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THERMODIFFUSION IN ELASTIC MATERIALS OF GRADE TWO

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The elastic solid of grade two subjected to the process of thermodiffusion is studied in the framework of the nonlinear mechanics of continua. Two independent sets of constitutive quantities are obtained: the first set results from the free energy density as a thermodynamic potential, and second one consists of thermodynamic fluxes. Constitutive equations for thermodynamic fluxes are restricted not only by the entropy inequality, but also by a number of additional conditions following from the entropy inequality. Expanding the free energy density into a quadratic polynomial and assuming particular forms of constitutive equations for entropy, heat and diffusion fluxes, the linearized field equations for displacements, temperature and concentration of diffused mass are derived.

1. INTRODUCTION

The problem of thermodiffusion in elastic solids with microstructure was studied in Paper [1]. In Paper [2] the nonlinear and linearized field equations for thermodiffusion in micropolar elastic materials were derived. In both papers the process of thermodiffusion was regarded as a sequence of nonuniformity of the temperature distribution in the body, thus in the absence of chemical reactions, that is, under the assumption of the validity of the law of mass conservation. In that sense the notion of thermodiffusion was primarly introduced by PODSTRIGACZ [3] and NOWACKI [4]. The same conception of thermodiffusion will be followed in the present paper.

2. EQUATIONS OF BALANCE

The theory of elastic materials of grade two was developed by many authors [5, 6]. The main purpose of constructing the theory of thermodiffusion for such materials is to find the density $\rho(X^K, t)$, the concentration of diffused mass $c(X^K, t)$, the temperature $\theta(X^K, t)$ and the motion $x^k(X^K, t)$ of material points of the body whose state in an arbitrary chosen reference configuration is characterized by the coordinates X^K and by the corresponding density $\rho_0(X^K)$, the uniform temperature field θ_0 and concentration $c_0(X^K)$.

Equations of balance of mass, linear momentum, moment of momentum and energy for the material of grade two may be written in the following local form:

$$\begin{aligned} &\dot\rho+\rho v^i_{,i}=0, \\ &t^{ij}_{,j}+\rho f^i=\rho\dot v^i, \\ &t^{[i,j]}+M^{ijk}_{,k}=\rho\Gamma^{ij}, \\ &\rho\dot u=t^{(ij)}\,v_{i,j}+M^{ijk}\,\omega_{ij,k}+q^i_{,i}+\rho h. \end{aligned} \tag{2.1}$$

In Eq. (2.1) the dots denote time derivatives with fixed X^K; $(\ldots)_{,k}$ represents derivatives with respect to the present position. The further following notations are used: t^{ij} nonsymmetric stress tensor, $M^{ijk}=-M^{jik}$ couple stress tensor, f^i body force density, u internal energy density, ρh body heat supply, q^i heat flux vector, Γ^{ij} inertia spin. The last quantity may be expressed in the form

$$\Gamma^{ij}=i^{lj}\,(\dot{\bar\omega}^i_{.l}+\omega^i_{.r}\,\omega^r_{.l}) \qquad (i^{kl}=I^{KL}\,x^k_{;K}\,x^l_{;L}) \tag{2.2}$$

with

$$\omega_{ij}=v_{[i,j]}, \qquad v^i=\dot x^i. \tag{2.3}$$

The term representing the contribution of couple stresses in Eq. $(2.1)_4$ may be written in the following way:

$$M^{ijk}\,\omega_{ij,k}=M^{ijk}\,v_{i,jk}=M^{i(jk)}\,\omega_{ij,k}. \tag{2.4}$$

Apart from that, the following decomposition of the couple stress tensor takes place

$$M^{ijk}=\mu^{ijk}+\frac{1}{3!}\,\delta^{ijk}_{lmn}\,m^{lmn} \tag{2.5}$$

with

$$\mu^{ijk}\,\varepsilon_{ijk}=0, \qquad M^{ijk}_{,jk}=\mu^{i(jk)}_{,jk}. \tag{2.6}$$

In this manner, instead of Eq. $(2.1)_4$ the equation of balance of energy will be presented in the form

$$\rho\dot u=t^{(ij)}\,v_{i,j}+\mu^{ijk}\,\omega_{ij,k}+q^i_{,i}+\rho h. \tag{2.7}$$

Assuming that there are no body sources of mass production and denoting by J^i the flux vector of diffused mass, we complete the set of equations of balance $(2.1)_{1-3}$ and (2.7) by the local equation of balance of diffused mass

$$\rho\dot c-J^i_{,i}=0. \tag{2.8}$$

From Eq. $(2.1)_1$ we find

$$\rho_0/\rho=\sqrt{g/G}\,\det\{x^k_{;K}\}. \tag{2.9}$$

Introducing

$$\dot{x}_{i,j}=\dot{x}_{i;K}X^K_{;j}, \qquad \dot{x}_{i,jk}=\dot{x}_{i;KL}X^K_{;j}X^L_{;k}+\dot{x}_{i;K}X^K_{;jk} \tag{2.10}$$

and eliminating the skew-symmetric part of the stress tensor by means of Eq. $(2.1)_3$, we write the remaining equations of balance in the form

$$\begin{gathered}[t^{(ij)}_{;K}-\mu^{i(jk)}_{;KL}X^L_{;k}+(\rho\Gamma^{ij})_{;K}]X^K_{;j}-\mu^{i(jk)}_{;K}X^K_{;jk}+\rho f^i=\rho\ddot{x}^i,\\ \rho\dot{u}=[t^{(ij)}X^K_{;j}+\mu^{i(jk)}X^K_{;jk}]\dot{x}_{i;K}+\mu^{i(jk)}X^K_{;j}\dot{x}_{i;KL}+q^i_{;K}X^K_{;i}+\rho h,\\ \rho\dot{c}-J^i_{;K}X^K_{;i}=0.\end{gathered} \tag{2.11}$$

In the case of an elastic solid, the stress tensor, the couple stress tensor, internal energy density, the heat flux vector and diffusion flux vector at time t are related through constitutive equations to the instant values of variables of state. In our case the response function describes the thermodynamic properties of the material depending on the instant values of the following variables:

$$x^i_{;K}, \quad x^i_{;KL}, \quad \theta, \quad \theta_{;K}, \quad c, \quad c_{;K}. \tag{2.12}$$

In that way we start with the following constitutive equations:

$$\begin{gathered}t^{(ij)}=t^{(ij)}(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}),\\ \mu^{i(jk)}=\mu^{i(jk)}(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}),\\ u=u(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}),\\ q^i=q^i(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}),\\ J^i=J^i(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}).\end{gathered} \tag{2.13}$$

Introducing the constitutive equations (2.13) in the equations of balance (2.11), we obtain a set of differential field equations for x^i, θ and c. Every particular solution of this set of equations represents a thermodynamic process in the body.

3. The entropy inequality

Possible forms of the constitutive equations (2.13) are restricted by the entropy inequality. Denoting by η the entropy density, Φ^i the entropy flux and σ the entropy production, we express the entropy balance and the entropy inequality in the form [7]

$$\rho\dot{\eta}-\Phi^i_{,i}-\frac{\rho h}{\theta}=\sigma, \qquad \sigma\geqslant 0, \tag{3.1}$$

where the last term on the lhs. of Eq. $(3.1)_1$ represents the supply of entropy due to the existence of body heat sources. The inequality $(3.1)_2$ must be satisfied in every thermodynamic process. In accordance with the principle of equipresence, the set of constitutive equations (2.13) will be completed by two additional equations:

$$\begin{gathered}\eta=\eta(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}),\\ \Phi^i=\Phi^i(x^i_{;K}, x^i_{;KL}, \theta, \theta_{;K}, c, c_{;K}).\end{gathered} \tag{3.2}$$

The equation of balance of linear momentum $(2.11)_1$ as well as the equation of energy balance $(2.11)_2$ include free terms in the form of body force density and heat supply. For an arbitrary choice of variables $\rho(X^K, t)$, $x^i(X^K, t)$, $t^{(ij)}(X^K, t)$, $\mu^{i(jk)}(X^K, t)$, $u(X^K, t)$, $q^i(X^K, t)$, these two equations may serve to determine the corresponding values of body forces and body heat supply. On the contrary, Eq. $(2.11)_3$ may be understood as a constraint on the thermodynamic fields which are admissible in the sense of the restriction $(3.1)_2$

From the inequality $(3.1)_2$ we may eliminate the term ρh using Eq. $(2.11)_2$. In that way we obtain

$$(3.3)\qquad -\rho(\dot{\psi}+\eta\dot{\theta})+[t^{(ij)}X^K_{;j}+\mu^{l(jk)}X^K_{;jk}]\dot{x}_{l;K}+\mu^{i(jk)}X^K_{;j}\dot{x}_{i;KL}+ \\ +X^K_{;i}(q^i_{;K}-\theta\Phi^i_{;K})\geqslant 0,$$

with the usual definition of the free energy density $\psi=u-\eta\theta$.

In order to overcome the restriction imposed on the process through the relation $(2.11)_3$, we multiply it with the Lagrangian multiplier μ and subtract it from the inequality (3.3). Proceeding in that way we obtain the following form of the dissipation inequality:

$$(3.4)\qquad -\rho(\dot{\psi}+\eta\dot{\theta}-\mu\dot{c})+[t^{(ij)}X^K_{;j}+\mu^{i(jk)}X^K_{;jk}]\dot{x}_{l;K}+\mu^{i(jk)}X^K_{;j}\dot{x}_{i;KL}+ \\ +X^K_{;i}(q^l_{;K}-\theta\Phi^i_{;K}-\mu J^i_{;K})\geqslant 0.$$

In calculating the derivatives included in the Eq. (3.4), we find e.g.

$$\dot{\psi}=\frac{\partial\psi}{\partial x_{l;K}}\dot{x}_{l;K}+\frac{\partial\psi}{\partial x_{l;KL}}\dot{x}_{l;KL}+\frac{\partial\psi}{\partial\theta}\dot{\theta}+\frac{\partial\psi}{\partial\theta_{;K}}\dot{\theta}_{;K}+\frac{\partial\psi}{\partial c}\dot{c}+\frac{\partial\psi}{\partial c_{;K}}\dot{c}_{;K},$$

$$q^i_{;K}=\frac{\partial q^i}{\partial x_{l;L}}x_{l;KL}+\frac{\partial q^i}{\partial x_{l;LM}}x_{l;KLM}+\frac{\partial q^i}{\partial\theta}\theta_{;K}+\frac{\partial q^i}{\partial\theta_{;L}}\theta_{;KL}+ \\ +\frac{\partial q^i}{\partial c}c_{;K}+\frac{\partial q^l}{\partial c_{;L}}c_{;KL}.$$

Inserting the expressions for the derivatives $\dot{\psi}$, $q^i_{;K}$, $\Phi^i_{;K}$, $J^i_{;K}$ in the inequality (3.4) we state that the relation obtained is linear in $\dot{x}_{l;KL}$, $\dot{\theta}$, $\dot{\theta}_{;K}$, $\dot{c}$, $\dot{c}_{;K}$, $x_{l;KLM}$, $\dot{x}_{l;K}$, $\theta_{;KL}$, $c_{;KL}$. The inequality must be valid for arbitrary values of variables and hence the following conditions must be fulfilled:

$$(3.5)\qquad \begin{aligned} & t^{(ij)}X^K_{;j}+\mu^{i(jk)}X^K_{;jK}-\rho\frac{\partial\psi}{\partial x_{i;K}}=0, \\ & \mu^{i(jk)}X^K_{;j}X^L_{;k}-\rho\frac{\partial\psi}{\partial x_{i;KL}}=0, \\ & \eta+\frac{\partial\psi}{\partial\theta}=0, \qquad \mu-\frac{\partial\psi}{\partial c}=0, \\ & \frac{\partial\psi}{\partial\theta_{;K}}=0, \qquad \frac{\partial\psi}{\partial c_{;K}}=0, \end{aligned}$$

(3.5) [cont.]

$$\left(\frac{\partial q^i}{\partial\theta_{;L}} - \theta\frac{\partial\Phi^i}{\partial\theta_{;L}} - \mu\frac{\partial J^i}{\partial\theta_{;L}}\right) X^K_{;i\,(K,L)}=0,$$

$$\left(\frac{\partial q^i}{\partial c_{;L}} - \theta\frac{\partial\Phi^i}{\partial c_{;L}} - \mu\frac{\partial J^i}{\partial c_{;L}}\right) X^K_{;i\,(K,L)}=0,$$

$$\left(\frac{\partial q^i}{\partial x_{l;LM}} - \theta\frac{\partial\Phi^i}{\partial x_{l;LM}} - \mu\frac{\partial J^i}{\partial x_{l;LM}}\right) X^K_{;i\,(K,L,M)}=0.$$

From Eqs. $(3.5)_{5,6}$ we conclude that the free energy density does not depend on temperature and concentration gradients, hence,

$$\psi=\psi\left(x^i_{;K}, x^i_{;KL}, \theta, c\right). \tag{3.6}$$

Moreover, on the ground of Eqs. $(3.5)_{1-4}$ we realize that the free energy density plays the role of the thermodynamic potential for the stress tensor, couple stress tensor, entropy density and for the quantity μ which we are going to denote as the chemical potential. The corresponding constitutive equations read:

$$t^{(ij)}=\rho g^{il}\left(\frac{\partial\psi}{\partial x^l_{;K}}\, x^j_{;K} + \frac{\partial\psi}{\partial x^l_{;KL}}\, x^j_{;KL}\right),$$

(3.7)
$$\mu^{i(jk)}=\rho g^{il}\frac{\partial\psi}{\partial x^l_{;KL}}\, x^j_{;K}\, x^K_{;L},$$

$$\eta=-\frac{\partial\psi}{\partial\theta}, \qquad \mu=\frac{\partial\psi}{\partial c}.$$

The lhs. of Eq. $(3.7)_1$ is symmetric in i, j and the same must hold for its right hand side. Further, taking into account the fact that $\mu^{ijk}=-\mu^{jik}$, we get $\mu^{(i(jk))}=0$. In view of these remarks the following conditions take place:

(3.8)
$$\left[g^{il}\left(\frac{\partial\psi}{\partial x^l_{;K}}\, x^j_{;K} + \frac{\partial\psi}{\partial x^l_{;KL}}\, x^j_{;KL}\right)\right]_{[i,j]}=0,$$

$$\left(g^{il}\frac{\partial\psi}{\partial x^l_{;KL}}\, x^j_{;K}\, x^k_{;L}\right)_{(i,j,k)}=0.$$

The conditions (3.8) represent a system of 13 partial differential equations. Considering these conditions, we realize that among 27 variables $x^i_{;K}$ and $x^i_{;KL}$, only 14 are independent. We choose the following independent measures of deformation:

$$E_{KL}=\frac{1}{2}\left(g_{kl}\, x^k_{;K}\, x^l_{;L}-G_{KL}\right), \qquad D_{KLM}=\frac{1}{2}\, g_{kl}\left(x^k_{;ML}\, x^l_{;K}-x^k_{;MK}\, x^l_{;L}\right) \tag{3.9}$$

and hence

$$\psi=\psi\left(E_{KL}, D_{KLM}, \theta, c\right). \tag{3.10}$$

The set of constitutive equations (3.7) reduces now to

$$\begin{aligned}
t^{(ij)} &= \rho\left(\frac{\partial\psi}{\partial E_{KL}}\, x^i_{;K}\, x^j_{;L} + 2\,\frac{\partial\psi}{\partial D_{KLM}}\, x^{(i}_{;K}\, x^{j)}_{;LM}\right),\\
\mu^{i(jk)} &= \rho\,\frac{\partial\psi}{\partial D_{KLM}}\, x^i_{;K}\, x^{(j}_{;L}\, x^{K)}_{;M},\\
\eta &= -\frac{\partial\psi}{\partial\theta}, \qquad \mu=\frac{\partial\psi}{\partial c}.
\end{aligned} \tag{3.11}$$

4. THERMODYNAMIC FLUXES

The stress tensor, the couple stress tensor, the entropy density and the chemical potential are, within the frame of this theory, derivable from the free energy density using Eqs. (3.11). The constitutive equations for q^i, Φ^i and J^i must, on the other side, obey the set of restrictive conditions $(3.5)_{7-9}$ as well as the inequality

$$X^K_{;i}\left[\left(\frac{\partial q^i}{\partial x_{l;L}} - \theta\,\frac{\partial\Phi^i}{\partial x_{l;L}} - \mu\,\frac{\partial J^i}{\partial x_{l;L}}\right) x_{l;KL} + \left(\frac{\partial q^i}{\partial\theta} - \theta\,\frac{\partial\Phi^i}{\partial\theta} - \mu\,\frac{\partial J^i}{\partial\theta}\right)\theta_{;K} + \right.$$
$$\left. + \left(\frac{\partial q^i}{\partial c} - \theta\,\frac{\partial\Phi^i}{\partial c} - \mu\,\frac{\partial J^i}{\partial c}\right) c_{;K}\right] \geqslant 0\,. \tag{4.1}$$

The relations $(3.5)_{7-9}$ may be satisfied by the following particular solutions:

$$\begin{aligned}
\Phi^i &= \frac{q^i}{\theta} - \frac{\mu J^i}{\theta},\\
q^i &= q^i\,(x_{l;LM},\, \theta_{;L},\, c_{;L}),\\
J^i &= J^i\,(x_{l;LM},\, \theta_{;L},\, c_{;L})
\end{aligned} \tag{4.2}$$

if the relation below is respected:

$$\frac{\partial}{\partial x_{l;LM}}\left(\frac{\partial\psi}{\partial c}\right) = 0\,. \tag{4.3}$$

This last relation restricts the form of the constitutive equation (3.15). Accounting Eqs. (4.2)–(4.3), we find that the entropy inequality reduces to

$$\theta J^i\left(\frac{\mu}{\theta}\right)_{;i} + q^i\,\frac{\theta_{,i}}{\theta} \geqslant 0\,. \tag{4.4}$$

Hereafter we will examine the sequel of particular solutions (4.2) together with the constitutive equations (3.11), on the form of field equations of the considered process.

5. ISOTROPIC SOLID. LINEARIZATION

In the case of an isotropic solid we introduce spatial tensors corresponding to those given by Eqs. (3.9)

$$(5.1)\qquad e_{ij}=\frac{1}{2}(g_{ij}-G_{KL}X^{K}_{;i}X^{L}_{;j}),\qquad d_{ijk}=\frac{1}{2}G_{KL}(X^{K}_{;jK}X^{L}_{;i}-X^{K}_{;j}X^{L}_{;iK}).$$

The corresponding constitutive equations for stress and couple stress tensors are

$$(5.2)\qquad \begin{aligned} t^{(ij)}&=\rho\left(\frac{\partial\psi}{\partial e_{ij}}-2\frac{\partial\psi}{\partial e_{jk}}e^{l}_{k}-2\frac{\partial\psi}{\partial d_{jkl}}d^{l}_{kl}-\frac{\partial\psi}{\partial d_{klj}}d^{i}_{kl}\right),\\ \mu^{i(jk)}&=\rho\left[\left(\frac{\partial\psi}{\partial d_{ijk}}+\frac{\partial\psi}{\partial d_{ikj}}\right)-e^{i}_{l}\left(\frac{\partial\psi}{\partial d_{ljk}}+\frac{\partial\psi}{\partial d_{lkj}}\right)\right].\end{aligned}$$

In the linearized theory, introducing the displacements

$$(5.3)\qquad u^{K}_{;k}=g^{K}_{k}-X^{K}_{;k},$$

we reduce the tensors (5.1) to the known expressions

$$(5.4)\qquad e_{ij}=u_{(i,j)},\qquad d_{ijk}=u_{[j,i]k}.$$

The corresponding constitutive equations are [6]

$$(5.5)\qquad t^{(ij)}=\rho\frac{\partial\psi}{\partial e_{ij}},\qquad \mu^{ijk}=2\rho\frac{\partial\psi}{\partial d_{ijk}}.$$

Instead of the tensor μ^{ijk}, we may introduce

$$(5.6)\qquad \mu^{k}_{l}=\frac{1}{2}\varepsilon_{ijl}\,\mu^{ijk},$$

given by

$$(5.7)\qquad \mu^{ij}=\rho\frac{\partial\psi}{\partial\kappa_{ij}}\qquad\left(\kappa_{ij}\overset{\text{df}}{=}\frac{1}{2}g_{im}\varepsilon^{klm}d_{klj}=\frac{1}{2}g_{im}\varepsilon^{klm}u_{l,kj}\right),$$

where Eq. (3.6) is to be replaced by

$$(5.8)\qquad \psi=\psi(e_{ij},\kappa_{ij},\theta,c).$$

We chose the reference state as an arbitrary state. Now we suppose that at the reference state the following conditions take place: $e_{ij}=0$, $\kappa_{ij}=0$, $\psi=0$, $t^{ij}=0$, $\mu^{ij}=0$, $\eta=0$ and $\mu=0$. Further, we introduce the following notations for the departure of temperature and concentration from their values at the reference state:

$$(5.9)\qquad T=\theta-\theta_0,\qquad C=c-c_0.$$

Assuming that during the process of thermodiffusion a small disturbance of the reference state takes place, that is,

$$(5.10)\qquad |T/\theta_0|\ll 1,\qquad |C/c_0|\ll 1,$$

we expand the free energy function in terms of its arguments in a quadratic polynomial of the form

$$(5.11)\qquad \rho\psi=\frac{1}{2}A^{ijkl}e_{ij}e_{kl}+\frac{1}{2}B^{ijkl}\kappa_{ij}\kappa_{kl}+c^{ij}e_{ij}T+D^{ij}c_{ij}C+ \\ +\frac{1}{2}mT^2+\frac{1}{2}nC^2+pTC,$$

where

$$(5.12)\qquad \begin{aligned} A^{ijkl}&=\lambda g^{ji}g^{kl}+\mu\left(g^{il}g^{jk}+g^{ik}g^{jl}\right),\\ B^{ijkl}&=4\mu l^2\left(g^{il}g^{jk}+\bar{\eta}g^{ik}g^{jl}\right),\\ C^{ij}&=-(3\lambda+2\mu)\,\alpha_T\,g^{ij},\\ D^{ij}&=-(3\lambda+2\mu)\,\alpha_C\,g^{ij}\end{aligned}$$

are isotropic tensors and m, n, p, λ, μ, l, $\bar{\eta}$, α_T, α_C are material constants. The total number of material constants in Eq. (5.11) is nine.

In that way we obtain the following linear constitutive equations:

$$(5.13)\qquad \begin{aligned} &t^{(ij)}=\lambda\left[e_I-\frac{3\lambda+2\mu}{\lambda}\left(\alpha_T T+\alpha_C C\right)\right]g^{ij}+2\mu e^{ij}\qquad (e_I=g_{ij}e^{ij});\\ &\mu^{ij}=2\mu l^2\left(\kappa^{ij}+\bar{\eta}\kappa^{ji}\right),\\ &\rho_0\,\eta=(3\lambda+2\mu)\,\alpha_T\,e_I-(mT+pC),\\ &\rho_0\,\mu=-(3\lambda+2\mu)\,\alpha_C\,e_I+(pT+nC).\end{aligned}$$

Linear constitutive equations for heat flux and flux of diffused mass are based on the expressions $(4.2)_{2,3}$. According to Eqs. (5.3) and (5.4), we replace the arguments $x_{l;LM}$, $\theta_{;L}$ and $C_{;L}$ by the following

$$(5.14)\qquad \begin{aligned} &x_{l;MK}\approx g^m_M\,g^k_K\,u_{l,mk},\\ &T_{;L}\approx g^l_L\,T_{,l},\qquad C_{;L}\approx g^l_L\,C_{;l}.\end{aligned}$$

Herefrom, in the case of an isotropic solid, linear constitutive equations for heat and diffusion fluxes may be written in the form

$$(5.15)\qquad \begin{aligned} q_i&=L_{11}\,T_{,i}-L_{12}\,C_{,i}+L_{13}\,e_{I,i},\\ J_i&=L_{21}\,T_{,i}+L_{22}\,C_{,i}+L_{23}\,e_{I,i}.\end{aligned}$$

According to the constitutive equations (5.13) and (5.15), the entropy inequality (4.4) reduces to the following expression:

$$(5.16)\qquad \left(L_{21}\,T_{,i}+L_{22}\,C_{,i}+L_{23}\,e_{I,i}\right)\left[pT^i_{,}+nC^i_{,}-(3\lambda+2\mu)\,\alpha_C\,e^i_{I,}\right]+ \\ +\frac{\rho_0}{\theta_0}\left(L_{11}\,T_{,i}+L_{12}\,C_{,i}+L_{13}\,e_{I,i}\right)T^i_{,}\geqslant 0.$$

The inequality must be fulfilled for arbitrary values of variables and herefrom the following relations between the coefficients L_{11}, L_{12}, ..., n, p, α_C must take place:

$$(5.17)\qquad \begin{aligned} & pL_{21}+\frac{\rho_0}{\theta_0}L_{11}>0, \quad nL_{21}+pL_{22}+\frac{\rho_0}{\theta_0}L_{12}=0,\\ & nL_{22}>0, \quad -(3\lambda+2\mu)L_{21}\alpha_C+pL_{23}+\frac{\rho_0}{\theta_0}L_{13}=0,\\ & -L_{23}\alpha_C>0, \quad -(3\lambda+2\mu)L_{22}\alpha_C+nL_{23}=0. \end{aligned}$$

Finally, we remark that the constant coefficients $L_{11}, L_{12}, \ldots, n, p, \alpha_C$ introduced in the linearized theory are useful only for small disturbances from the reference state and their values are connected with the reference state.

6. FIELD EQUATIONS

The linearized field equations for thermodiffusion in an elastic solid of grade two consist of equations of balance (2.1) and (2.8) as well as of constitutive equations (5.13) and (5.15). On the rhs. of Eq. $(2.1)_3$ there is the inertia term Γ^{ij}. Regarding material points of the continuum as infinitesimal spheres, the following value for inertia spin, as its first approximation, may be obtained:

$$(6.1)\qquad \Gamma^{ij}=\frac{1}{2}I\frac{\partial^2}{\partial t^2}(u^{i,j}-u^{j,i}),$$

and hence

$$\rho\Gamma^{ij}_{,j}=\frac{\rho_0 I}{2}(\nabla^2\ddot{u}^i-\ddot{e}_{r}{}^{,i}) \quad \text{with} \quad \rho\approx\rho_0(l-e_I)\approx\rho_0.$$

Accounting Eq. (6.1), we may represent both equations of balance $(2.1)_{2,3}$ in the form of the following equation:

$$(6.2)\qquad t^{(ij)}_{,j}-\mu^{ijk}_{,jk}-\frac{\rho_0 I}{2}(\ddot{e}_{I,}{}^{i}-\nabla^2\ddot{u}^i)-\rho_0\ddot{u}^i+\rho f^i=0.$$

The remaining equations of balance are

$$(6.3)\qquad \begin{aligned} & \rho_0\dot{\psi}=t^{(ij)}\dot{u}_{i,j}+\mu^{ijk}\dot{u}_{i,jk}-\rho_0(\theta\dot{\eta}+\rho\dot{\theta})+q^i_{,i}+\rho h,\\ & \rho_0\dot{c}=J^i_{,i}. \end{aligned}$$

Thus the definitive form of linearized field equations for the process of thermodiffusion in an isotropic solid of grade two is

$$(6.4)\qquad \begin{aligned} & (\lambda+\mu)u^k_{,k}{}^{i}+\mu\nabla^2u^i+\mu l^2(\nabla^2\nabla^2u^i-\nabla^2u^k_{,k}{}^{i})+\frac{\rho_0 I}{2}(\nabla^2\ddot{u}^i-\ddot{u}^k_{,k}{}^{i}-\ddot{u}^i)=\\ & \qquad\qquad =(3\lambda+2\mu)(\alpha_T T^i_{,}+\alpha_C C^i_{,})-\rho f^i,\\ & (3\lambda+2\mu)\alpha_T\theta_0\dot{u}^k_{,k}-\theta_0(m\dot{T}+p\dot{C})=L_{11}\nabla^2T+L_{12}\nabla^2C+L_{13}\nabla^2u^k_{,k}+\rho h,\\ & \rho_0\dot{C}=L_{21}\nabla^2T+L_{22}\nabla^2C+L_{23}\nabla^2u^k_{,k}. \end{aligned}$$

The three equations of motion $(6.4)_1$, the equation of the temperature field $(6.4)_2$ and the equation of the field of concentration $(6.4)_3$ represent the complete system of differential equations of the problem.

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Streszczenie

TERMODYFUZJA W MATERIAŁACH SPRĘŻYSTYCH DRUGIEGO RZĘDU

W ramach nieliniowej teorii ośrodków ciągłych rozważono materiał sprężysty drugiego rzędu poddany procesowi termodyfuzji. Otrzymano dwa niezależne układy wielkości konstytutywnych: jedne wynikają z gęstości energii swobodnej jako potencjału termodynamicznego, a drugie są strumieniami termodynamicznymi. Równania konstytutywne strumieni termodynamicznych ograniczone są nie tylko przez nierówność entropii lecz także przez szereg warunków dodatkowych wynikających z tej nierówności. Rozwijając funkcję gęstości energii swobodnej w wielomian kwadratowy i zakładając pewne szczególne postacie równań konstytutywnych dla entropii oraz strumieni ciepła i dyfuzji, wyprowadza się zlinearyzowane równania pola dla przemieszczeń, temperatury i koncentracji dyfundującej masy.

Резюме

ТЕРМОДИФФУЗИЯ В УПРУГИХ МАТЕРИАЛАХ ВТОРОГО ПОРЯДКА

В рамках нелинейной теории сплошных сред рассмотрен упругий материал второго порядка подвергнутый процессу термодиффузии. Получены две независимые системы определяющих величин: один вытекают из плотности свободной энергии, как термодинамического потенциала, а вторые являются термодинамическими потоками. Определяющие уравнения термодинамических потоков ограничены не только неравенством энтропии, но также рядом дополнительных условий, вытекающих из этого неравенства. Разлагая функцию плотности свободной энергии в квадратный многочлен и предполагая некоторые частные виды определяющих уравнений для энтропии, а также потоков тепла и диффузии, выводятся линеаризованные уравнения поля для перемещений, температуры и концентрации диффундирующей массы.

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