ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 28, 3, 487—500, 1980
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

SOLUTION OF THE DIFFERENTIAL EQUATIONS GOVERNING THE EQUILIBRIUM OF A SKEW-CURVED BEAM

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The present paper deals with the closed form solution of the differential equations, with variable coefficients, governing the equilibrium of a skew or planar-curved beam· This problem is associated with a system or twelve coupled linear differential equations of first order, the closed form solution of which was achieved through tow analytical methods.

Notation

Throughout the text we use a global right-handed coordinate system $Oxyz$ and the Frenet trihedron $\mathbf{A}\,(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{b})$ corresponing to any point $\mathbf{A}$ of the center line of the curve. Einstein's summations convention is implied for all repeated indices. Also, dots are used to designate differentiations with respect to an arbitrary parameter u and the superscirpt "T" to indicate the transpose of a vector.

Nomenclature

$\boldsymbol{r}\,(u)=[\mathbf{x}\,(u), \mathbf{y}\,(u), \mathbf{z}\,(u)]^T$	position vector,
$\mathbf{K}\,(u)=[(\ddot{\mathbf{x}}^2+\ddot{\mathbf{y}}^2+\ddot{\mathbf{z}}^2-\ddot{\mathbf{S}}^2)^{\frac{1}{2}}/\dot{\mathbf{S}}^2]^{-1}$	radius of curvature,
$\boldsymbol{\tau}\,(u)=[\dot{\boldsymbol{r}}\,(\ddot{\boldsymbol{r}}\times\dddot{\boldsymbol{r}})/(\mathbf{K}^2\,\dot{\mathbf{S}}^6)]^{-1}$	radius of torsion,
$\mathbf{S}$	arc length of the curve,
$\boldsymbol{l}=[\mathbf{t}_1\,(u), \mathbf{t}_2\,(u), \mathbf{t}_3\,(u)]^T$	unit-tangent vector pointing to the direction of the increase of arc S,
$\boldsymbol{n}=[\mathbf{n}_1\,(u), \mathbf{n}_2\,(u), \mathbf{n}_3\,(u)]^T$	unit-normal vector pointing to the center of curvature,
$\boldsymbol{b}=[\mathbf{b}_1\,(u), \mathbf{b}_2\,(u), \mathbf{b}_3\,(u)]^T$	unit-binormal vector, defined in such a way that the corresponding Frenet trihedron is a right handed system,
$\boldsymbol{R}=[\mathbf{T}_t\,(u), \mathbf{T}_n\,(u), \mathbf{T}_b\,(u)]^T$	vector of internal forces,
$\boldsymbol{M}=[\mathbf{M}_t\,(u), \mathbf{M}_n\,(u), \mathbf{M}_b\,(u)]^T$	vector of internal moments,
$\boldsymbol{Y}=[\boldsymbol{\psi}_t\,(u), \boldsymbol{\psi}_n\,(u), \boldsymbol{\psi}_b\,(u)]^T$	vector of rotations,
$\boldsymbol{W}=[\mathbf{W}_t\,(u), \mathbf{W}_n\,(u), \mathbf{W}_b\,(u)]^T$	vector of deflections,
$\boldsymbol{g}=[\mathbf{q}_t\,(u), \mathbf{q}_n\,(u), \mathbf{q}_b\,(u)]^T$	vector of external continuous force,
$\boldsymbol{m}=[\mathbf{m}_t\,(u), \mathbf{m}_n\,(u), \mathbf{m}_b\,(u)]^T$	vector of external continuous moment,

$p=[0, \lambda_0 \Delta t_b/h_b, \lambda_0 \Delta t_n/h_n]^T$ vector of temperature differences,
$w=[\lambda_0 t_s, 0, 0]^T$ vector of temperature of the centroidal axis,
$\Delta t_b=t_0-t_u, \Delta t_n=t_r-t_l$ constant temperature differences of the limits of the cross-section with respect to the axes b and n,
t_s temperature of the centroidal axis,
E, G moduli of elasticity and shear,
λ_0 coefficient of thermal expansion,
I_t torsional moment of inertia,
I_n, I_b moments of inertia of the cross-section about the axes of n and b,
λ_1, λ_2 coefficients depending on the shape of the cross-section,
h_0, h_b maximum dimensions of the cross-section parallel to the axes of n and b,
F cross-section area.

1. Introduction

The problem of the linear elastic analysis of skew or planar-curved beams under static loading has attracted the interest of many researchers. The differential equations governing the equilibrium of a skew or planar-curved arc in terms of generalized forces and displacements have been presented in various studies [1] to [6]; however, these equations were applied only in special cases, such as helicoidal girders and circular arcs. In these cases the resulting differential equations are linear with the constant coefficients since the radii of curvature and torsion are constant. Also the stiffness approach and the flexibility method have been used by many investigators [7] to [16] in analysing skew or planar-curved beams with constant radii of curvature and torsion. Particularly, in reference [16] based on the principle of virtual work the flexibility matrix of a skew-curved beam is obtained. Finally, Bergman [17], McManus [18] and Washizu [19] proposed approximate methods for the determination of internal forces and displacement components of the afore--mentioned problem.

The present investigation deals with the development of a closed form solution of the differential equations (with variable coefficients) governing the equilibrium of any arbitrary skew-curved beam. The equilibrium equations of a skew arc element based on the linear elastic analysis are given in references [1] and [6]; they are, generally, associated with a system of twelve coupled linear differential equations of first order with variable coefficients, parameterized to the arc length S. In this paper these equations are converted to a new, arbitrary parameter u and, in the sequel, the closed form solution is obtained through tow different analytical methods. The first method consists in decoupling the previous equations of first order so that a new system of linear differential equations of third order with variable coefficients derives, the closed form solution of which can be obtained. The second method is a straight integration of the coupled linear differential equations using the matrix algebra. The main difference between the two previous methods is that in the first one mechanical knowledge was necessary, while in the second mathematical analysis was the tool of solution. Finally, the methodology which is followed in the first method is succesfully applied to the case of planar curved beams.

2. Mathematical formulation

Consider a skew-curved beam of uniform cross-section, whose centroidal axis is defined in a global coordinate system $Oxyz$ by the position vector $\boldsymbol{r}(u)$ and the radii of curvature and torsion $\mathbf{K}(u)$ and $\boldsymbol{\tau}(u)$ respectively; consider also at an arbitrary point of the beam the Frenet trihedron $\mathbf{A}(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{b})$. The differential equations governing the equilibrium of a skew-arc element (Fig. 1) expressed in the arbitrary parameter u are:

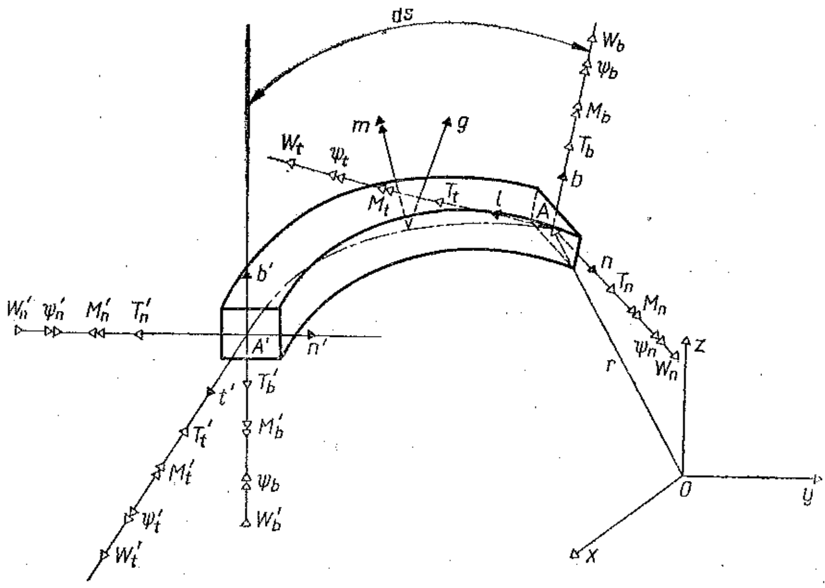

Fig. 1. Geometry and sign convention of a skew-curved arc under arbitrary loading.

$$\frac{1}{\dot{S}}\dot{R} = \mathsf{A}\,R + g,$$

(2.1)

$$\frac{1}{\dot{S}}\dot{M} = \mathsf{A}\,M + \mathsf{B}\,R + m;$$

$$\frac{1}{\dot{S}}\dot{Y} = \mathsf{A}\,Y + \mathsf{C}\,M + p,$$

(2.2)

$$\frac{1}{\dot{S}}\dot{W} = \mathsf{A}\,W + \mathsf{B}\,Y + \mathsf{D}\,R + w,$$

where

$$\mathsf{A}=[\mathbf{a}_{ij}]=\begin{Bmatrix} 0 & 1/\mathbf{K} & 0 \\ -1/\mathbf{K} & 0 & 1/\tau \\ 0 & -1/\tau & 0 \end{Bmatrix}, \quad \mathsf{B}=[\mathbf{b}_{ij}]=\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{Bmatrix},$$

$$\mathsf{C}=[\mathbf{c}_{ii}]=\begin{Bmatrix} -1/GI_t & 0 & 0 \\ 0 & -1/EI_n & 0 \\ 0 & 0 & -1/EI_b \end{Bmatrix},$$

$$\mathsf{D}=[\mathbf{d}_{ii}]=\begin{Bmatrix} -1/EF & 0 & 0 \\ 0 & -1/\lambda_1 GF & 0 \\ 0 & 0 & -1/\lambda_2 GF \end{Bmatrix}.$$

We remark that the vectors $\boldsymbol{R}$, $\boldsymbol{M}$, $\boldsymbol{Y}$, $\boldsymbol{W}$, $\boldsymbol{g}$, $\boldsymbol{m}$, $\boldsymbol{p}$,$\boldsymbol{w}$ and matrix A are continuous and continuously differentiable functions of u. Note also that axes of Frenet trihedron coincide with the principal axes of inertia of the cross-section.

In the sequel we make use of the well-known Frenet relations:

$$\dot{\boldsymbol{l}}=\frac{\dot{\mathbf{S}}}{\mathbf{K}}\boldsymbol{n}, \quad \dot{\boldsymbol{b}}=-\frac{\dot{\mathbf{S}}}{\tau}\boldsymbol{n}, \quad \dot{\boldsymbol{n}}=\frac{\dot{\mathbf{S}}}{\tau}\boldsymbol{b}-\frac{\dot{\mathbf{S}}}{\mathbf{K}}\boldsymbol{l}. \tag{2.3}$$

The solution of the system of vectorial equations (2.1) and (2.2), can be achieved through the following two analytical methods.

3. Fundamental solutions

3.1. *First method*

Equations (2.1) and (2.2), after differentiations and appropriate rearrangements, lead to the following differential equations:

$$\begin{gathered} f_1(u)\dddot{\mathbf{T}}_t+f_2(u)\ddot{\mathbf{T}}_t+f_3(u)\dot{\mathbf{T}}_t+f_4(u)\mathbf{T}_t=\dot{\mathbf{S}}^3 a_1(u), \\ \mathbf{T}_n=\mathbf{K}\left[\frac{1}{\dot{\mathbf{S}}}\dot{\mathbf{T}}_t-\mathbf{q}_t\right], \\ \mathbf{T}_b=\tau\left[\frac{1}{\dot{\mathbf{S}}}\dot{\mathbf{T}}_n+\frac{1}{\mathbf{K}}\mathbf{T}_t-\mathbf{q}_n\right]; \end{gathered} \tag{3.1}$$

$$\begin{gathered} f_1(u)\dddot{\mathbf{M}}_t+f_2(u)\ddot{\mathbf{M}}_t+f_3(u)\dot{\mathbf{M}}_t+f_4(u)\mathbf{M}_t=\dot{\mathbf{S}}^3 a_2(u), \\ \mathbf{M}_n=\mathbf{K}\left[\frac{1}{\dot{\mathbf{S}}}\dot{\mathbf{M}}_t-\mathbf{m}_t\right], \\ \mathbf{M}_b=\tau\left[\frac{1}{\dot{\mathbf{S}}}\dot{\mathbf{M}}_n+\frac{1}{\mathbf{K}}\mathbf{M}_t-\mathbf{T}_b-\mathbf{m}_n\right]; \end{gathered} \tag{3.2}$$

$$f_1(u)_t \dddot{\psi}_t + f_2(u)\ddot{\psi}_t + f_3(u)\dot{\psi}_t + f_4(u)\psi_t = \dot{\mathbf{S}}^3 a_3(u),$$

$$(3.3)\qquad \psi_n = \mathbf{K}\left[\frac{1}{\dot{\mathbf{S}}}\dot{\psi}_t - \mathbf{c}_{11}\mathbf{M}_t\right],$$

$$\psi_b = \boldsymbol{\tau}\left[\frac{1}{\dot{\mathbf{S}}}\dot{\psi}_n + \frac{1}{\mathbf{K}}\psi_t - \mathbf{c}_{22}\mathbf{M}_n - \lambda_0\frac{\Delta t_b}{h_b}\right];$$

$$f_1(u)\dddot{\mathbf{W}}_t + f_2(u)\ddot{\mathbf{W}}_t + f_3(u)\dot{\mathbf{W}}_t + f_4(u)\mathbf{W}_t = \dot{\mathbf{S}}^3 a_4(u),$$

$$(3.4)\qquad \mathbf{W}_n = \mathbf{K}\left[\frac{1}{\dot{\mathbf{S}}}\dot{\mathbf{W}}_t + \mathbf{d}_{11}\mathbf{T}_t - \lambda_0 t_s\right],$$

$$\mathbf{W}_b = \boldsymbol{\tau}\left[\frac{1}{\dot{\mathbf{S}}}\dot{\mathbf{W}}_n + \frac{1}{\mathbf{K}}\mathbf{W}_t - \psi_b - \mathbf{d}_{22}\mathbf{T}_n\right],$$

where

$$f_1(u) = \boldsymbol{\tau}\mathbf{K},$$

$$f_2(u) = (\boldsymbol{\tau}\mathbf{K})^{\cdot} + \boldsymbol{\tau}\dot{\mathbf{K}},$$

$$(3.5)\qquad f_3(u) = \dot{\mathbf{S}}\left(\boldsymbol{\tau}\frac{\dot{\mathbf{K}}}{\dot{\mathbf{S}}}\right)^{\cdot} + \dot{\mathbf{S}}^2\left(\frac{\boldsymbol{\tau}}{\mathbf{K}} + \frac{\mathbf{K}}{\boldsymbol{\tau}}\right) - \frac{\ddot{\mathbf{S}}}{\dot{\mathbf{S}}}f_2(u) - \frac{\dddot{\mathbf{S}}}{\dot{\mathbf{S}}}f_1(u),$$

$$f_4(u) = \dot{\mathbf{S}}^2\left(\frac{\boldsymbol{\tau}}{\mathbf{K}}\right)^{\cdot}$$

and

$$a_1(u) = \left[\frac{\boldsymbol{\tau}}{\dot{\mathbf{S}}}(\mathbf{K}\mathbf{q}_t)^{\cdot}\right]^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \frac{\mathbf{K}}{\boldsymbol{\tau}}\mathbf{q}_t + (\boldsymbol{\tau}\mathbf{q}_n)^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \mathbf{q}_b,$$

$$a_2(u) = \left[\frac{\boldsymbol{\tau}}{\dot{\mathbf{S}}}(\mathbf{K}\mathbf{m}_t)^{\cdot}\right]^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \frac{\mathbf{K}}{\boldsymbol{\tau}}\mathbf{m}_t + (\boldsymbol{\tau}\mathbf{m}_n)^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \mathbf{m}_b + (\boldsymbol{\tau}\mathbf{T}_b)^{\cdot}\frac{1}{\dot{\mathbf{S}}} - \mathbf{T}_n,$$

$$(3.6)\qquad a_3(u) = \mathbf{d}_{11}\left[\frac{\boldsymbol{\tau}}{\dot{\mathbf{S}}}(\mathbf{K}\mathbf{M}_t)^{\cdot}\right]^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \mathbf{d}_{11}\frac{\mathbf{K}}{\boldsymbol{\tau}}\mathbf{M}_t + \mathbf{d}_{22}(\boldsymbol{\tau}\mathbf{M}_n)^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \mathbf{d}_{33}\mathbf{M}_b + \lambda_0\frac{\Delta t_b}{h_b}\dot{\boldsymbol{\tau}}\frac{1}{\dot{\mathbf{S}}} + \lambda_0\frac{\Delta t_n}{h_n},$$

$$a_4(u) = \mathbf{d}_{11}\left[\frac{\boldsymbol{\tau}}{\dot{\mathbf{S}}}(\mathbf{K}\mathbf{T}_t)^{\cdot}\right]^{\cdot}\frac{1}{\dot{\mathbf{S}}} + \mathbf{d}_{11}\frac{\mathbf{K}}{\boldsymbol{\tau}}\mathbf{T}_t + \mathbf{d}_{22}(\boldsymbol{\tau}\mathbf{T}_n)^{\cdot}\frac{1}{\dot{\mathbf{S}}} + (\boldsymbol{\tau}\psi_b)^{\cdot}\frac{1}{\dot{\mathbf{S}}} - \psi_n + \mathbf{d}_{33}\mathbf{T}_b + \frac{\mathbf{K}}{\boldsymbol{\tau}}\lambda_0 t_s.$$

It is worthwhile remarking that each equation from Eqs. $(3.1)_1$ to (3.4) can be solved once the solution of its preceding equations is known. Thus, Eqs. (3.4) can be solved if $a_4(u)$ is known, which presupposes that Eqs. (3.1), (3.2) and (3)3) have been integrated. Basing on the previous assumptions, the determination of the unknown vectors $\boldsymbol{R}$, $\boldsymbol{M}$, $\boldsymbol{Y}$ and $\boldsymbol{W}$ can be achieved throughout the solution of a linear differential equation with variable coefficients of the form:

$$(3.7)\qquad f_1(u)\dddot{g}(u)+f_2(u)\ddot{g}(u)+f_3(u)\dot{g}(u)+f_4(u)g(u)=k(u),$$

where $g(u)$ and its derivatives denote one of the functions $\mathbf{T}_t$, $\mathbf{M}_t$, $\boldsymbol{\psi}_t$, $\mathbf{W}_t$, while $k(u)$ one of the functions $\dot{\mathbf{S}}^3 a_j(u)$ $(j=1, 2, 3, 4,)$.

The general integral of the homogeneous differential equation of Eq. (3.7) is given by:

$$g_0(u)=c_i g_i(u) \qquad (i=1, 2, 3),$$

where $g_i(u)$ are three linearly independent solutions and c_i integrated constants. For the determination of the previous three independent solutions the following observation is made; it is known that the homogeneous equation (3.7) describes the response of a skew-curved beam subjected to a static loading applied at its ends. Consider that this beam is subjected at one end to three unit generalized forces $\mathbf{T}_t$, $\mathbf{T}_n$, $\mathbf{T}_b$ in directions parallel to those corresponding to the Frenet trihedron; then the projections of these forces on the tangent (normal or binormal) to the center line at an arbitrary point of the beam should satisfy the homogeneous differential equation (3.7); consequently $\mathbf{t}_i(u)$ $(\mathbf{n}_i(u)$ or $\mathbf{b}_i(u))$, $i=1, 2, 3$, are three independent solutions of this equation. This is clearly shown by inserting $\mathbf{t}_i$ into the foregoing equation, and by using the relations (2.3). The above procedure is also valid for the generalized forces $\mathbf{M}_t$, $\mathbf{M}_n$, $\mathbf{M}_b$ and generalized displacements $\boldsymbol{\psi}_t$, $\boldsymbol{\psi}_n$, $\boldsymbol{\psi}_b$ and $\mathbf{W}_t$, $\mathbf{W}_n$, $\mathbf{W}_b$. The particular integral $g_p(u)$ Eq. (3.7) depending on the function $k(u)$, is determined by the method of variation of constants.

Thus the particular integral is given by

$$(3.8)\qquad g_p(u)=\mathbf{t}_i(u)\int_u k(u)\,\mathbf{b}_i(u)\,du \qquad (i=1, 2, 3)$$

and the general integral of Eq. (3.7) by the formulae

$$(3.9)\qquad g(u)=c_i\,\mathbf{t}_i(u)+\mathbf{t}_i(u)\int_u k(u)\,\mathbf{b}_i(u)\,du.$$

One may observe that the second member of Eq. (3.1) is a function of the external loading only and is therefore known. Unlike this case the second members of Eqs. $(3.2)_1$, $(3.3)_1$ and $(3.4)_1$ are functions of the external loading together with generalized forces and displacements. In order to separate these quantities in the right-hand sides of Eqs. $(3.1)_1$, $(3.2)_1$, $(3.3)_1$ and $(3.4)_1$, we can write

$$F_{1,\dot{S}^3 a_{11}}, \qquad F_{1,\dot{S}^3 a_2}=F_{1,\dot{S}^3 a_{21}}+F_{1,\dot{S}^3 a_{22}},$$
$$F_{1,\dot{S}^3 a_3}=F_{1,\dot{S}^3 a_{31}}+F_{1,\dot{S}^3 a_{32}}, \qquad F_{1,\dot{S}^3 a_4}=F_{1,\dot{S}^3 a_{41}}+F_{1,\dot{S}^3 a_{42}},$$

where $\dot{\mathbf{S}}^3 a_{11}$, $\dot{\mathbf{S}}^3 a_{21}$, $\dot{\mathbf{S}}^3 a_{31}$ and $\dot{\mathbf{S}}^3 a_{41}$ refer to the external loading, while $\dot{\mathbf{S}}^3 a_{22}$, $\dot{\mathbf{S}}^3 a_{32}$ and $\dot{\mathbf{S}}^3 a_{42}$ to the generalized forces and displacements.

At this point it is convenient to give the expression of the functional $F_{1,\dot{S}^3 a_{jk}}$ ($j=1, ..., 4-k=1, 2$) defined on the corp of real numbers as follows:

$$F_{1,\dot{S}^3 a_{jk}} = g_p(u) = \mathbf{t}_i(u) \int_u \dot{S}^3(u)\, a_{jk}(u)\, \mathbf{b}_i(u)\, du, \tag{3.10}$$

where a_j denotes one of the functions a_{11}, a_{21}, a_{31}, a_{41} or a_{22}, a_{32}, a_{42}. Moreover, it is convenient to introduce the functionals

$$\begin{aligned} F_{2,\dot{S}^3 a_{jk}} &= \mathbf{n}_i(u) \int_u \dot{S}^3(u)\, a_{jk}(u)\, \mathbf{b}_i(u)\, du, \\ F_{3,\dot{S}^3 a_{jk}} &= \mathbf{b}_i(u) \int_u \dot{S}^3(u)\, a_{jk}(u)\, \mathbf{b}_i(u)\, du \end{aligned} \tag{3.11}$$

satisfying the following relations:

$$\begin{aligned} \dot{F}_{1\,\dot{S}^3 a_{jk}} &= \frac{\dot{S}}{\mathbf{K}} F_{2,\dot{S}^3 a_{jk}}, \\ \dot{F}_{2,\dot{S}^3 a_{jk}} &= -\frac{\dot{S}}{\mathbf{K}} F_{1,\dot{S}^3 a_{jk}} + \frac{\dot{S}}{\boldsymbol{\tau}} F_{3,\dot{S}^3 a_{jk}}, \\ \dot{F}_{3,\dot{S}^3 a_{jk}} &= \dot{S}^4 a_{jk} - \frac{\dot{S}}{\boldsymbol{\tau}} F_{2,\dot{S}^3 a_{jk}}. \end{aligned} \tag{3.12}$$

Now, by means of the relations (2.3) to (3.6), of the general integral (3.9) and using the relations (3.10) to (3.12), one may readily obtain the expressions for the generalized internal forces and displacements $\boldsymbol{R}$, $\boldsymbol{M}$, $\boldsymbol{Y}$, $\boldsymbol{W}$ at any point of the center line of the skew-curved beam. Thus the following matrix equation can be obtained:

$$\begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{M} \\ \boldsymbol{Y} \\ \boldsymbol{W} \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \{0\} & \{0\} & \{0\} \\ \mathsf{E} & \mathbf{G} & \{0\} & \{0\} \\ \mathsf{F} & \mathsf{H} & \mathbf{G} & \{0\} \\ \mathsf{I} & \mathsf{J} & \mathsf{E} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{C}_\alpha \\ \mathbf{C}_\beta \\ \mathbf{C}_\gamma \\ \mathbf{C}_\varepsilon \end{bmatrix} + \begin{bmatrix} \boldsymbol{R}_0 \\ \boldsymbol{M}_0 \\ \boldsymbol{Y}_0 \\ \boldsymbol{W}_0 \end{bmatrix}, \tag{3.13}$$

in which $\mathbf{C}_\alpha$, $\mathbf{C}_\beta$, $\mathbf{C}_\gamma$, $\mathbf{C}_\varepsilon$, $\boldsymbol{R}_0$, $\boldsymbol{M}_0$, $\boldsymbol{Y}_0$, $\boldsymbol{W}_0$ are submatrices of dimensions (3×1) and $\mathbf{G}$, E, H, F, J, I, $\{0\}$ are submatrices of dimensions (3×3). The analytical expressions of the previous matrices is given in the Appendix. Note that the submatrix $\mathbf{G}$ is always non-singular; this is so as it can easily be shown using the relations (2.3) that $\mathbf{G}^{-1}=\mathbf{G}^T$ and $\det \mathbf{G}=1$.

3.2. Second method

The four vectorial equations (2.1) and (2.2) are all of the general form:

$$\dot{\mathfrak{r}} = \dot{S}\, \mathsf{A}\, \mathfrak{r} + \dot{S}\, \mathfrak{f}, \tag{3.14}$$

where $\mathfrak{r}$ denotes one of the vectors $\boldsymbol{R}$, $\boldsymbol{M}$, $\boldsymbol{Y}$, $\boldsymbol{W}$ while $\mathfrak{f}$ one of the vectors $\boldsymbol{g}$, $\mathsf{B}\,\boldsymbol{R}+\boldsymbol{m}$, $\mathsf{C}\,\boldsymbol{M}+\boldsymbol{p}$, $\mathsf{B}\,\boldsymbol{Y}+\mathsf{D}\,\boldsymbol{R}+\boldsymbol{w}$. Equation (3.14) is completely solved if a fundamental matrix $\mathbf{G}$ of dimensions (3×3) of the corresponding homogeneous system is known;

in fact, if **G** is known, the particular integral of Eq. (3.14) can be obtained by the method of variation of constants.

The homogeneous system of the vectorial equation (3.14) is

$$\dot{\mathfrak{r}} = \dot{S}\,\mathsf{A}\,\mathfrak{r}, \tag{3.15}$$

Also, the vectorial equations Frenet (2.3) may be written under the form of a matrix equation as

$$\dot{\mathfrak{v}} = \dot{S}\,\mathsf{A}\,\mathfrak{v}, \tag{3.16}$$

where

$$\dot{\mathfrak{v}} = \dot{S}\,(\dot{\boldsymbol{l}}, \dot{\boldsymbol{n}}, \dot{\boldsymbol{b}})^T.$$

Finally, Eqs. (3.15) and (3.16) lead to the following relations:

$$\mathfrak{r} = \left(\exp \int_u \dot{S}\,\mathsf{A}\,du\right) \mathbf{C}, \quad \mathfrak{v} = \left(\exp \int_u \dot{S}\,\mathsf{A}\,du\right) \mathbf{C}', \tag{3.17}$$

where

$$\mathbf{C} = (c_1, c_2, c_3)^T, \quad \mathbf{C}' = (c_1', c_2', c_3')^T$$

are vectors of arbitrary constants. Through the comparison of the relations (3.17) we infer that the matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3 \\ \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$$

is a fundamental matrix of Eq. (3.17), that is **GC** $[\mathbf{C} = (c_1, c_2, c_3)^T]$ is the general solution of the homogeneous equation (3.15). This is clearly shown by inserting **GC** into the previous equation. Hence the general solution of Eq. (3.14) is given by the formulae

$$\mathfrak{r} = \mathbf{GC} + \mathbf{G} \int_u \dot{S}\,\mathbf{G}^T \mathfrak{f}\,du. \tag{3.18}$$

Finally, the solution of the system of Eqs. (2.1) and (2.2) may be derived under the form of a matrix equation as

$$\begin{bmatrix} R \\ M \\ Y \\ W \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \{0\} & \{0\} & \{0\} \\ \mathsf{K} & \mathbf{G} & \{0\} & \{0\} \\ \mathsf{L} & \mathsf{M} & \mathbf{G} & \{0\} \\ \mathsf{N} & \mathsf{P} & \mathsf{K} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{C}_\alpha \\ \mathbf{C}_\beta \\ \mathbf{C}_\gamma \\ \mathbf{C}_\varepsilon \end{bmatrix} + \begin{bmatrix} R_0' \\ M_0' \\ Y_0' \\ W_0' \end{bmatrix}, \tag{3.19}$$

where the submatrices **G**, K, L, M, N, P, $\{0\}$ are of dimensions (3×3), while the submatrices $\mathbf{C}_\alpha$, $\mathbf{C}_\beta$, $\mathbf{C}_\gamma$, $\mathbf{C}_\varepsilon$, R_0', M_0', Y_0', W_0' of dimensions (3×1). The analytical expression of the foregoing matrices is given in the Appendix.

4. Application to the case of a planar-curved beam

Introducing into Eqs. (2.1) and (2.2) the value for the torsion of the curve $\mathbf{1/\tau=0}$, the respective differential equations (3.1) to (3.6) become for the case of a planar-curved beam:

(4.1)
$$\left[\mathbf{K}\frac{\dot{\mathbf{T}}_t}{\dot{\mathbf{S}}}\right]^{\cdot}+\frac{\dot{\mathbf{S}}}{\mathbf{K}}\mathbf{T}_t=\dot{\mathbf{S}}\,a_1(u),$$
$$\mathbf{T}_n=\mathbf{K}\left[\frac{\dot{\mathbf{T}}_t}{\dot{\mathbf{S}}}-\mathbf{q}_t\right],$$
$$\mathbf{T}_b=\int_u \dot{\mathbf{S}}\mathbf{q}_b\,du+c_1;$$

(4.2)
$$\left[\mathbf{K}\frac{\dot{\mathbf{M}}_t}{\dot{\mathbf{S}}}\right]^{\cdot}+\frac{\dot{\mathbf{S}}}{\mathbf{K}}\mathbf{M}_t=\dot{\mathbf{S}}\,a_2(u),$$
$$\mathbf{M}_n=\mathbf{K}\left[\frac{\dot{\mathbf{M}}_t}{\dot{\mathbf{S}}}\cdot\mathbf{m}_t\right],$$
$$\mathbf{M}_b=\int_u \dot{\mathbf{S}}(-\mathbf{T}_n+\mathbf{m}_b)\,du+c_2;$$

(4.3)
$$\left[\mathbf{K}\frac{\dot{\psi}_t}{\dot{\mathbf{S}}}\right]^{\cdot}+\frac{\dot{\mathbf{S}}}{\mathbf{K}}\psi_t=\dot{\mathbf{S}}\,a_3(u),$$
$$\psi_n=\mathbf{K}\left[\frac{\dot{\psi}_t}{\dot{\mathbf{S}}}-c_{11}\mathbf{M}_t\right],$$
$$\psi_b=\int_u \dot{\mathbf{S}}\left(c_{33}\mathbf{M}_b-\lambda_0\frac{\Delta t_b}{h_b}\right)du+c_3;$$

(4.4)
$$\left[\mathbf{K}\frac{\dot{\mathbf{W}}_t}{\dot{\mathbf{S}}}\right]^{\cdot}+\frac{\dot{\mathbf{S}}}{\mathbf{K}}\mathbf{W}_t=\dot{\mathbf{S}}\,a_4(u),$$
$$\mathbf{W}_n=\mathbf{K}\left[\frac{\dot{\mathbf{W}}}{\dot{\mathbf{S}}}-\mathbf{d}_{11}\mathbf{T}_t-\lambda_0 t_s\right],$$
$$\mathbf{W}_b=-\int_u \dot{\mathbf{S}}(_n-\mathbf{d}_{33}\mathbf{T}_b)\,du+c_4,$$

where

(4.5)
$$a_1(u)=\frac{1}{\dot{\mathbf{S}}}(\mathbf{K}\mathbf{q}_t)^{\cdot}+\mathbf{q}_n,$$
$$a_2(u)=\frac{1}{\dot{\mathbf{S}}}(\mathbf{K}\mathbf{m}_t)^{\cdot}+\mathbf{m}_n+\mathbf{T}_b,$$
$$a_3(u)=c_{11}(\mathbf{K}\mathbf{M}_t)^{\cdot}\frac{1}{\dot{\mathbf{S}}}+c_{22}\mathbf{M}_n+\lambda_0\frac{\Delta t_n}{h_n},$$
$$a_4(u)=\mathbf{d}_{11}(\mathbf{K}\mathbf{T}_t)^{\cdot}\frac{1}{\dot{\mathbf{S}}}+\mathbf{d}_{22}\mathbf{T}_n+\psi_b+\lambda_0 t_s\dot{\mathbf{K}}\frac{1}{\dot{\mathbf{S}}}.$$

Following the procedure outlined in the previous section we integrate the differential equation

$$\left[\frac{\mathbf{K}}{\dot{\mathbf{S}}}\dot{g}(u)\right]^{\cdot}+\frac{\dot{\mathbf{S}}}{\mathbf{K}}g(u)=k(u), \tag{4.6}$$

where $g(u)$ and its derivatives denote one of the functions $\mathbf{T}_t$, $\mathbf{M}_t$, ψ_t, $\mathbf{W}_t$, and $k(u)$ one of the functions $\dot{\mathbf{S}}a_j(u)$ $(j=1, \ldots, 4)$. Using the Frenet relations for the case of a planar curve expressed by

$$\dot{l}=\frac{1}{\mathbf{K}}n\dot{\mathbf{S}}, \qquad \dot{n}=-\frac{1}{\mathbf{K}}l\dot{\mathbf{S}}$$

we can readily prove that the homogeneous equation of the self-adjoint differential equation (4.6) has a solution of the form

$$g_0(u)=c_1\,\mathbf{t}_1(u)+c_2\,\mathbf{t}_2(u).$$

We remark in passing that a self-adjoint equation with coefficients expressing the curvature and the radius of curvature of a planar curve has always a closed form solution.

One may prefer to determine the particular integral of Eq. (4.6) by introducing the Green function (see reference [20]). Thus the general solution of the previous equations can be written as

$$g(u)=u\tilde{g}-v\bar{g}+F_k,$$

where

$$u=\frac{\bar{\mathbf{t}}_2\,\mathbf{t}_1(u)-\bar{\mathbf{t}}_1\,\mathbf{t}_2(u)}{\tilde{\mathbf{t}}_1\,\bar{\mathbf{t}}_2-\bar{\mathbf{t}}_1\,\tilde{\mathbf{t}}_2}, \qquad v=\frac{\tilde{\mathbf{t}}_2\,\mathbf{t}_1(u)-\tilde{\mathbf{t}}_1\,\mathbf{t}_2(u)}{\tilde{\mathbf{t}}_1\,\bar{\mathbf{t}}_2-\bar{\mathbf{t}}_1\,\tilde{\mathbf{t}}_2}$$

and

$$F_k=(\tilde{\mathbf{t}}_1\,\bar{\mathbf{t}}_2-\bar{\mathbf{t}}_1\,\tilde{\mathbf{t}}_2)\left[u(u)\int_0^u v(\zeta)\,k(\zeta)\,d\zeta+v(u)\int_u^{\tilde{u}} u(\zeta)\,k(\zeta)\,d\zeta\right].$$

Note that the symbols ~ and — are introduced to characterise functions for the values $u=0$ and $u=\tilde{u}$ in the space $[0, \tilde{u}]$.

5. Conclusions

In this investigation two analytical methods for establishing the closed form solution of the twelve doupled linear differential equations of first order with variable coefficients, governing the equilibrium of a skew of planar-curved beam, are developed.

The main feature of the first method consists in decoupling these equations and, using mechanical knowledge, in finding the closed form solution of the new ones. The second method is a straight integration of the twelve coupled linear differential equations by means of matrix algebra. Finally, using both methods the determination of the generalized internal forces and displacements at an arbitrary point of the center-line of the beam can easily be achieved.

Appendix

The analytical expression for the matrices of the formulae (3.13) is given below:

$$\mathbf{C}_\alpha=\{c_j\},\quad \mathbf{C}_\beta=\{c_{j+3}\},\quad \mathbf{C}_\gamma=\{c_{j+6}\},\quad \mathbf{C}_\varepsilon=\{c_{j+9}\},$$

$$\mathbf{G}=\{\mathbf{G}_{ij}\}:\quad \mathbf{G}_{1j}=\mathbf{t}_j,\quad \mathbf{G}_{2j}=\mathbf{n}_j,\quad \mathbf{G}_{3j}=\mathbf{b}_j;$$

$$\mathsf{E}=\{\mathbf{E}_{ij}\}:\quad \mathbf{E}_{1j}=F_{1,\dot{S}^3\nu_j},\quad \mathbf{E}_{2j}=F_{2,\dot{S}^3\nu_j},\quad \mathbf{E}_{3j}=F_{3,\dot{S}^3\nu_j}-\tau\mathbf{b}_j;$$

$$\mathsf{F}=\{\mathbf{F}_{ij}\}:\quad \mathbf{F}_{1j}=F_{1,\dot{S}^3\sigma_j},\quad \mathbf{F}_{2j}=F_{5,\dot{S}^3\sigma_j}-\mathbf{K}\mathbf{c}_{11}F_{1,\dot{S}^3\nu_j},$$
$$\mathbf{F}_{3j}=F_{3,\dot{S}^3\sigma_j}-\mathbf{c}_{11}(\mathbf{K}F_{1,\dot{S}^3\nu_j})^{\cdot}-\mathbf{c}_{22}\,\tau F_{2,\dot{S}^3\nu_j};$$

$$\mathsf{H}=\{\mathbf{H}_{ij}\}:\quad \mathbf{H}_{1j}=F_{1,\dot{S}^3\rho_j},\quad \mathbf{H}_{2j}=F_{2,\dot{S}^3\rho_j}-\mathbf{K}\mathbf{c}_{11}\mathbf{t}_j,$$
$$\mathbf{H}_{3j}=F_{3,\dot{S}^3\rho_j}-\mathbf{c}_{11}(\mathbf{K}\mathbf{t}_j)^{\cdot}-\mathbf{c}_{22}\,\tau\mathbf{n}_j;$$

$$\mathsf{I}=\{\mathbf{I}_{ij}\}:\quad \mathbf{I}_{1j}=F_{1,\dot{S}^3\omega_j},\quad \mathbf{I}_{2j}=F_{2,\dot{S}^3\omega_j}+\mathbf{K}\mathbf{d}_{11}\mathbf{n}_j,$$
$$\mathbf{I}_{2j}=F_{3,\dot{S}^3\omega_j}+\mathbf{d}_{11}(\mathbf{K}\mathbf{n}_j)^{\cdot}-\tau\,[F_{3,\dot{S}^3\sigma_j}-\mathbf{c}_{11}(\mathbf{K}F_{1\,\dot{S}^3\nu_j})^{\cdot}-\mathbf{c}_{22}\,\tau F_{2,\dot{S}^3\nu_j}];$$

$$\mathsf{J}=\{\mathbf{J}_{ij}\}:\quad \mathbf{J}_{1j}=F_{1,\dot{S}^3\varphi_j},\quad \mathbf{J}_{2j}=F_{2,\dot{S}^3\varphi_j},$$
$$\mathbf{J}_{3j}=F_{3,\dot{S}^3\varphi_j}-\tau\,[F_{3,\dot{S}^3\rho_j}-\mathbf{c}_{11}(\mathbf{K}\mathbf{t}_j)^{\cdot}-\mathbf{c}_{22}\,\tau\mathbf{n}_j]$$

and

$$R_0=\begin{bmatrix}F_{1,\dot{S}^3 a_{11}}\\ \zeta_0\\ \mu_0\end{bmatrix},\qquad M_0=\begin{bmatrix}F_{1,\dot{S}^3\nu_0}\\ F_{2,\dot{S}^3\nu_0}-\mathbf{K}\mathbf{m}_t\\ F_{3,\dot{S}^3\nu_0}-\tau(\mu_0+\mathbf{m}_n)\end{bmatrix},$$

$$Y_0=\begin{bmatrix}F_{1,\dot{S}^3(a_{31}+\rho_0)}\\ F_{5,\dot{S}^3(a_3\ +\rho_0)}-\mathbf{c}_{11}\,\mathbf{K}F_{1,\dot{S}^3\nu_0}\\ F_{3,\dot{S}^3(a_{31}+\rho_0)}-\mathbf{c}_{11}(\mathbf{K}F_{1,\dot{S}^3\nu_0})^{\cdot}-\mathbf{c}_{22}\,\tau(F_{2,\dot{S}^3\nu_0}-\mathbf{K}\mathbf{m}_t)-\tau\lambda_0\,\Delta t_b/h_b\end{bmatrix},$$

$$W_0=\begin{bmatrix}F_{1,\dot{S}^3(a_{41}+v_0)}\\ F_{2,\dot{S}^3(a_{41}+v_0)}+\mathbf{K}\mathbf{d}_{11}\,\mu_0-\mathbf{K}\lambda_0\,t_s\\ F_{3,\dot{S}^3(a_{41}+v_0)}+\mathbf{d}_{11}(\mathbf{K}\mu_0)^{\cdot}-\lambda_0\,t_s\,\dot{\mathbf{K}}-\tau\,[F_{3,\dot{S}^3(a_{31}+\rho_0)}-\mathbf{c}_{11}(\mathbf{K}F_{1,\dot{S}^3\nu_0})]-\\ -\mathbf{c}_{22}\,\tau(F_{2,\dot{S}^3\nu_0}-\mathbf{K}\mathbf{m}_t-\tau\lambda_0\,\Delta_b/h_b)-\tau\mathbf{d}_{22}\,\zeta_0\end{bmatrix},$$

where

$$\nu_j=\frac{1}{\dot{S}}(\tau\mathbf{b}_j)^{\cdot}-\mathbf{n}_j,$$

$$\rho_j=\frac{\mathbf{d}_{11}}{\dot{S}}\left[\frac{\tau}{\dot{S}}(\mathbf{K}\mathbf{t}_j)^{\cdot}\right]^{\cdot}+\frac{\mathbf{d}_{11}\mathbf{K}}{\tau}\mathbf{t}_j+\frac{\mathbf{d}_{22}}{\dot{S}}(\tau\mathbf{n}_j)^{\cdot}+\mathbf{d}_{33}\,\mathbf{b}_j,$$

$$\sigma_j=\frac{\mathbf{d}_{11}}{\dot{\mathbf{S}}}\left[\frac{\tau}{\dot{\mathbf{S}}}(\mathbf{K}F_{1,\dot{\mathbf{S}}^3\nu_j})^{\cdot}\right]^{\cdot}+\frac{\mathbf{d}_{11}\mathbf{K}}{\tau}\mathbf{t}_j+\frac{\mathbf{d}_{22}}{\dot{\mathbf{S}}}(\tau F_{2,\dot{\mathbf{S}}^2\nu_j})^{\cdot}+\mathbf{d}_{33}(F_{3,\dot{\mathbf{S}}^3\nu_j}-\tau\mathbf{b}_j),$$

$$\varphi_j=\frac{1}{\dot{\mathbf{S}}}[\tau(F_{3,\dot{\mathbf{S}}^3\rho_j}-\mathbf{c}_{11}(\mathbf{K}\mathbf{t}_j)'-\mathbf{c}_{22}\tau\mathbf{n}_j)]^{\cdot}-F_{2,\dot{\mathbf{S}}^3\rho_j}+\mathbf{K}\mathbf{c}_{11}\mathbf{t}_j,$$

$$\omega_j=\rho_j+\frac{1}{\dot{\mathbf{S}}}[\tau(F_{3,\dot{\mathbf{S}}^3\sigma_j}-\mathbf{c}_{11}(\mathbf{K}F_{1,\dot{\mathbf{S}}^3\nu_j})^{\cdot}-\mathbf{c}_{22}\tau F_{2,\dot{\mathbf{S}}^3\nu_j})]^{\cdot}-\dot{F}_{2,\dot{\mathbf{S}}^3\sigma_j}+\mathbf{K}\mathbf{c}_{11}F_{1,\dot{\mathbf{S}}^3\nu_j},$$

$$\zeta_0=F_{2,\dot{\mathbf{S}}^3a_{11}}-\frac{\mathbf{K}}{\dot{\mathbf{S}}}\mathbf{q}_t,$$

$$\mu_0=F_{3,\dot{\mathbf{S}}^3a_{11}}-\tau\left[\frac{1}{\dot{\mathbf{S}}}\left(\frac{\mathbf{K}}{\dot{\mathbf{S}}}\mathbf{q}_t\right)^{\cdot}+\mathbf{q}_n\right],$$

$$\nu_0=a_{21}-\frac{1}{\dot{\mathbf{S}}}(\tau\mu_0)^{\cdot}-\zeta_0,$$

$$\rho_0=\frac{\mathbf{d}_{11}}{\dot{\mathbf{S}}}\left[\frac{\tau}{\dot{\mathbf{S}}}(\mathbf{K}F_{1,\dot{\mathbf{S}}^3\nu_0})^{\cdot}\right]^{\cdot}+\frac{\mathbf{d}_{11}\mathbf{K}}{\tau}F_{1,\dot{\mathbf{S}}^3\nu_0}+\frac{\mathbf{d}_{22}}{\dot{\mathbf{S}}}(\tau F_{2,\dot{\mathbf{S}}^3\nu_0}-\mathbf{K}\mathbf{m}_t)^{\cdot}+$$
$$+\mathbf{d}_{33}[F_{3,\dot{\mathbf{S}}^3\nu_0}-\tau(\mu_0+\mathbf{m}_n)],$$

$$v_0=\frac{\mathbf{d}_{11}}{\dot{\mathbf{S}}}\left[\frac{\tau}{\dot{\mathbf{S}}}(\mathbf{K}F_{1,\dot{\mathbf{S}}^3\nu_0})^{\cdot}\right]^{\cdot}+\frac{\mathbf{d}_{11}\mathbf{K}}{\tau}F_{1,\dot{\mathbf{S}}^3a_{11}}+\frac{\mathbf{d}_{22}}{\dot{\mathbf{S}}}(\tau\zeta_0)^{\cdot}+$$
$$+\frac{1}{\dot{\mathbf{S}}}[\tau(F_{3,\dot{\mathbf{S}}^3(a_{31}+\rho_0)}-\mathbf{c}_{11}(\mathbf{K}F_{1,\dot{\mathbf{S}}^3\nu_0})^{\cdot}-\mathbf{c}_{22}\tau(F_{2,\dot{\mathbf{S}}^3\nu_0}-\mathbf{K}\mathbf{m}_t)-\tau\lambda_0\,\Delta t_b/h_b]^{\cdot}-$$
$$-F_{2,\dot{\mathbf{S}}^3(a_{31}+\rho_0)}+\mathbf{K}\mathbf{c}_{11}F_{1,\dot{\mathbf{S}}^3\nu_0}+\mathbf{d}_{33}\mu_0,$$

$j=1, 2, 3.$

The anlytical expression for the matrices of the formulae (3.19) is given below:

$$\mathsf{K}=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T\mathsf{B}du,\quad \mathsf{M}=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T\mathsf{C}\mathbf{G}du,$$

$$\mathsf{L}=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T\mathsf{C}\mathsf{K}du,\quad \mathsf{P}=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T\mathsf{B}\mathsf{M}du,$$

$$\mathsf{N}=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T\mathsf{B}\mathsf{L}du+\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T\mathsf{D}\mathbf{G}du,\quad R_0'=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T g du,$$

$$M_0'=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T(\mathsf{B}R_0'+m)\,du,\quad Y_0'=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T(\mathsf{C}M_0'+p)\,du,$$

$$W_0'=\mathbf{G}\int_u\dot{\mathbf{S}}\mathbf{G}^T(\mathsf{B}Y_0'+\mathsf{D}R_0'+w)\,du.$$

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Streszczenie

ROZWIĄZANIE UKŁADU RÓŻNICZKOWYCH RÓWNAŃ RÓWNOWAGI PRZESTRZENNIE ZAKRZYWIONEJ BELKI

W pracy otrzymano w postaci zamkniętej rozwiązanie układu równań różniczkowych o zmiennych współczynnikach opisujących równowagę belki zakrzywionej w przestrzeni lub płaszczyźnie. Problem ten sprowadza się do układu dwunastu sprzężonych liniowych równań różniczkowych pierwszego rzędu który scałkowano metodami analitycznymi.

Резюме

РЕШЕНИЕ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ РАВНОВЕСИЯ ПРОСТРАНСТВЕННО ИСКРИВЛЕННОЙ БАЛКИ

В работе получено, в замкнутом виде, решение системы дифференциальных уравнений с переменными коэффициентами, описывающей равновесие искривленной балки в пространстве или на плоскости. Эта проблема сводится к системе двенадцати сопряженных, линейных дифференциальных уравнений первого порядка, которая проинтегрирована аналитическими методами.

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Received May 4, 1979.