson, *Optimal design of vibrating beams under axal compression*, Arch. Mech., **24**, 1029, 1972.
5. N. V. Banichuk and B. L. Karihaloo, *Minimum-weight design of multipurpose cylindrical bars*, Int. J. Solids Structures, **12**, 267, 1976.
6. R. Parbery and B. L. Karihaloo, *Minimum-weight design of hollow cylinders for given lower bound on torsional and flexural rigidities*, Int. J. Solids Structures, **13**, 1271, 1977.
7. W. Prager and R. I. Shield, *Optimal design of multi-purpose structures*, Int. J. Solids Structures, **4**, 469, 1968.

Streszczenie

OPTYMALNE PROJEKTOWANIE WIELOZADANIOWYCH PRZEKŁADKOWYCH BELKO-KOLUMN

W pracy rozpatruje się problem projektowania zakończonych sworzniami elementów, o minimalnym ciężarze, które w pewnych sytuacjach muszą działać jako belki, a w innych jako kolumny. Przestrzeń więzów projektowania podzielona jest na rejony, w których projekt optymalny określony jest przez jeden z dwu wymagań projektowanych lub też przez obydwa jednocześnie. Metoda jest zilustrowana przykładem elementu o liniowym związku pomiędzy sztywnością i polem przekroju poprzecznego (masą). Przedyskutowano też oszczędność osiągniętą przez optymalizację.

Резюме

ОПТИМАЛЬНОЕ ПРОЕКТИРОВАНИЕ УНИВЕРСАЛЬНЫХ ПРОКЛАДОЧНЫХ БАЛКО-КОЛОНН

В работе рассматривается проблема проектирования оконченных штырями элементов, с минимальным весом, которые в некоторых ситуациях должны действовать как балки, а в других как колонны. Пространство связей проектирования разделено на два района, в которых оптимальный проект определен каким-нибудь из двух проектных требований или же обома требованиями одновременно. Метод иллюстрирован примером элемента с линейным соотношением между жесткостью и полем поперечного сечения (массой). Обсуждена тоже экономия достигнутая путем оптимизации.

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