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EDDY VISCOSITY IN ACCELERATED AND RETARDED CASCADE-WAKE FLOWS

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The paper presents the results of both theoretical and experimental analyses of the eddy viscosity evolution behind the row of symmetrical bodies in a cascade flow with a longitudinal pressure gradient having a constant value in a downstream direction. The investigations have proved that the eddy viscosity coefficient decreases decidedly with the growth of the coordinate $\bar{x}_1$. For $\bar{x}_1 > 11$ it appears to be the increasing function of the longitudinal pressure gradient.

Notations

l aerodynamical pitch of the cascade flow,
p, p_t, q static, total and dynamic pressures,
t cascade pitch,
U_i mean velocity components,
U_∞ undisturbed flow velocity upstream of the cascade plate,
U_s space average value of the cascade-flow velocity,
u_i turbulence velocity components,
$\rho \overline{u_i u_j}$ turbulence stresses, turbulence stress tensor,
x_i Cartesian coordinates,
$\underline{\alpha}$ relative aerodynamical pitch of the cascade flow,
β parameter of mean flow nonuniformity,
γ dimensionless average velocity,
η dimensionless coordinate ($\eta = x_2/l$),
ρ fluid density,
ν_T eddy viscosity coefficient,
ε_0 initial turbulence of oncoming flow.

1. Introduction

In spite of many investigations carried out in the domain of turbulence, the problems of turbulent flows continue to be of major interest in a number of scientific laboratories.

One of the particular questions in this vast and interesting field of knowledge is undoubtedly the evolution of turbulent wake flows. Although the wake problem related to the single isolated body has been the subject of a great deal of scientific research, this topic is not covered by a large bibliography and literature which would discuss significantly the evolution of the velocity behind the arbitrary airfoil cascade.

In particular, there is a distinct lack of analysis concerning the influence exerted on the macro- and microstructure of the cascade-wake flow by the longitudinal pressure gradient with a constant value in a downstream direction.

The development of the wake shape behind a single isolated body has been the object of studies of numerous scientists. As an example we can quote here the works of SCHLICHTING [1], GÖRTLER [2], REINCHARDT [3], KOVASZNAY [4], and above all TOWNSEND [5, 6, 7, 8] whose studies, especially in the period 1940–60, considerably enlarged the actual state of knowledge of the structure of turbulent shear flows.

Far less numerous are the works dealing with the flow pattern behind the cascade of aerofoil blades. One of the first investigations in this field was undertaken by OLSSON [9] who, on the basis of the mixing length concept, derived the relationships describing the process of mean velocity field equalization in cascade flow without any longitudinal pressure gradient.

The evolution of the cascade flow was also studied by STEWART [10], SATO [11] as well as by TAMAKI and OSHIMA [12], authors of experimental analysis of the turbulence structure in a flow behind the row of parallel rods. The empirical data obtained by Sato pointed out that the distributions of turbulent shear stresses in any flow cross-section were similar to the variations of the mean velocity gradient; this suggested the mutual proportionality of the quantities mentioned above.

Consequently, in all the planes perpendicular to the main flow, the coefficient of eddy viscosity ν_T was nearly constant, slightly increasing with the growth of the distance from the rods.

On the contrary, the experiment of TAMAKI and OSHIMA [12] found ν_T to be a decreasing function of the coordinate x_1, approximately according to the relation $\nu_T \sim x_1^{-1}$.

Discussion on this subject was continued in the works [13] and [14], dealing with the development of the wake flow behind the cascade of symmetrical bodies. The theoretical considerations were based here on the assumption that the overall velocity field may be described by means of two velocity scales, different for the mean and for the turbulent motion. Moreover, the eddy viscosity coefficient was found to be constant across the wake and to decay in the downstream direction according to the relation $\nu_T = c x_1^{-1}$, with the parameter c dependent on the turbulence intensity of oncoming flow.

An interesting study of a flow field behind the grid of parallel bars was presented by KLATT [15]. In his analysis of the relations between the production and dissipation of turbulence, Klatt stated that the production was noticeable in a close vicinity of the grid only. Eddy viscosity in a flow cross-section was expressed by the Fourier series and supposed to be a power function of a distance from the bars.

The presented rewiev states that none of the quoted papers has analysed to what extent the eddy viscosity in the cascade-wake flow was affected by the longitudinal pressure gradient.

The first attempt at an empirical penetration of the cascade-wake flow problem in complicance with the longitudinal pressure gradient was undertaken in the work [16]. In particular, it was found that the negative pressure gradient accelerates the mean velocity field equalization and damps the turbulent fluctuations of the flowing medium.

The presented paper, being the sequence of the previous one, is devoted to the theoretical and experimental analyses of the eddy viscosity evolution in the cascade-wake flows with a constant value of $\partial p(x_1)/\partial x_1$ in a downstream direction.

2. Theoretical considerations

The starting point for the theoretical description of the type of flow considered here is the equation of turbulent motion in which the influence of compressibility as well as molecular viscosity may be disregarded as negligible. On the basis of the assumption mentioned above, this equation can be presented in the form

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}\left(\overline{u_i u_j}\right). \tag{2.1}$$

The flow behind the cascade of symmetrical bodies may be treated as a two-dimensional one ($U_3=0$). Moreover, when the coordinate system is assumed in accordance with Fig. 1, the component U_1 of the mean velocity becomes predominant ($U_2/U_1 \ll 1$).

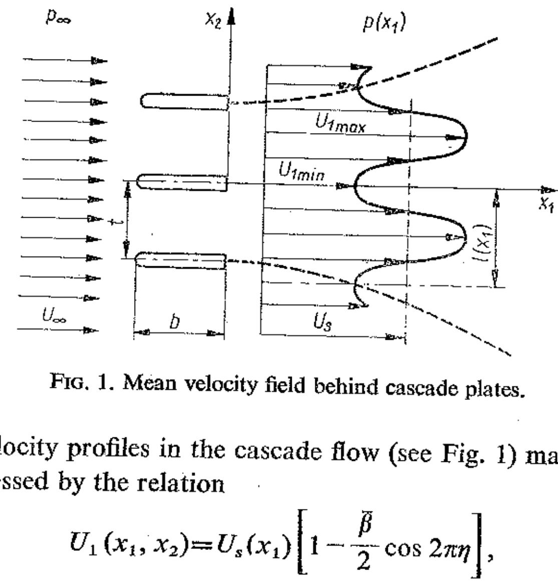

FIG. 1. Mean velocity field behind cascade plates.

The mean velocity profiles in the cascade flow (see Fig. 1) may be, according to [9] or [14], expressed by the relation

$$U_1(x_1, x_2) = U_s(x_1)\left[1 - \frac{\beta}{2}\cos 2\pi\eta\right], \tag{2.2}$$

where the parameter of the mean-flow nonuniformity $\bar{\beta}$ is defined as

$$\bar{\beta}(x_1) = \frac{U_{max} - U_{min}}{U_s}. \tag{2.3}$$

The velocity U_s from Eqs. (2.2) and (2.3) can be acquired from the continuity equation according to the formula

$$U_s t = \int_{-l/2}^{l/2} U_1(x_1, x_2)\, dx_2 \tag{2.4}$$

and stands for the space-average value of flow velocity behind the cascade plates.

The term in Eq. (2.1) including the longitudinal pressure gradient may be replaced by the suitable combination of the velocity field components, what demands, however, additional analysis of the type of flow under consideration.

It is quite simple to solve the problem in the case of a boundary layer or in the wake flow behind a single, isolated body where, in the outer region characterized by the undisturbed flow velocity U_∞, the velocity field may be treated as a potential one and where the formula

$$p_t(x_1)=p(x_1)+\frac{\rho}{2}U_\infty^2(x_1)=\text{const}$$

is to be fulfilled.

In the cascade flow, however, due to total pressure losses, the total pressure $p_t(x_1)$ cannot be treated as an absolute constant value along any of the streamlines. Undisputed experimental evidence observed in a number of cascade flows proves that there is a faster equalization of the static than the total pressure in the downstream direction. Thus, at a relatively small distance behind the plates, the static pressure distribution becomes practically independent on the coordinate x_2. In such circumstances the space-average value of the total pressure (p_{tm}) may be determined in several different ways, depending on the assumptions taken into account:

$$p_{tm}^{*}=p+\frac{\rho}{2}U_s^2(x_1),$$

$$(2.5)\qquad p_{tm}^{**}=p+\frac{\rho}{2l(x_1)}\int\limits_0^{l(x_1)}U_1^2(x_1,x_2)\,dx_2=p+\frac{\rho}{2}U_s^2\left(1+\frac{1}{8}\bar{\beta}^2\right),$$

$$p_{tm}^{***}=p+\frac{\rho}{2U_s\,l(x_1)}\int\limits_0^{l(x_1)}U_1^3(x_1,x_2)\,dx_2=p+\frac{\rho}{2}U_s^2\left(1+\frac{3}{8}\bar{\beta}^2\right).$$

It is to be noticed that all the formulae quoted above differ from each other only by the factor comprising $\bar{\beta}^2$, the influence of which decreases rapidly with the growth of the coordinate x_1. Since in the considered region of flow ($\bar{x}>9$) the value of $3/8\bar{\beta}^2$ does not exceed 0.00375, this allows us to accept the expression (2.6) and assume that the quantity p_{tm}^* is practically independent of the coordinate x_1:

$$(2.6)\qquad p_{tm}(x_1)=p_{tm}^*=p(x_1)+0.5\rho\,U_s^2(x_1)=\text{const}$$

what yields

$$(2.7)\qquad \frac{1}{\rho}\frac{\partial p}{\partial x_1}=-U_s\frac{dU_s}{dx_1}.$$

In the two-dimensional flow analysed here, the coefficient of eddy viscosity in conformity with Boussinesq concept may be defined as

$$(2.8)\qquad \nu_T=-\frac{\overline{u_1u_2}}{\partial U_1/\partial x_2}$$

and assumed to have a constant value across the wake flow (cf. [7, 11, 14]).

Substituting Eqs. (2.6) and (2.7) in Eq. (1.1), the latter, after bilateral multiplication by $x_2\,dx_2$, can be presented in the integral form

$$\int_0^{l/2}\left(U_1\frac{\partial U_1}{\partial x_1}+U_2\frac{\partial U_1}{\partial x_2}-U_s\frac{dU_s}{dx_1}\right)x_2\,dx_2=\nu_T\int_0^{l/2}\frac{\partial^2 U_1}{\partial x_2^2}x_2\,dx_2\,. \tag{2.9}$$

The lateral component U_2 of mean velocity may be evaluated from the continuity equation for two-dimensional, incompressible flow as

$$U_2=\int_0^{x_2}(-\partial U_1/\partial x_1)\,dx_2. \tag{2.10}$$

Applying the relations (2.2) and (2.10) in Eq. (1.9), one may easily find the expression

$$l^2\frac{dU_s}{dx_1}\left(4+\frac{\pi^2\beta}{8}-\frac{\pi^2}{2}\right)+U_s l^2\frac{d\beta}{dx_1}\left(\frac{1}{\beta}+\frac{\pi^2}{8}\right)+ +U_s l\frac{dl}{dx_1}\left(2-\frac{\pi^2}{2}-\frac{\pi^2\beta}{16}\right)=-4\pi^2\nu_T, \tag{2.11}$$

which, after some simplifications, leads to the asymptotic equation

$$l^2\frac{dU_s}{dx_1}-\frac{U_s l^2}{\beta}\frac{d\beta}{dx_1}+3U_s l\frac{dl}{dx_1}=4\pi^2\nu_T. \tag{2.12}$$

Introducing here the quantities

$$\begin{aligned}&\alpha(x_1)=l(x_1)/t,\\ &\gamma(x_1)=U_s(x_1)/U_\alpha,\\ &\bar{x}_1=x_1/t\end{aligned} \tag{2.13}$$

and keeping in mind that $\alpha\gamma=$, Eq. (2.12) can be expressed in the dimensionless form

$$\frac{1}{\beta\gamma}\frac{d\beta}{d\bar{x}_1}+\frac{2}{\gamma^2}\frac{d\gamma}{d\bar{x}_1}=-\nu_T\frac{4\pi^2}{U_\infty t}. \tag{2.14}$$

If, furthermore, the reduced static pressure is defined by

$$\bar{p}=\frac{p-p_\infty}{0.5\rho U_\infty^2} \tag{2.15}$$

then, after utilizing the relation (2.6), the longitudinal pressure gradient will be expressed as

$$\Phi=\frac{d\bar{p}}{d\bar{x}_1}=-2\gamma\frac{d\gamma}{d\bar{x}_1}. \tag{2.16}$$

Applying this formula in Eq. (2.14), one can derive the equation

$$\frac{d\beta}{\beta}=\left(\frac{\Phi}{\gamma^2}-\frac{4\pi^2\gamma}{U_\infty t}\nu_T\right)d\bar{x}_1 \tag{2.17}$$

which, when integrated within the limits $\bar{x}_1, (\bar{x}_1)_0$ and $\bar{\beta}, \bar{\beta}_0$ leads finally to the relation

$$(2.18) \qquad \bar{\beta}=\bar{\beta}_0 \exp\left[\int_{(\bar{x}_1)_0}^{\bar{x}_1}\left(\frac{\Phi}{\gamma^2}-\frac{4\pi^2\gamma}{U_\infty t}\nu_T\right)d\bar{x}_1\right]$$

expressing in the most general case the parameter of the mean flow nonuniformity $\bar{\beta}$ as an implicit function of the quantities Φ, γ and ν_T. Since the latter depend in a certain unknown way on the coordinate $\bar{x}$, it is impossible to obtain the analytical solution of Eq. (2.18) in a closed form.

Let us assume, however, that along any streamline: $\Phi=\partial\bar{p}/\partial\bar{x}_1=\text{const}$ and that, according to experimental evidence, the parameter $\bar{\beta}$ may be treated as the power function of the distance from the cascade

$$(2.19) \qquad \frac{\bar{\beta}}{\bar{\beta}_0}=\left[\frac{\bar{x}_1}{(\bar{x}_1)_0}\right]^{-\kappa}.$$

Substituting Eq. (2.19) in Eq. (2.17), the result may be expressed in the form

$$(2.20) \qquad \frac{\nu_T}{U_\infty t}=\frac{1}{4\pi^2\gamma}\left(\frac{\Phi}{\gamma^2}+\frac{\kappa}{\bar{x}_1}\right),$$

which describes the eddy viscosity evolution in a wake flow with the longitudinal pressure gradient having the constant value in a downstream direction. By applying the last relation, it is possible to determine the eddy viscosity coefficient in a relatively simple way by using the measurements of the mean-velocity field parameters, which makes the "metrological" solution of the problem easier.

It should be noticed here that the longitudinal pressure gradient $\partial\bar{p}/\partial\bar{x}_1$ may be realized by the flow spreading in the $x_1\,x_3$ plane as well. In such a case $l(x)=t=\text{const}$ and after neglecting the third term in the left-hand side of Eq. (2.12), one can easily find

$$(2.21) \qquad \frac{\nu_T}{U_\infty t}=\frac{\gamma}{4\pi^2}\left(\frac{\kappa}{\bar{x}_1}-\frac{\Phi}{2\gamma^2}\right).$$

To be precise, the relations (2.20) and (2.21) are valid only at a sufficiently far distance behind the cascade, where the approximations made during their derivation may be assumed to be fully justified.

Nevertheless, even in the nearer wake region the above formula gives a quick, though not so accurate estimation of the eddy viscosity coefficient.

In particular, for cascade flows without the longitudinal pressure gradient ($\Phi=0$ and $\gamma=1$) Eq. (2.20) is directly transformed into the relation

$$(2.22) \qquad \frac{\nu_T}{U_\infty t}=\frac{\kappa}{4\pi^2}\bar{x}_1^{-1}$$

already given in the quoted paper [14].

3. Organization of experiment

In order to verify the theoretical considerations previously presented, a suitable program of experimental studies was set up and realized in an open-circuit, subsonic wind-tunnel in a wake flow behind the row of parallel flat plates. The constant value of a longitudinal pressure gradient downstream of the plates was secured by means of the elastic, suitably formed walls of a measuring channel, the cross-section of which could be varied approximately according to the relation

$$A=A_2\left\{(A_2/A_1)^2-[(A_2/A_1)^2-1]\frac{x_1}{L}\right\}^{-1/2}. \tag{3.1}$$

It enables us to obtain the static pressure and its gradient from the equations

$$\bar{p}=\frac{p-p_\infty}{0.5\rho U_\infty^2}=[1-(A_2/A_1)^{-2}]\frac{x_1}{L} \tag{3.2}$$

and

$$\frac{d\bar{p}}{d(x_1/L)}=1-(A_2/A_1)^{-2}. \tag{3.3}$$

In order to estimate the influence of the inflow turbulence level, the experiment was performed with two different values of the initial turbulence intensity $\varepsilon_0=0.9$ and 8.2%.

The test stand and the measuring equipment applied have been described in more detail in the paper [16].

Since the relative error of the Reynolds shear stresses $\overline{u_1 u_2}$ was estimated to be at the level of 10%, the uncertainty of the eddy viscosity coefficient could be evaluated to be of the order of 15%.

4. Experimental results

As it has been previously mentioned, the main aim of the investigations was to check to what extent the cascade flow evolution was affected by the longitudinal pressure gradient of a constant value in a downstream direction. The initial stand testing showed that the linear dependences $\bar{p}=\bar{p}(\bar{x}_1)$ were achieved with satisfactory accuracy and, consequently, the condition $\Phi=\partial\bar{p}/\partial\bar{x}_1=\text{const}$ could be fulfilled exactly.

The picture of a mean flow evolution is shown in Fig. 2 where for five different values of the pressure gradient Φ the maximum and minimum velocities of a cascade flow have been plotted in a dimensionless form against the coordinate $\bar{x}_1$.

The influences of $\partial\bar{p}/\partial\bar{x}_1$ is manifested not only by the retardation or acceleration of the mean flow but also by the changes of intensity of mean velocity field equalization. It is easy to notice that with increasing distance from the cascade plates, the mean flow nonuniformity decays more rapidly in accelerated than in retarded flows.

Moreover, the experimental data confirm the validity of a power functional relation (2.19) — see also [16] — and prove that the exponent $\varkappa$ (Fig. 3) is a function of both the pressure gradient Φ and the initial turbulence intensity ε_0.

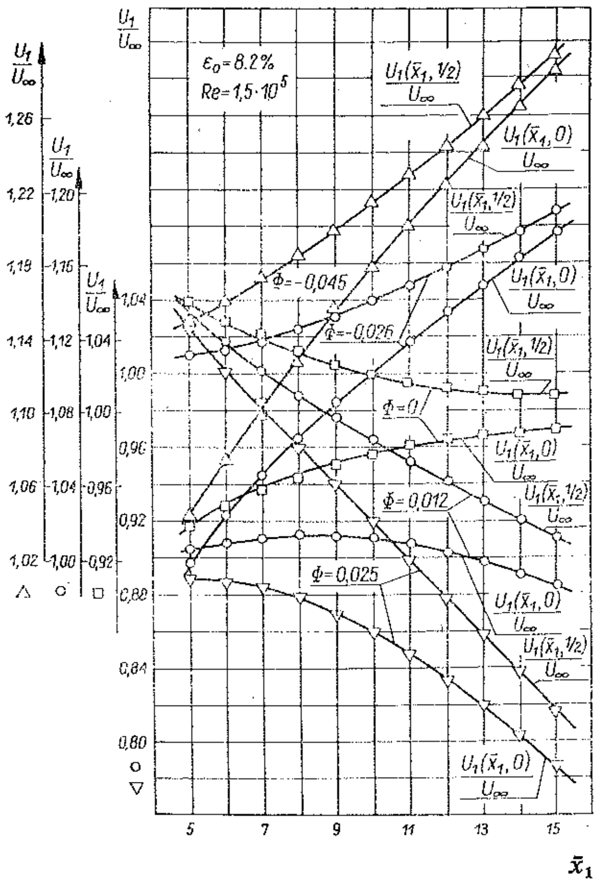

FIG. 2. Evolution of mean velocity in a downstream direction.

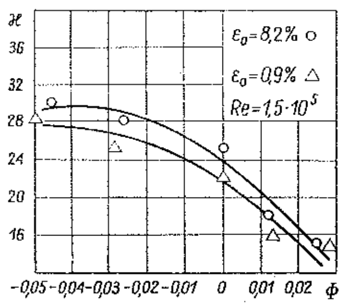

FIG. 3. Exponent $\varkappa$ in terms of relative pressure gradient Φ.

The evolution of normal $(\overline{u_1^2}, \overline{u_2^2})$ and shear $(\overline{u_1 u_2})$ turbulent stresses is cited after [16] in Fig. 4. As it can be seen, all the components of the turbulent stress-tensor are strongly affected by the longitudinal pressure gradient. It is worth noticing that at $\partial\bar{p}/\partial\bar{x}_1=\text{const}$ the turbulent shear stresses decay in a downstream direction more rapidly than the normal ones; this fact points at the overall tendency towards the state of isotropy noted in the majority of turbulent free flows.

Functional dependence between the eddy viscosity coefficient ν_T and the longitudinal pressure gradient Φ is presented in Fig. 5. The empirical results obtained during the experiment were calculated from the Eq. (2.8) and compared with the results obtained from the theoretical equation (2.20) previously derived.

It is easily seen that in the nearest control plane $x_1=11$ considered here, the maximum difference between the values given by the both methods does not exceed

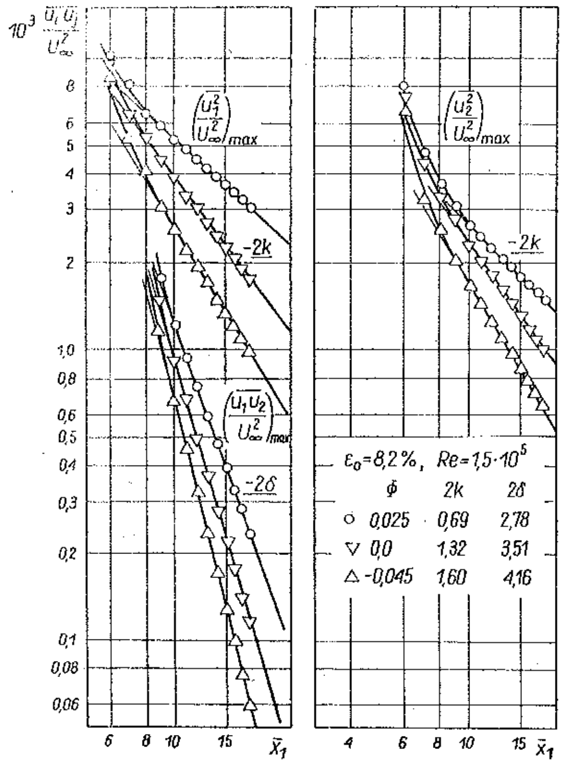

FIG. 4. Evolution of turbulence stresses in a downstream direction.

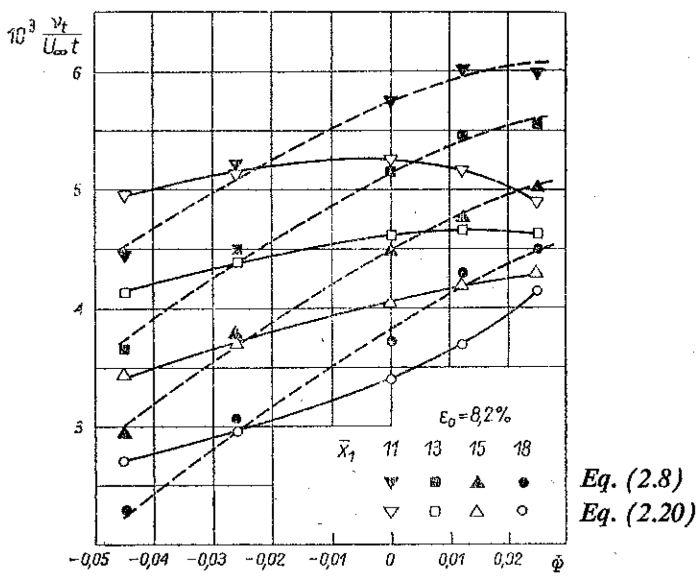

FIG. 5. Eddy viscosity coefficient as a function of longitudinal pressure gradient.

20%. However, it should be remembered that the above relation based on theoretical considerations is of an asymptotic character what means that its accuracy increases in larger distances from the cascade where approximation made when Eq. (2.20) was derived have a great precision. Beginning from the distance $\bar{x}_1 > 15$, the relative discrepancy of both methods does not exceed 10% and falls completely within the limits of measurement accuracy which, for the quantity ν_T, has been estimated at bout 15%. This statement verifies the validity of the functional relation

$$\nu_T = \nu_T [\Phi, \kappa(\Phi, \varepsilon_0)]$$

formulated before and, at the same time, confirms the correctness of the theoretical analysis of the phenomena considered here. Further analysis of the data showed in Fig. 5 leads, moreover, to the conclusion that in the far region of the cascade flow the eddy viscosity coefficient appears to be an increasing function of the longitudinal pressure gradient.

5. Concluding remarks

Theoretical considerations and the experimental data presented in this paper confirm the existence of the influence exerted by the longitudinal pressure gradient on the evolution of the overall flow pattern behind the cascade of symmetrical bodies. In particular, in the type of flow considered, the eddy viscosity coefficient may be evaluated with satisfactory accuracy from the formula (2.20) which enables the value of ν_T to be directly obtained only from the measurements of the mean flow parameters. Moreover, in the region $\bar{x}_1 > 11$, eddy viscosity appears to be an increasing function of the longitudinal pressure gradient.

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STRESZCZENIE

LEPKOŚĆ WIROWĄ W PRZYSPIESZONYCH I OPÓŹNIONYCH STRUGACH ZAPALISADOWYCH

Praca przedstawia rezultaty teoretycznej i eksperymentalnej analizy ewolucji lepkości wirowej za palisadą ciał symetrycznych w strugach z podłużnym gradientem ciśnienia o stałej wzdłuż przepływu wartości.

Badania wykazały, że współczynnik lepkości wirowej maleje ze wzrostem współrzędnej $\bar{x}_1$, stając się jednocześnie (dla $\bar{x}_1 > 11$) rosnącą funkcją podłużnego gradientu ciśnienia.

Резюме

ВИХРЕВАЯ ВЯЗКОСТЬ В УСКОРЯЕМЫХ И В ЗАМЕДЛЯЕМЫХ ЗАРЕШЕТОЧНЫХ СТРУЯХ

Работа представляет результаты теоретического и экспериментального анализа эволюции вихревой вязкости за решеткой симметрических тел в струях с продольным градиентом давления о постоянном значении вдоль течения.

Исследования показали, что коэффициент вихревой вязкости убывает с ростом координаты $\bar{x}_1$, становясь одновременно (для $\bar{x}_1 > 11$) возрастающей функцией продольного градиента давления.

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