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STRESS CONCENTRATIONS IN NON-HOMOGENEOUS ELASTIC LAYER WEAKENED BY CRACKS

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The paper presents the statical problem of stress concentrations around the tips of two semi-infinite cracks situated within an infinite, non-homogenous elastic layer. The conditions of antiplane state of strain are assumed to be satisfied. The outer boundaries of the layer are rigidly clamped, while the crack surfaces are loaded by prescirібed forces.

The application of the complex exponential Fourier transform reduces the problem of determining the stress intensity factors at the crack tips to the solution of a corresponding system of Wiener-Hopf equations. By assuming the cracks to be located symmetrically with respect to the interface between the two elastic materials, the system of equations splits up into two separate Wiener-Hopf equations; their exact solutions are derived.

Stress intensity factors at the two crack tips are determined and several particular cases are discussed. The solution is illustrated by an example in which the displacements are prescribed along the boundaries of the layer, the crack surfaces being free from loading.

1. Formulation of the problem

Let us consider an infinite elastic layer of thickness 2h, its middle surface separating two infinite and homogeneous layers characterized by different elastic properties. Assume each of these layers to contain a semi-infinite crack, both cracks being located symmetrically with respect to the interface between the layers (Fig. 1).

It is additionally assumed that the system is subject to an antiplane state of strain. This means that the displacement $w(x, y)$ is the only identically non-vanishing component of the displacement vector expressed in the rectangular coordinate system (x, y, z). In the absence of body forces the Lamé equilibrium equations reduce to the single equation

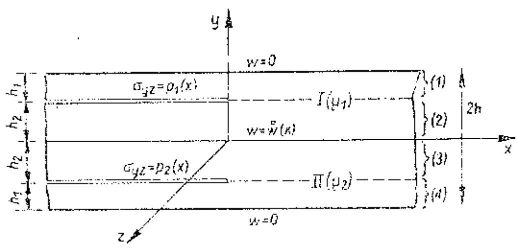

Fig. 1.

$$\nabla^2 w(x, y)=0. \tag{1.1}$$

The identically non-vanishing stress components are expressed in the form

$$\sigma_{xz}(x, y)=\mu\frac{\partial w}{\partial x}, \quad \sigma_{yz}(x, y)=\mu\frac{\partial w}{\partial y}, \tag{1.2}$$

μ being the Lamé constant.

The paper is aimed at determining the σ_{yz} stress intensity factors at the tips of the cracks under the assumption that the layer boundaries $y=\pm h$ are rigidly clamped, and the crack surfaces are loaded by arbitrary forces (Fig. 1).

The boundary conditions of the problem considered are the following:

$$
\begin{aligned}
&w(x,\pm h)=0 && \text{for} \quad |x|<\infty,\\
&\sigma_{yz}(x, h_2)=p_1(x) && \text{for} \quad x<0, \qquad (1.3)\\
&\sigma_{yz}(x,-h_2)=p_2(x) && \text{for} \quad x<0.
\end{aligned}
$$

The problem will be solved by means of the complex integral Fourier transform [1]

$$
\begin{aligned}
F(\alpha, y)&=\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x, y)\, e^{i\alpha x}\, dx, \qquad (1.4)\\
f(x, y)&=\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty+i\tau_0}^{\infty+i\tau_0} F(\alpha, y)\, e^{-i\alpha x}\, d\alpha.
\end{aligned}
$$

Here $\alpha=\sigma+i\tau$, the region of regularity of $F(\alpha, y)$ is a certain strip $\{\tau_- <\mathrm{Im}\alpha<\tau_+, |\mathrm{Re}\alpha|<\infty\}$, and the path of integration in Eq. $(1.4)_2$ is situated in the region of regularity of $F(\alpha, y)$, i.e. $\tau_-<\tau_0<\tau_+$. Account will also be taken of the fact that the function $F(\alpha, y)$ expressed by Eq. $(1.4)_1$ may be represented, within its regularity region, in the form [2]

$$
F(\alpha, y)=F^-(\alpha, y)+F^+(\alpha, y), \qquad (1.5)
$$

where

$$
\begin{aligned}
F^-(\alpha, y)&=\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{0} f(x, y)\, e^{i\alpha x}\, dx, \qquad (1.6)\\
F^+(\alpha, y)&=\frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} f(x, y)\, e^{i\alpha x}\, dx.
\end{aligned}
$$

The functions $F^-(\alpha, y)$ and $F^+(\alpha, y)$ are analytic functions of the complex variable α in the respective half-planes $\mathrm{Im}\alpha<\tau_+$ and $\mathrm{Im}\alpha>\tau_-$.

Applying the integral F-transform (1.4) to the relations (1.1) and (1.2) we obtain

$$
\begin{aligned}
&\frac{d^2 W(\alpha, y)}{dy^2}-\alpha^2 W(\alpha, y)=0,\\
&\Sigma_{xz}(\alpha, y)=-i\mu\alpha W(\alpha, y), \qquad (1.7)\\
&\Sigma_{yz}(\alpha, y)=\mu\frac{\partial W(\alpha, y)}{\partial y}.
\end{aligned}
$$

The solutions of Eq. $(1.7)_1$ is now written in terms of hyperbolic functions to yield the following expressions for the F-transforms of the displacement w and stresses σ_{xz}, σ_{yz}:

$$(1.8)\qquad \begin{aligned} W(\alpha, y) &= A(\alpha)\,\mathrm{sh}\alpha y + B(\alpha)\,\mathrm{ch}\alpha y, \\ \Sigma_{xz}(\alpha, y) &= -i\mu\alpha\,[A(\alpha)\,\mathrm{sh}\alpha y + B(\alpha)\,\mathrm{ch}\alpha y], \\ \Sigma_{yz}(\alpha, y) &= \mu\alpha\,[A(\alpha)\,\mathrm{ch}\alpha y + B(\alpha)\,\mathrm{sh}\alpha y]. \end{aligned}$$

The unknown functions $A(\alpha)$ and $B(\alpha)$ occurring in these formulae must be determined from the corresponding boundary conditions.

2. Solution of the boundary value problem

In order to solve the problem of a layer with clamped edges formulated above, let us cut layer along the plane separating the two different materials and impose the condition of equal (as yet unknown) displacements $w(x) = \mathring{w}(x)$ at the newly-created surfaces (Fig. 1).

The solution of the initial problem may now be composed of solutions of the problems I and II concerning homogeneous layers containing single cracks.

The boundary conditions of the problems I and II are now complemented by the corresponding conditions of continuity of the displacement and stress vectors:

Problem I

$$(2.1)\qquad \begin{aligned} w(x, h) &= 0 && \text{for} \quad |x| < \infty, \\ w(x, 0) &= \mathring{w}(x) && \text{for} \quad |x| < \infty, \\ \sigma_{yz}(x, h_2) &= p_1(x) && \text{for} \quad x < 0, \\ [\![w(x, h_2)]\!] &= 0 && \text{for} \quad x > 0, \\ [\![\sigma_{yz}(x, h_2)]\!] &= 0 && \text{for} \quad |x| < \infty. \end{aligned}$$

Problem II

$$(2.2)\qquad \begin{aligned} w(x, -h) &= 0 && \text{for} \quad |x| < \infty, \\ w(x, 0) &= \mathring{w}(x) && \text{for} \quad |x| < \infty, \\ \sigma_{yz}(x, -h_2) &= p_2(x) && \text{for} \quad x < 0, \\ [\![w(x, -h_2)]\!] &= 0 && \text{for} \quad x > 0, \\ [\![\sigma_{yz}(x, -h_2)]\!] &= 0 && \text{for} \quad |x| < \infty. \end{aligned}$$

In addition, at the intereference between the layers the condition of continuity of the stress vector must be satisfied,

$$(2.3)\qquad [\![\sigma_{yz}(x, 0)]\!] = 0 \quad \text{for} \quad |x| < \infty.$$

The symbol $[\![f(x, h)]\!]$ denotes the jump of $f(x, y)$ at the plane (x, y) defined as follows:

$$[\![f(x, h)]\!] = \lim_{y \to h+} f(x, y) - \lim_{y \to h-} f(x, y).$$

Applying the standard method of solution of such problems, let us cut the layers corresponding to the problems I and II along the planes containing the cracks. In view of the conditions (2.1)–(2.3) we obtain the following boundary conditions for the two problems considered:

Problem I

$$\overset{1}{w}(x, h)=0 \quad \text{for} \quad |x|<\infty,$$

$$\overset{2}{w}(x, 0)=\overset{\circ}{w}(x) \quad \text{for} \quad |x|<\infty,$$

(2.4)
$$\overset{1}{\sigma}_{yz}(x, h_2)=\overset{2}{\sigma}_{yz}(x, h_2)=p_1(x) \quad \text{for} \quad x<0,$$

$$\overset{1}{w}(x, h_2)=\overset{2}{w}(x, h_2) \quad \text{for} \quad x>0,$$

$$\overset{1}{\sigma}_{yz}(x, h_2)=\overset{2}{\sigma}_{yz}(x, h_2) \quad \text{for} \quad x>0.$$

Problem II

$$\overset{3}{w}(x, 0)=\overset{\circ}{w}(x) \quad \text{for} \quad |x|<\infty,$$

$$\overset{4}{w}(x, -h)=0 \quad \text{for} \quad |x|<\infty,$$

(2.5)
$$\overset{3}{\sigma}_{yz}(x, -h_2)=\overset{4}{\sigma}_{yz}(x, -h_2)=p_2(x) \quad \text{for} \quad x<0,$$

$$\overset{3}{w}(x, -h_2)=\overset{4}{w}(x, -h_2) \quad \text{for} \quad x>0,$$

$$\overset{3}{\sigma}_{yz}(x, -h_2)=\overset{4}{\sigma}_{yz}(x, -h_2) \quad \text{for} \quad x>0$$

and

(2.6)
$$\overset{2}{\sigma}_{yz}(x, 0)=\overset{3}{\sigma}_{yz}(x, 0) \quad \text{for} \quad |x|<\infty,$$

the upper indices $m=1, 2, 3, 4$ denoting the displacement and stress σ_{yz} in the respective layers 1–4 (Fig. 1).

Performing the F-transforms (1.4) on the functions (2.4)–(2.5) and using the necessary relations (1.8), the first three boundary conditions (2.4)–(2.5) yield the following results:

Problem I

(2.7)
$$\overset{1}{W}(\alpha, h_2)= -\frac{th\alpha h_1}{\mu_1 \alpha}\overset{1}{\Sigma}_{yz}(\alpha, h_2),$$

$$\overset{2}{W}(\alpha, h_2)=\frac{th\alpha h_2}{\mu_1 \alpha}\overset{2}{\Sigma}_{yz}(\alpha, h_2)+\frac{1}{ch\alpha h_2}\overset{\circ}{W}(\alpha).$$

Problem II

(2.8)
$$\overset{3}{W}(\alpha, -h_2)= -\frac{\text{th}\alpha h_2}{u_2 \alpha}\overset{3}{\Sigma}_{yz}(\alpha, -h_2)+\frac{1}{\text{ch}\,\alpha h_2}\overset{\circ}{W}(\alpha),$$

$$\overset{4}{W}(\alpha, -h_2)=\frac{\text{th}\alpha h_1}{\mu_2 \alpha}\overset{4}{\Sigma}_{yz}(\alpha_1 -h_2).$$

Here

(2.9)
$$\overset{2}{\Sigma}_{yz}(\alpha, 0)=\frac{1}{\operatorname{ch}\alpha h_2}\overset{2}{\Sigma}_{yz}(\alpha, h_2)-\mu_1\alpha\mathring{W}(\alpha)\operatorname{th}\alpha h_2,$$
$$\overset{3}{\Sigma}_{yz}(\alpha, 0)=\frac{1}{\operatorname{ch}\alpha h_2}\overset{3}{\Sigma}_{yz}(\alpha, -h_2)+\mu_2\alpha\mathring{W}(\alpha)\operatorname{th}\alpha h_2.$$

The expressions (2.7) and (2.8) are now subtracted from each other, and the function $\mathring{W}(\alpha)$ in Eq. (2.9) is found by means of Eqs. (2.6 writing the π-transforms in the form (1.5, and using the conditions of continuity of the displacement and stress vectors between the individual layers, we obtain the system of equations

(2.10)
$$\overset{*}{W}^{-}(z, h^*)=-\frac{h\operatorname{sh}z}{\mu_1 z\operatorname{ch}zh^*\operatorname{ch}z(1-h^*)}\Sigma_{yz}(z, h^*)-\frac{\mathring{W}(z)}{\operatorname{ch}zh^*},$$
$$\overset{*}{W}^{-}(z, -h^*)=-\frac{h\operatorname{sh}z}{\mu_2 z\operatorname{ch}zh^*\operatorname{ch}z(1-h^*)}\Sigma_{yz}(z, -h^*)+\frac{\mathring{W}(z)}{\operatorname{ch}zh^*},$$
$$\mathring{W}(z)=\frac{h}{(\mu_1+\mu_2)z\operatorname{sh}zh^*}[\Sigma_{yz}(z, h^*)-\Sigma_{yz}(z, -h^*)],$$

Here

(2.11)
$$\overset{*}{W}^{-}(z, h^*)=\overset{1}{W}^{-}(z, h^*)-\overset{2}{W}^{-}(z, h^*),$$
$$\overset{*}{W}^{-}(z, -h^*)=\overset{3}{W}^{-}(z, -h^*)-\overset{4}{W}^{-}(z, -h^*),$$
$$z=\alpha h, \quad h^*=h_2/h, \quad h=h_1+h_2.$$

The system (2.10) represents a system of Wienei-Hopf integral equations; after certain algebraic transformations and by eliminating $\mathring{W}(\alpha)$ from the first two equations, the system may be decomposed into two single Wiener-Hopf equations:

(2.12)
$$\Phi^{-}(z)=-H(z)[\Phi^{+}(z)+S_1^{-}(z)],$$
$$\Psi^{-}(z)=-K(z)[\Psi^{+}(z)+S_2^{-}(z)].$$

Here

(2.13)
$$H(z)=\frac{\operatorname{sh}z}{z\operatorname{ch}zh^*\operatorname{ch}z(1-h^*)},$$
$$K(z)=\frac{\operatorname{ch}z}{z\operatorname{sh}zh^*\operatorname{ch}z(1-h^*)}.$$

The unknown functions $\Phi^{\pm}(z)$, $\Psi^{\pm}(z)$ are determined as follows:

$$\begin{aligned}
\Phi^-(z) &= \frac{1}{h}[\overset{*}{W}^-(z, h^*) + \overset{*}{W}^-(z, -h^*)],\\
\Psi^-(z) &= \frac{1}{h}[\mu_1 \overset{*}{W}^-(z, h^*) - \mu_2 \overset{*}{W}^-(z, -h^*)],\\
\Phi^+(z) &= \frac{1}{\mu_1}\Sigma_{yz}^+(z, h^*) + \frac{1}{\mu_2}\Sigma_{yz}^+(z, -h^*),\\
\Phi^-(z) &= \Sigma_{yz}^+(z, h^*) - \Sigma_{yz}^-(z, -h^*),
\end{aligned} \tag{2.14}$$

and the known, functions $S_m^-(z)$, $(m=1, 2)$, are represented in the form

$$\begin{aligned}
S_1^-(z) &= \frac{1}{u_1}\Sigma_{yz}^-(z, h^*) + \frac{1}{\mu_2}\Sigma_{yz}^-(z, -h^*),\\
S_2^-(z) &= \Sigma_{yz}^-(z, h^*) - \Sigma_{yz}^-(z, -h^*).
\end{aligned} \tag{2.15}$$

In order to determine the regions of regularity of Eqs. (2.12) let us observe that $H(z)$ is regular in the strip $\{|\text{Im } z| < \delta, |\text{Re } z| < \infty\}$, and the function $K(z)$ is regular in the strip $\{-\gamma < \text{Im } z < 0, |\text{Re } z| < \infty\}$ or $\{0 < \text{Im } z < \gamma, |\text{Re } z| < \infty\}$, where

$$\delta = \begin{cases} \dfrac{\pi}{2(1-h^*)} & \text{for} \quad 0 < h^* \leqslant \dfrac{1}{2}, \\[2ex] \dfrac{\pi}{2h^*} & \text{for} \quad \dfrac{1}{2} \leqslant h^* < 1; \end{cases} \qquad \gamma = \begin{cases} \dfrac{\pi}{2(1-h^*)} & \text{for} \quad 0 < h^* \leqslant \dfrac{2}{3}, \\[2ex] \dfrac{\pi}{h^*} & \text{for} \quad \dfrac{2}{3} \leqslant h^* < 1. \end{cases}$$

The criteria of applicability of the F-transforms (1.4) and certain physical considerations yield the conclusion that there exists such a non-negative real number $\tau_1 < \delta$ that the functions $\Phi^-(z)$, $\Psi^-(z)$, $S_m^-(z)$ and $\Phi^+(z)$ $\Psi^+(z)$ possess common regions of regularity and, namely, the respective half-planes $\text{Im } z < 0$ and $\text{Im } z > -\tau_1$. This result and also the earlier statement concerning the regularity regions of $H(z)$ and $K(z)$ enable us to determine the common strip of regularity of both Eqs. (2.12),

$$\Omega: \{-\delta < -\tau_1 < \text{Im } z < 0, \ |\text{Re } z| < \infty\}.$$

Let us now pass to the standard method of solution for eqs. (2.2) by means of factorization [2] and observe that the functions (2.13) may be represented in the form

$$\begin{aligned}
H(z) &= \frac{1}{\pi^2} H^-(z)\, H^+(z),\\
K(z) &= \frac{1}{h^*} K^-(z)\, K^+(z).
\end{aligned} \tag{2.16}$$

Here

$$H^+(z)=\frac{\Gamma\left(\frac{1}{2}-\frac{izh^*}{\pi}\right)\Gamma\left[\frac{1}{2}-\frac{iz(1-h^*)}{\pi}\right]e^{\frac{i\beta z}{\pi}}}{\Gamma\left(1-\frac{iz}{\pi}\right)},$$

(2.17)
$$K^+(z)=\frac{\Gamma\left(1-\frac{izh^*}{\pi}\right)\Gamma\left[\frac{1}{2}-\frac{iz(1-h^*)}{\pi}\right]e^{\frac{i\beta z}{\pi}}}{\Gamma\left(\frac{1}{2}-\frac{iz}{\pi}\right)},$$

$$H^-(z)=H^+(-z), \qquad K^-(z)=\frac{1}{z^2}K^+(-z),$$

$$\beta=\ln\left[(h^*)^{h^*}(1-h^*)^{1-h^*}\right].$$

The functions $H^{\pm}(z)$ are regular and non-zero in the respective half-planes $\mathrm{Im}\,z>-\delta$ and $\mathrm{Im}\,z<\delta$; the function $K^-(z)$ is regular and non-zero in the half-plane $\mathrm{Im}\,z<0$, and the function $K^+(z)$ is non-zero in the half-plane $\mathrm{Im}\,z>-\gamma$ and is regular within it except the point at infinity which is the branchpoint.

The functions (2.16) are substituted in Eqs. (2.12) to yield for $z\in\Omega$

(2.18)
$$-\pi^2\frac{\Phi^-(z)}{H^-(z)}=H^+(z)\,\Phi^+(z)+E_1(z),$$

$$-h^*\frac{\Psi^-(z)}{K^-(z)}=K^+(z)\,\Psi^+(z)+E_2(z).$$

Here

(2.19)
$$E_1(z)=H^+(z)\,S_1^-(z), \qquad E_2(z)=K^+(z)\,S_2^-(z).$$

The functions $E_m(z)$ are regular within the common strip $\Omega_1:\{-\delta<\mathrm{Im}\,z<\tau_*, |\mathrm{Re}\,z|<\infty\}$, τ_* being so selected as to satisfy the conditions of existence of the F-transforms of $p_1(x)$, $p_2(x)$. The functions $E_m(z)$ determined in this manner may be represented in the region $\{-\delta<\varepsilon_-<\mathrm{Im}\,z<\varepsilon_+<\tau_*, |\mathrm{Re}\,z|<\infty\}$ in the form [2]

(2.20)
$$E_m(z)=E_m^+(z)-E_m^-(z),$$

where the functions

(2.21)
$$E_m^+(z)=\frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon_-}^{\infty+i\varepsilon_-}\frac{E_m(\zeta)}{\zeta-z}\,d\zeta, \qquad E_m^-(z)=\frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon_+}^{\infty+i\varepsilon_+}\frac{E_m(\zeta)}{\zeta-z}\,d\zeta,$$

are regular in the respective half-planes $\mathrm{Im}\,z>-\delta$ and $\mathrm{Im}\,z<\tau_*$.

On using Eq. (2.20), the relations (2.18) may be written in the form

$$-\pi^2\frac{\Phi^-(z)}{H^-(z)}+E_1^-(z)=H^+(z)\,\Phi^+(z)+E_1^+(z),$$

$$-h^*\frac{\Psi^-(z)}{K^-(z)}+E_2^-(z)=K^+(z)\,\Psi^+(z)+E_2^+(z).$$

whence, on the basis of the generalized Liouville theorem, the solution of the Wiener--Hopf equations (2.12) is obtained:

$$\Phi^-(z)=\frac{1}{\pi^2}H^-(z)\,E_1^-(z),\qquad \Phi^+(z)=-\frac{E_1^+(z)}{H^+(z)},$$

(2.22)

$$\Psi^-(z)=-\frac{1}{h^*}K^-(z)\,[a_0-E_2^-(z)],\qquad \Psi^+(z)=\frac{a_0-E_2^+(z)}{K^+(z)}.$$

The functions $\Phi^-(z)$, $\Psi^-(z)$ and $\Phi^+(z)$, $\Psi^+(z)$ are regular in the respective half-planes $\operatorname{Im} z<0$ and $\operatorname{Im} z>-\tau_1$ and a_0 is a certain constant to be determined from the condition of equalibrium of external forces.

In the considered case of a layer with rigidly clamped edges ($y=\pm h$) the equilibrium of external loads requires that

$$\int_{-\infty}^{\infty}\sigma_{yz}(x,h)\,dx-\int_{-\infty}^{\infty}\sigma_{yz}(x,-h)\,dx=0, \tag{2.23}$$

and in view of the F-transform (1.4) this condition may be replaced with

$$\Sigma_{yz}(0,h)-\Sigma_{yz}(0,-h)=0. \tag{2.24}$$

On the other hand, Eqs. (1.8) may be used to prove that

$$\Sigma_{yz}(\alpha,\pm h)=\frac{\Sigma_{yz}(\alpha,\pm h_2)}{\operatorname{ch}\alpha(h-h_2)}$$

and then the relation (2.24) may be written in the form

$$\Sigma_{yz}(0,h^*)-\Sigma_{yz}(0,-h^*)=0.$$

Thus, by means of Eqs. $(2.14)_4$ and $(2.15)_2$ the final form of the equilibrium condition (2.23) is obtained:

$$\Psi^+(0)+S_2^-(0)=0. \tag{2.25}$$

If the loading of the crack surfaces is such that the point $z=0$ belongs to the strip Ω_1 representing the regularity region of $E_m(z)$, i.e. when $\tau_*>0$, then, in view of regularity of the functions $\Psi^+(z)$ and $S_2^-(z)$ in the region Ω_1, the expressions in Eq. (2.25) make sense and no additional assumptions concerning the loads applied to the edges are necessary.

On the contrary, if the loadings are such that the point $z=0$ does not belong to the region Ω_1, i.e. when $\tau_*=0$, then the expressions in Eq. (2.25) exist only under the condition that $S_2^-(z)$ assumes a finite value at $z=0$. This requirement leads, due to the F-transform definition (1.4), to an additional condition on the crack boundary loading and, namely, to the condition of convergence of the integral

$$\int_{-\infty}^{0}[\sigma_{yz}(x,h_2)-\sigma_{yz}(x,-h_2)]\,dx. \tag{2.26}$$

Making use of Eqs. $(2.22)_4$ and of the fact that $K^+(0)=1$, we obtain from Eq. (2.25) in the general case the relation

$$a_0=E_2^+(0)-S_2^-(0) \quad (\tau_*\geqslant 0). \tag{2.27}$$

If $\tau_*>0$, Eqs. (2.19) and (2.20) may be used to write the constant a_0 in a simpler form

$$a_0=E_2^-(0) \quad (\tau_*>0). \tag{2.28}$$

Once the solution of the Wiener-Hopf Eqs. (2.12) is known, we may proceed to determine the F-transforms of the displacement jumps at the crack surfaces, and also of the stresses σ_{yz} along the crack extensions. In the general case, on the basis of Eqs. (2.14) and (2.15), they are expressed by the formulae

$$\begin{aligned}
\overset{*}{W}{}^-(z,h^*)&=\frac{h\mu_2}{\mu_1+\mu_2}\left[\Phi^-(z)+\frac{1}{\mu_2}\Psi^-(z)\right],\\
\overset{*}{W}{}^-(z,-h^*)&=\frac{h\mu_1}{\mu_1+\mu_2}\left[\Phi^-(z)-\frac{1}{\mu_1}\Psi^-(z)\right],\\
\Sigma_{yz}^+(z,h^*)&=\frac{\mu_1\mu_2}{\mu_1+\mu_2}\left[\Phi^+(z)+\frac{1}{\mu_2}\Psi^+(z)\right],\\
\Sigma_{yz}^+(z,-h^*)&=\frac{\mu_1\mu_2}{\mu_2+\mu_2}\left[\Phi^+(z)-\frac{1}{\mu_1}\Psi^-(z)\right],
\end{aligned} \tag{2.29}$$

the functions $\Phi^\pm(z)$ and $\Psi^\pm(z)$ being defined by Eqs. (2.22).

If the cracks are loaded identically, i.e. if $\sigma_{yz}(x,h_2)=\sigma_{yz}(x,-h_2)$, the expressions (2.29) are considerably simplified in view of $\Psi^\pm(z)=0$.

$$\begin{aligned}
&\overset{*}{W}{}^-(z,h^*)=\frac{h}{\mu_1}\bar{\Phi}^-(z), \qquad \overset{*}{W}{}^-(z,-h^*)=\frac{\mu_1}{\mu_2}\overset{*}{W}{}^-(z,h^*),\\
&\Sigma_{yz}^+(z,\pm h^*)=\bar{\Phi}^+(z).
\end{aligned} \tag{2.30}$$

Here

$$\begin{aligned}
&\bar{\Phi}^-(z)=\frac{1}{\pi^2}H^-(z)\,\chi^-(z), \qquad \bar{\Phi}^+(z)=-\frac{\chi^+(z)}{H^+(z)},\\
&\chi^\pm(z)=\frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon_\mp}^{+\infty+i\varepsilon_\mp}\frac{\chi(\zeta)}{\zeta-z}\,d\zeta, \qquad \chi(z)=H^+(z)\,\Sigma_{yz}^-(z,h^*).
\end{aligned} \tag{2.31}$$

From Eqs. (2.30) it follows that in the case of identical loading of both cracks, the stresses σ_{yz} along their extensions are also identical and do not depend on the material constants, i.e. they are the same as in the case of a homogeneous layer.

3. Stress intensity factors

The application of the inverse F-transforms to the formulae (2.29) leads to a purely formal solution of the problem since, due to a rather complex form of the integrands, the integrations prescribed cannot be performed analytically. However, use can be made of the Abel theorem [3] which enables us to predict the behaviour of the function $f(x)$ at $x \to \pm 0$ and $x \to \pm\infty$ once the behaviour of the transform $F^{\pm}(\alpha)$ at $|\alpha| \to \infty$ and $|\alpha| \to 0$ is known. Applying this procedure to Eqs. (2.29), we determine the values which are of principal interest from the point of view of the crack stability theory and, namely, the stress intensity factors at the crack tips and the crack displacements at the tips.

First of all let us observe that the functions $E_m^{\pm}(z)$ given by Eqs. (2.20) may, in view of regularity of $E_m(z)$ in Ω_1, be represented in the form

$$E_m^{\pm}(z) = -\frac{1}{z}\left[B_m - \frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon_{\mp}}^{\infty+i\varepsilon_{\mp}} \frac{\zeta E_m(\zeta)}{\zeta - z}\, d\zeta\right], \tag{3.1}$$

where

$$B_m = \frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon_-}^{\infty+i\varepsilon_-} E_m(\zeta)\, d\zeta. \tag{3.2}$$

Returning in Eqs. (2.29) to the variables $\alpha = z/h$ and using the corresponding formulae (2.17), (2.22) and (3.1), the first terms of expansion of the function (2.29) into a power series in the neighbourhood of the point $|\alpha| = \infty$ are

$$\begin{aligned} &\overset{*}{W}{}^{-}(\alpha, h^*) = -\frac{\sqrt{2i}N_1}{\mu_1}\frac{1}{\alpha\sqrt{\alpha}}, \qquad \overset{*}{W}{}^{-}(\alpha, -h^*) = -\frac{\sqrt{2i}N_2}{\mu_2}\frac{1}{\alpha\sqrt{\alpha}}, \\ &\Sigma_{yz}^{+}(\alpha^+, h^*) = \frac{N_1}{\sqrt{-2i}}\frac{1}{\sqrt{\alpha}}, \qquad \Sigma_{yz}^{+}(\alpha, -h^*) = \frac{N_2}{\sqrt{-2i}}\frac{1}{\sqrt{\alpha}}. \end{aligned} \tag{3.3}$$

Here

$$\begin{aligned} N_1 &= -\frac{\mu_1\mu_2}{\sqrt{h}(\mu_1+\mu_2)}\left[\frac{iB_1}{\pi} - \frac{a_0}{\mu_2\sqrt{h^*}}\right], \\ N_2 &= -\frac{\mu_1\mu_2}{\sqrt{h}(\mu_1+\mu_2)}\left[\frac{iB_1}{\pi} + \frac{a_0}{\mu_1\sqrt{h^*}}\right]. \end{aligned} \tag{3.4}$$

The application of the Abel theorem mentioned earlier and of Eqs. (3.3) makes it possible to derive the formulae for the crack tip displacements and the stress distribution at the tips

$$\begin{aligned} &[\![w(x,h^*)]\!] = \frac{4N_1}{\mu_1}\sqrt{-x}, \quad [\![w(x,-h^*)]\!] = \frac{4N_2}{\mu_2}\sqrt{-x} \quad \text{for} \quad x \to (-0), \\ &\sigma_{yz}(x,h^*) = \frac{N_1}{\sqrt{x}}, \qquad \sigma_{yz}(x,-h^*) = \frac{N_2}{\sqrt{x}} \qquad \text{for} \quad x \to (+0). \end{aligned} \tag{3.5}$$

According to the generally accepted definition, the stress intensity factors at the upper and lower tips of the cracks are the values

$$K_{\mathrm{III}}^{(m)} = \sqrt{2\pi}\, N_m \qquad (m=1,2), \tag{3.6}$$

N_m are given by Eqs. (3.4) and the index III refers to the antiplane state of strain (Mode III). Using that definition and Eqs. (3.4), we determine the sum and difference of the stress intensity factors at the crack tips:

$$\begin{aligned} K_{\mathrm{III}}^{(1)} + K_{\mathrm{III}}^{(2)} &= -\sqrt{\frac{2\pi}{h}}\,\frac{\mu_1\mu_2}{\mu_1+\mu_2}\left[\frac{2iB_1}{\pi} - \frac{\mu_1-\mu_2}{\mu_1\mu_2\sqrt{h^*}}a_0\right], \\ K_{\mathrm{III}}^{(1)} - K_{\mathrm{III}}^{(2)} &= \sqrt{\frac{2\pi}{hh^*}}\,a_0. \end{aligned} \tag{3.7}$$

It follows that the difference of the stress intensity factors is a function of crack loadings and of the distance between them, and is independent of the material constants.

In the case of a homogeneous layer ($\mu=\mu_1=\mu_2$), the sum and difference of the stress intensity factors are calculated from Eqs. (3.7):

$$\begin{aligned} \bar{K}_{\mathrm{III}}^{(1)} + \bar{K}_{\mathrm{III}}^{(2)} &= -\sqrt{\frac{2}{\pi h}}\, i\bar{B}_1, \\ \bar{K}_{\mathrm{III}}^{(1)} - \bar{K}_{\mathrm{III}}^{(2)} &= \sqrt{\frac{2\pi}{hh^*}}\,a_0, \end{aligned} \tag{3.8}$$

with the notations

$$\begin{aligned} \bar{B}_1 &= \frac{1}{2\pi i}\int_{-\infty+i\varepsilon_-}^{\infty+i\varepsilon_-} H^+(\zeta)\,\bar{S}_1^-(\zeta)\,d\zeta, \\ \bar{S}_1^-(z) &= \Sigma_{yz}^-(z,h^*) + \Sigma_{yz}^-(z,-h^*). \end{aligned} \tag{3.9}$$

It follows that in the case of a homogeneous layer the stress intensity factors are independent of the Lamé constant μ, and its difference is the same as in the case of a non-homogeneous layer.

The results obtained here and concerning both the homogeneous and non-homogeneous layers indicate that it should be possible to select the crack loadings so as to obtain any prescribed value of the ratio of the stress intensity factors produced by the loadings.

1. In the case of crack surface loadings which lead to

$$a_0 = 0, \tag{3.10}$$

in both cases of non-homogeneous and homogeneous layers the stress intensity factors at both tips are equal to each other; in a non-homogeneous layer

$$K_{\mathrm{III}} = K_{\mathrm{III}}^{(1)} = K_{\mathrm{III}}^{(2)} = -\sqrt{\frac{2}{\pi h}}\,\frac{\mu_1\mu_2}{\mu_1+\mu_2}\,iB_1, \tag{3.11}$$

and in a homogeneous layer

$$\bar{K}_{\mathrm{III}}=\bar{K}^{(1)}_{\mathrm{III}}=\bar{K}^{(2)}_{\mathrm{III}}=-\frac{i\bar{B}_1}{\sqrt{2\pi h}}. \tag{3.12}$$

This is also true in the case when both cracks are loaded identically since then $S_2^-(z)=$ $=0$ what, in view of Eqs. (2.21), is a sufficient condition for the constant a_0 to vanish.

On assuming $\sigma_{yz}(x, h_2)=\sigma_{yz}(x, -h_2)$ for $x<0$ we obtain from Eqs. (3.11) and (3.12) the stress intensity factor K_{III} independent from the material constants μ_1, μ_2 and equal to the factor $\bar{K}_{\mathrm{III}}$ corresponding to a homogeneous layer,

$$K_{\mathrm{III}}=\bar{K}_{\mathrm{III}}=-\sqrt{\frac{2}{\pi h}}\,iB, \tag{3.13}$$

where

$$B=\frac{1}{2\pi i}\int_{-\infty+i\varepsilon_-}^{\infty+i\varepsilon_-} H^+(\zeta)\,\Sigma^-_{yz}(\zeta, h^*)\,d\zeta. \tag{3.14}$$

2. If the crack loads are such that

$$B_1=0, \tag{3.15}$$

then the stress intensity factors in a non-homogeneous layer are equal to

$$K^{(1)}_{\mathrm{III}}=\sqrt{\frac{2\pi}{hh^*}}\,\frac{\mu_1}{\mu_1+\mu_2}\,a_0, \qquad \mu_2\,K^{(1)}_{\mathrm{III}}=-\mu_1\,K^{(2)}_{\mathrm{III}}, \tag{3.16}$$

while in a homogeneous layer

$$\bar{K}^{(1)}_{\mathrm{III}}=\sqrt{\frac{\pi}{2hh^*}}\,a_0, \qquad \bar{K}^{(1)}_{\mathrm{III}}=-\bar{K}^{(2)}_{\mathrm{III}}. \tag{3.17}$$

The condition (3.15) is satisfied also in the case when the crack surface loadings are such that $\mu_2\,\sigma_{yz}(x, h_2)=-\mu_1\,\sigma_{yz}(x, -h_2)$.

3. If the crack loads are such that

$$\frac{iB_1}{\pi}=\frac{\mu_1-\mu_2}{2\mu_1\mu_2\sqrt{h^*}}\,a_0, \tag{3.18}$$

then in the case of a non-homogeneous layer the stress intensity factors are equal to

$$K^{(1)}_{\mathrm{III}}=\sqrt{\frac{\pi}{2hh^*}}\,a_0, \qquad K^{(1)}_{\mathrm{III}}=-K^{(2)}_{\mathrm{III}}. \tag{3.19}$$

4. If the crack loads are such that

$$\frac{iB_1}{\pi}=\frac{a_0}{\mu_2\sqrt{h^*}} \quad \text{or} \quad \frac{iB_1}{\pi}=-\frac{a_0}{\mu_1\sqrt{h^*}}, \tag{3.20}$$

then one of the stress intensity factors vanishes and either

$$K_{\rm III}^{(1)}=0, \quad K_{\rm III}^{(2)}=-\sqrt{\frac{2\pi}{hh^*}}\,a_0, \tag{3.21}$$

or

$$K_{\rm III}^{(1)}=\sqrt{\frac{2\pi}{hh^*}}\,a_0, \quad K_{\rm III}^{(2)}=0. \tag{3.22}$$

4. Particular cases

To illustrate the results obtained let us assume the crack surfaces to be loaded by the following stresses:

$$\begin{aligned} \sigma_{yz}(x, h_2)&=p_1\exp(\lambda_1 x) \quad \text{for} \quad x<0,\\ \sigma_{yz}(x, -h_2)&=p_2\exp(\lambda_2 x) \quad \text{for} \quad x<0, \end{aligned} \tag{4.1}$$

with $p_m=\text{const.}$, $\lambda_m>0$ $m=1, 2$.

The F-transforms applied to the functions (4.1) yield the results

$$\Sigma_{yz}^-(z, h^*)=\frac{hp_1}{i\sqrt{2\pi}}\frac{1}{z-i\lambda_1^*}, \quad \Sigma_{yz}^-(z, -h^*)=\frac{hp_2}{i\sqrt{2\pi}}\frac{1}{z-i\lambda_2^*},$$

whence, in view of Eqs. (2.15),

$$S_1^-(z)=\frac{h}{i\sqrt{2\pi}}\left[\frac{p_1}{\mu_1}\frac{1}{z-i\lambda_1^*}+\frac{p_2}{\mu_2}\frac{1}{z-i\lambda_2^*}\right],$$

$$S_2^-(z)=\frac{h}{i\sqrt{2\pi}}\left[\frac{p_1}{z-i\lambda_1^*}-\frac{p_2}{z-i\lambda_2^*}\right],$$

with the notation $\lambda_m^*=h\lambda_m$.

In the case considered here $\tau_*=\min(\lambda_1^*, \lambda_2^*)>0$ and so, due to Eqs. (2.21), (2,28), (3.2) and (3.7), the stress intensity factors are given by the formulae

$$\begin{aligned} K_{\rm III}^{(1)}&=-\frac{K^+(i\lambda_1^*)}{\lambda_1^*(1+\mu^*)}\sqrt{\frac{h}{h^*}}\{p_1[\mu^*+\omega(i\lambda_1^*)]-p_2\kappa\mu^*[1-\omega(i\lambda_2^*)]\},\\ K_{\rm III}^{(2)}&=-\frac{K^+(i\lambda_1^*)}{\lambda_1^*(1+\mu^*)}\sqrt{\frac{h}{h^*}}\{p_2\kappa[1+\mu^*\omega(i\lambda_2^*)]-p_1[1-\omega(i\lambda_1^*)]\}. \end{aligned} \tag{4.2}$$

Here

$$\begin{aligned} \kappa&=\frac{\lambda_1^* K^+(i\lambda_2^*)}{\lambda_2^* K^+(i\lambda_1^*)}, \quad \mu^*=\frac{\mu_1}{\mu_2},\\ \omega(t)&=-\frac{it\sqrt{h^*}\,H^+(t)}{\pi K^+(t)}. \end{aligned} \tag{4.3}$$

In addition, $0<\omega(it)<1$ for $t>0$, and the functions $K^+(t)$, $H^+(t)$ are given by Eqs. (2.17).

In the case when the parameters p_m and λ_m characterizing the boundary condition (4.1) are so selected that

$$\frac{p_1}{p_2}=\kappa\mu^*\frac{1-\omega(i\lambda_2^*)}{\mu^*+\omega(i\lambda_1^*)}$$

or

$$\frac{p_1}{p_2}=\kappa\frac{1+\mu^*\omega(i\lambda_2^*)}{1-\omega(i\lambda_1^*)},$$

the condition $(3.20)_1$ or $(3.20)_2$ is satisfied and in view of Eqs. (2.21) or (2.22) the stress intensity factors are equal to

$$K_{III}^{(1)}=0,$$
$$K_{III}^{(2)}=-\frac{p_2K^+(i\lambda_2^*)}{\lambda_2^*}\sqrt{\frac{h}{h^*}}\frac{\omega(i\lambda_1^*)+\mu^*\omega(i\lambda_2^*)}{\mu^*+\omega(i\lambda_1^*)} \tag{4.4}$$

or

$$K_{III}^{(1)}=-\frac{p_1K^+(i\lambda_1^*)}{\lambda_1^*}\sqrt{\frac{h}{h^*}}\frac{\omega(i\lambda_1^*)+\mu^*\omega(i\lambda_2^*)}{1+\mu^*\omega(i\lambda_2^*)}, \tag{4.5}$$
$$K_{III}^{(2)}=0.$$

In the case of the parameters p_m and λ_m selected so that

$$\frac{p_1}{p_2}=\kappa,$$

κ being defined by Eq. $(4.3)_1$, the condition (3.10) is fulfilled, and the stress intensity factors at both crack tips are equal to each other. These factors are obtained from Eqs. (3.11):

$$K_{III}=-\frac{p_2\sqrt{h}}{\pi(1+\mu^*)}[\kappa H^+(i\lambda_1^*)+\mu^*H^+(i\lambda_2^*)]. \tag{4.6}$$

The condition (3.10) is also fulfilled when the crack surfaces are subject to identical loads. The assumption

$$\sigma_{yz}(x,\pm h_2)=p\exp(\lambda x)\quad \text{for}\quad x<0,$$

leads, in view of Eqs. (3.13), (3.14) or (4.6), with $\kappa=1$, to the stress intensity factor at both crack tips:

$$K_{III}=-\frac{p\sqrt{h}}{\pi}H^+(i\lambda^*),\qquad \lambda^*=h\lambda. \tag{4.7}$$

If both cracks are loaded uniformly by equal stresses, i.e. $\sigma_{yz}(x,\pm h)=p$ for $x<0$, then $\tau_*=0$ and from Eqs. (3.13), (3.14) or (4.7), with $\lambda^*=0$, we obtain

$$K_{III}=-p\sqrt{h}. \tag{4.8}$$

This result will now be used to determine the stress intensity factors at the crack tips in a non-homogeneous elastic layer at the edges of which constant displacements

$w(x, \pm h) = \pm w_0$ are prescribed, the crack surfaces being free from stresses . 2a).

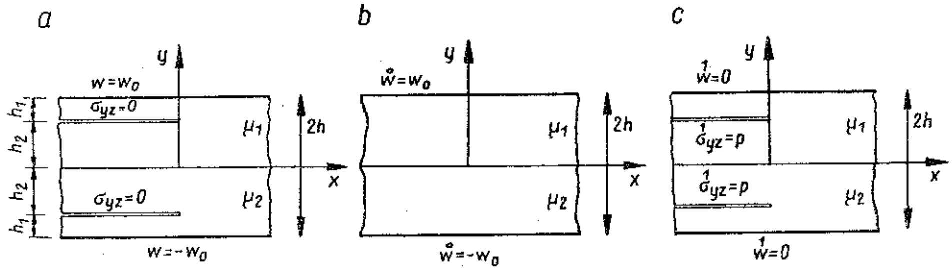

FIG. 2.

Using the superposition principle the solution sought for may be composed of the solution corresponding to a solid layer with prescribed displacements at the boundaries $\overset{0}{w}(x, \pm h) = \pm w_0$ (Fig. 2b), and another solution corresponding to a layer constaining two cracks loaded by stress $\overset{1}{\sigma}_{yz}(x, \pm h) = -\overset{0}{\sigma}_{yz}(x, \pm h)$ (Fig. 2c); the layer is rigidly clamped at the edges.

The solution of the problem shown in Fig. 2b has the form

$$\overset{0}{w}(x, y) = \frac{w_0(\mu_1 - \mu_2)}{\mu_1 + \mu_2} - \frac{2w_0}{\mu_1 + \mu_2} \frac{y}{h} \begin{cases} \mu_2 & \text{for} \quad 0 \leqslant y \leqslant h, \\ \mu_1 & \text{for} \quad -h \leqslant y \leqslant 0, \end{cases}$$

$$\overset{0}{\sigma}_{xz}(x, y) = 0,$$

$$\overset{0}{\sigma}_{yz}(x, y) = \frac{2\mu_1 \mu_2}{\mu_1 + \mu_2} \frac{w_0}{h}$$

and so

$$\overset{1}{\sigma}_{yz}(x, \pm h_2) = p = -\frac{2\mu_1 \mu_2}{\mu_1 + \mu_2} \frac{w_0}{h}. \tag{4.9}$$

Substitution of the function (4.9) in Eq. (4.8) yields the stress intensity factors at the cracks tips in the original configuration (Fig. 2a); they are equal to

$$K_{III} = \frac{2\mu_1 w_0}{\sqrt{h}\,(1 + \mu^*)}.$$

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Streszczenie

KONCENTRACJA NAPRĘŻEŃ W NIEJEDNORODNEJ WARSTWIE SPRĘŻYSTEJ OSŁABIONEJ SZCZELINAMI

W pracy rozważa się statyczne zagadnienie koncentracji naprężeń wokół wierzchołków dwóch półnieskończonych szczelin znajdujących się w nieskończonej i niejednorodnej warstwie sprężystej. Zakłada się, że spełnione są warunki antypłaskiego stanu odkształcenia. Poza tym przyjęto, że brzegi warstwy są sztywno zamocowane, na powierzchniach zaś obu szczelin dane są różne obciążenia.

Stosując zespoloną transformację całkową Fouriera, problem poszukiwania współczynników intensywności naprężenia w wierzchołkach szczelin sprowadza się do rozwiązania odpowiedniego układu równań typu Wienera-Hopfa. Zakładając, że szczeliny są usytuowane symetrycznie względem płaszczyzny rozdzielającej dwa różne materiały sprężyste, rozdziela się układ równań na dwa niezależne równania Wienera-Hopfa, dla których podano ścisłe rozwiązania.

Wyznaczono współczynniki intensywności naprężenia w obu wierzchołkach szczelin oraz przedyskutowano szereg przypadków szczególnych. Dla ilustracji otrzymanych wyników rozwiązano przykład, w którym przyjęto, że na brzegach szczelin dane jest przemieszczenie, powierzchnie zaś obu szczelin wolne są od obciążeń.

Резюме

КОНЦЕНТРАЦИЯ НАПРЯЖЕНИЙ В НЕОДНОРОДНОМ УПРУГОМ СЛОЕ ОСЛАБЛЕННОМ ЩЕЛЯМИ

В работе рассматривается статическая задача концентрации напряжений вокруг вершин двух полубесконечных щелей, находящихся в бесконечном и неоднородном упругом слое. Предпологается, что удовлетворены условия антиплоского деформационного состояния. Кроме этого принято, что границы слоя жестко закреплены, на поверхностях же обоих щелей даны разные нагрузки.

Применяя комплексное интегральное преобразование Фурье, задача нахождения коэффициентов интенсивности напряжения в вершинах щелей сводится к решению соответствующей системы уравнений типа Винера-Хопфа. Предполагая, что щели помещены симметричным образом по отношению к плоскости разделяющей два разных упругих материала, система уравнений разделяется на два независимых уравнения Винера-Хопфа, для которых даются точные решения.

Определены коэффициенты интенсивности напряжения в обоих вершинах щелей, а также обсужден ряд частных случаев. Для иллюстрации полученных результатов решен пример, в котором принимается, что на границах щелей дано перемещение, поверхности же обоих щелей свободны от нагрузок.

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