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THE FIRST HALF-TURN OF AN INERTIAL VIBRATOR

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The paper concerns the starting of an inertial vibrator which, due to a high value of static unbalance, is unable to perform the first half-turn in the gravitational field of force. The problem plays an essential role in the selection of driving units for the wide class of over–resonance machines.

On the basis of the Pontryagin's Maximum Principle the problem was formulated in terms of a dynamic optimisation. Thus, the driving moment performs the function of control – which is being looked for – while equations of motion and start-end conditions related to the position of a vibrator, act as constrains equations and boundary conditions. The possible simplifications in description of the motion of the vibrator, as well as the influence of the variable asynchronous motor driving moment on the optimal solutions, are discussed in the paper. On the basis of the work-energy equivalence principle, the minimum values of the driving moment – which warrants the performance of the first half-turn of the vibrator at a given number of the moment switch-overs – were determined. The problem of thermal loads of the motor during the first half-turn was also considered. The theoretical results were confirmed by the computer simulations.

Key words: starting, vibratory machine, dynamic optimisation.

1. Introduction

The problem of selecting the driving unit power is very important for a wide class of over-resonance vibrating machines, such as: conveyers, screens, tables, grids etc. The demand for the driving moment for these machines is related to the need of overcoming two critical states. The first state concerns the necessity to surmount the gravitational force of an unbalanced mass during the first half-turn, while the second state corresponds to overcoming the rapidly increased resistance to motion during passing through the resonance zone. The estimated – in such a way – driving unit power exceeds, in many cases, several times the power demand for realization of the in-coming steady state. In the steady state, the energy dissipation occurs due to the technological or transporting processes, due to overcoming the resistance to the motion of vibrators and elements of machine suspension, or losses related to the driving moment transmission. The problem has been repeatedly pointed out and discussed in scientific papers. For example, the paper by Banaszewski [2], describes the start of the one- and two-mass vibrator at the given moment of force and the resistances from the rolling friction and bearing seal. Michalczyk [8] determined the minimum value of the moment of force:

$$M_{\min} = 0.7246\, meg\,, \tag{1.1}$$

which allows to perform the first half-turn when the motor starts from the arrested lowest position (me – static unbalance of a vibrator, g – gravitational acceleration). However, the approaches applied in the papers mentioned above were restricted to the vibrator model presented as a pendulum – at the assumption that the value of the driving moment as well as its sense are constant.

2. Problem formulated in terms of the Maximum Principle

Let us consider the problem in a more general way. Thus, *let us find the form of the moment acting on the vibrator shaft, which will allow to move the centre of its mass from the lower to upper position – without imposing at that moment any limitations concerning the value and direction of operation.* It is obvious that there is an infinite number of such forms; however, only some of them can are of practical significance. The selection of the criterion deciding upon the chosen solution is an open problem. However, it should be taken into account that not all criteria are useful. E.g. an instinctively appearing criterion related to minimalisation of the work being done by the driving moment, when applied to the loss-less model of a physical pendulum, will not provide the expected results. The optimal solution can be easily guessed since it is related to the state, in which the kinetic energy is zeroing at the upper position of the vibrator. But again – such a state can be obtained by an unlimited number of measures. This diversity of solutions requires either imposing of an additional condition on the criterion or the task formulation in the form of a multi-optimisation expression.

The most reasonable and convenient criterion seems to be the time-minimal criterion, which:

1. Provides directly an explicit solution.
2. Due to the time decrease in which the driving motor is present within the range of short-circuit currents, it directly causes lowering of the heat produced in rotor windings.

Thus the origin for further considerations becomes the form quality functional (2.1) as well as the equations of motion of the vibrating machine (2.2) – the model of which is presented in Fig. 1.

$$T = \int\limits_{t=0}^{t=t_k} dt \to \min\,, \tag{2.1}$$

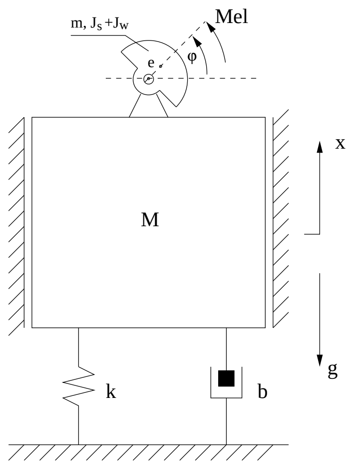


FIG. 1. The vibrating machine model.

$$
\begin{aligned}
&(M+m)\ddot{x} + me\ddot{\varphi}\cos(\varphi) - me\dot{\varphi}^2\sin(\varphi) + b\dot{x} + kx = 0, \\
&\qquad\qquad (J_s + me_w^2 + J_w)\ddot{\varphi} + me\ddot{x}\cos(\varphi) = M_{el} - mge\cos(\varphi),
\end{aligned}
\tag{2.2}
$$

J_s – central moment of the vibrator inertia, J_w – axial moment of the motor rotor inertia.

The problem presented hereby can be formulated and solved by means of the theory of calculus of variations, based on the Pontryagin's Maximum Principle. Due to the reasons of the presentation clarity, two variants of the problem will be discussed in the paper. At first the simplified variant – in which mutual interactions of the machine body and the vibrator are disregarded – will be presented. In this variant, the mechanical system becomes a physical pendulum subjected to the influence of the gravitational field and to the moment – which is being looked for. Later on, the second variant, that takes into account the previously omitted interactions and which is based on Eqs. (2.3) – will be discussed.

2.1. Simplified variant

On the basis of equations of motion of the pendulum (2.3) being subjected to the influence of $M(t)$ moment and referring to the Maximum Principle [4], the Hamiltonian function – due to the criterion (2.1) – takes the form given by Eq. (2.4).

$$
\begin{aligned}
&\frac{d\omega}{dt} = \frac{1}{J_{zr}}\left(M(t) - mge\sin(\varphi)\right), \\
&\frac{d\varphi}{dt} = \omega,
\end{aligned}
\tag{2.3}
$$

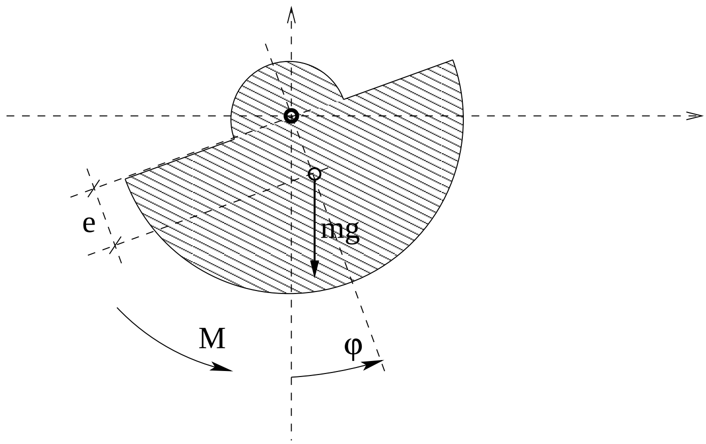


FIG. 2. Inertial vibrator model presented in the form of a physical pendulum.

$$H = \frac{\Psi_1}{J_{zr}}\left(M(t) - mge\sin(\varphi)\right) + \omega\Psi_2 - 1. \tag{2.4}$$

The Hamiltonian function linear dependence on the $M(t)$ value implicates immediately the form of the moment maximising the Hamiltonian along the optimal trajectory to the trajectory contained within the boundary of variability of the force moment. Thus, limiting the $M(t)$ moment to the set:

$$M(t) \subset [-M_0, +M_0] \tag{2.5}$$

we may observe that:

$$M(t) = M_0 \cdot \operatorname{sgn}(\Psi_1), \tag{2.6}$$

where M_0 is given value.

On the grounds of the conditions necessary for existing of the Hamiltonian extremum, the system (2.3) can be supplemented with equations for the coupled functions Ψ_1, Ψ_2:

$$\begin{aligned} \frac{d\Psi_1}{dt} &= -\frac{\partial H}{\partial \omega} = -\Psi_2, \\ \frac{d\Psi_2}{dt} &= -\frac{\partial H}{\partial \varphi} = \frac{mge\cos(\varphi)\Psi_1}{J_{zr}}. \end{aligned} \tag{2.7}$$

In turn, due to the time-minimal variant of the Maximum Principle, the final positions of coupled functions should fulfil the transversality conditions, which – in a general form [1] – can be written as:

$$\bar{\Psi}^* = \sum_{\alpha=1}^{m} k_\alpha \frac{\partial g_\alpha[\bar{x}^*, t_k^*]}{\partial \bar{x}^*[t_k^*]}, \tag{2.8}$$

where $g_\alpha[x^*(t_k), t_k^*]$ – equations imposed on the motion coordinates at time $t = t_k^*$.

In the problem discussed hereby, we have only one condition imposed on the final positions of the motion coordinates:

$$g_1 : \varphi^*(t_k^*) - \pi = 0 \tag{2.9}$$

and on its basis we determine:

$$\begin{aligned} \Psi_1^*(t_k^*) &= k_1 \frac{\partial g_1}{\partial \omega} = 0, \\ \Psi_2^*(t_k^*) &= k_1 \frac{\partial g_1}{\partial \varphi} = k_1, \end{aligned} \tag{2.10}$$

where k_1 – certain constant.

The fact of explicit independence of the final conditions of the time imposes one additional condition on the Hamilton's function: zeroing of its value along the optimal trajectory.

Thus, supplementing the final conditions (2.10) with conditions at the start:

$$\begin{aligned} \omega^*(0) &= 0, \\ \varphi^*(0) &= 0, \end{aligned} \tag{2.11}$$

and condition of zeroing of the Hamiltonian e.g. at $t = t_k$:

$$H(t_k^*) = 0 \tag{2.12}$$

we obtain the set of dependences needed for an explicit solution of systems (2.3) and (2.7).

The task being considered here belongs to the so-called two-point boundary value problems, which in a general case cannot be solved by traditional methods of numerical integration and requires a special approach. However, the preliminary analysis provides already some interesting conclusions.

Using the Eq. (2.7) we obtain the following equation:

$$\frac{d^2\Psi_1}{dt^2} + \frac{mge}{J_{zr}} \Psi_1 \cos(\varphi) = 0 \tag{2.13}$$

which, for small values of φ angle, becomes a homogeneous differential equation with constant coefficients. One can state – on its basis – that the time form of the moment is a switch-over type function with the switching frequency being equal to the double frequency of the pendulum free vibrations:

$$f_0 = \frac{1}{\pi} \sqrt{\frac{mge}{J_{zr}}} \,. \tag{2.14}$$

The system (2.7) – for any chosen value of angle φ – can be solved e.g. by application of the gradient shooting method [6].

Applying such an approach, the problem was solved for two values of moment M_0, namely for $M_0 = 16.47$ [Nm] and $M_0 = 65.88$ [Nm]. The values $J_{zr} = 1.98$ [kg m^2], $m_w = 382.3$ [kg], $e = 0.04$ [m] were assumed for the pendulum.

The solution for the first M_0 value is presented in Fig. 3. As it can be seen from the graph, the moment changes its sign eight times and finally, the angle

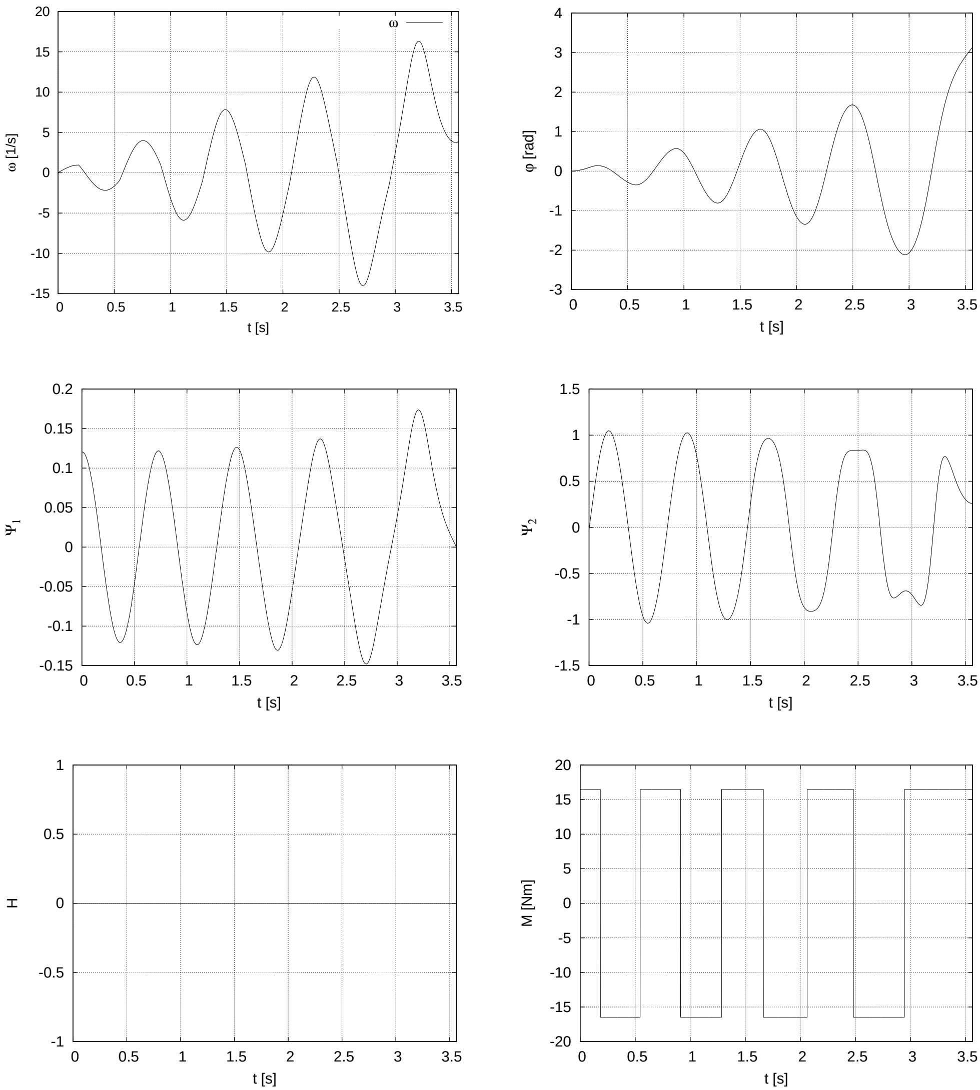


FIG. 3. Graphical presentation of solutions for the set of Eqs. (2.3) and (2.7), for $M_0 = 16.47$ [Nm].

coordinate φ reaches the value π. In accordance with Eq. (2.6), the time when the coupled coordinate Ψ_1 passes zero indicates the switch-over time and equals successively: 0.185 [s], 0.548 [s], 0.914 [s], 1.286 [s], 1.666 [s], 2.064 [s], 2.484 [s] and 2.947 [s].

The solution for the second M_0 value (being 4 times larger then the previous one) is presented in Fig. 4. In this case the moment changes its sign only once at $t = 0.36$ [s].

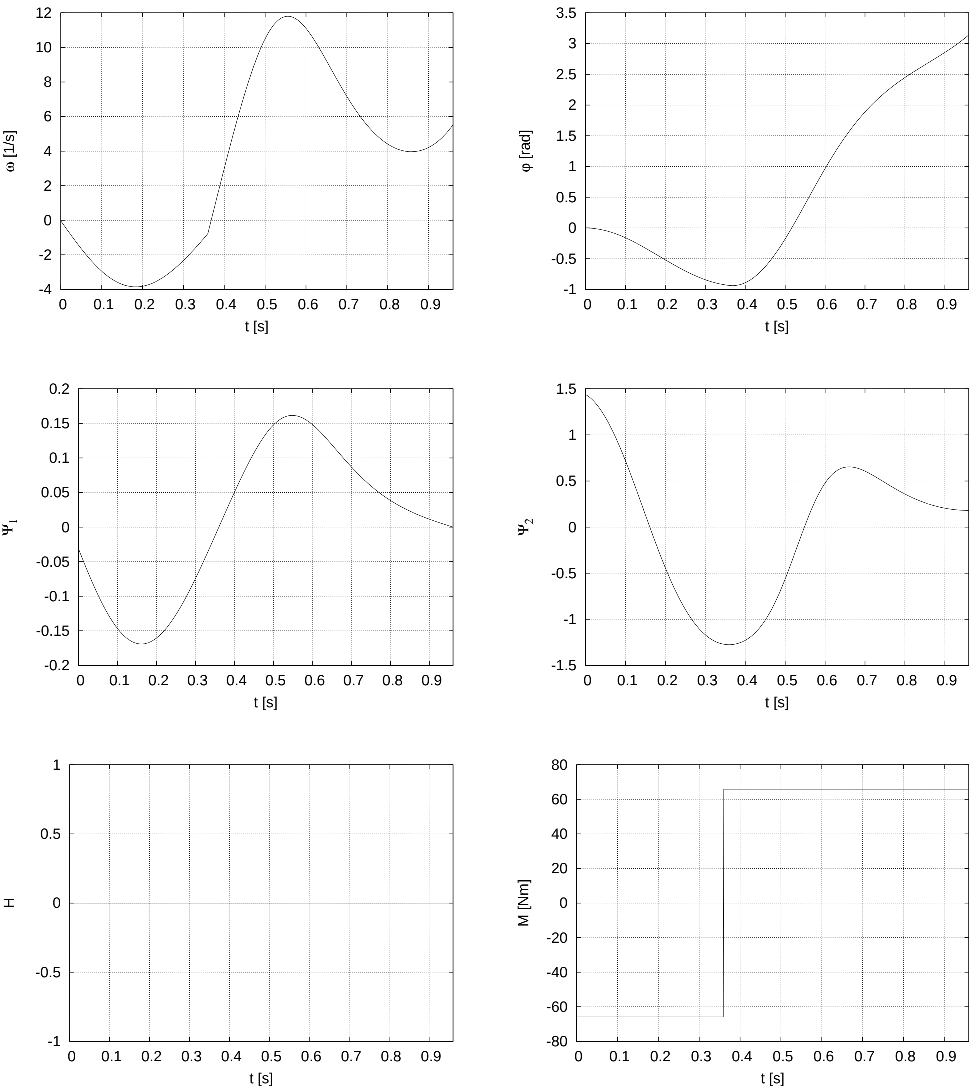


Fig. 4. Graphical presentation of solutions for the set of Eqs. (2.3) and (2.7), for $M_0 = 65.88$ [Nm].

2.2. *Full variant*

Proceeding in a similar fashion as in the previous chapter, we determine the Hamilton's function, which after rearrangement of Eqs. (2.2) to a standard form (2.15)

$$
\begin{aligned}
\frac{dv}{dt} &= \frac{(J_w + me^2)(me\omega^2 \sin(\varphi) - kx - bv - Mg - mg)}{\Delta} \\
&\qquad - \frac{me\cos(\varphi)(Mel - mge\cos(\varphi))}{\Delta}, \\
\frac{d\omega}{dt} &= \frac{-me\cos(\varphi)(me\omega^2 \sin(\varphi) - kx - bv - Mg - mg)}{\Delta} \\
&\qquad + \frac{(M+m)(Mel - mge\cos(\varphi))}{\Delta}, \\
\frac{dx}{dt} &= v, \\
\frac{d\varphi}{dt} &= \omega, \\
\Delta &= (M+m)(J_w + me^2) - (me\cos(\varphi))^2
\end{aligned}
\tag{2.15}
$$

assumes the following form:

$$
H = \Psi_1 \frac{dv}{dt} + \Psi_2 \frac{d\omega}{dt} + \Psi_3 \frac{dx}{dt} + \Psi_4 \frac{d\varphi}{dt} - 1 \ . \tag{2.16}
$$

On the basis of Eq. (2.16) and criterion (2.1) we determine the formulae for derivatives of coupled functions (2.17):

$$
\begin{aligned}
\frac{d\Psi_1}{dt} &= \frac{(J_w + me^2) b\Psi_1 - me\cos(\varphi) b\Psi_2}{\Delta} - \Psi_3, \\
\frac{d\Psi_2}{dt} &= \frac{-2(J_w + me^2) me\omega \sin(\varphi)\Psi_1 + 2m^2e^2\cos(\varphi)\omega\sin(\varphi)\Psi_2}{\Delta} - \Psi_4, \\
\frac{d\Psi_3}{dt} &= \frac{(J_w + me^2) k\Psi_1 - me\cos(\varphi) k\Psi_2}{\Delta},
\end{aligned}
\tag{2.17}
$$

(2.17) [cont.]
$$\begin{aligned}\frac{d\Psi_4}{dt} = &-\Big[(J_w + me^2)me\omega^2\cos(\varphi) + me\sin(\varphi)(Mel - mge\cos(\varphi)) \\ &\qquad - m^2e^2g\cos(\varphi)\sin(\varphi)\Big]\frac{\Psi_1}{\Delta} \\ &+ 2\Big[(J_w + me^2)(me\omega^2\sin\varphi - kx - bv - Mg - mg) \\ &\qquad - me\cos\varphi(Mel - mge\cos\varphi)\Big]\frac{\Psi_1 m^2e^2\cos\varphi\sin\varphi}{\Delta^2} \\ &- \Big[me\sin\varphi(me\omega^2\sin\varphi - k - bv - Mg - mg) - m^2e^2(\cos\varphi)^2\omega^2 \\ &\qquad + (m+M)mge\sin\varphi\Big]\frac{\Psi_2}{\Delta} \\ &+ \Big[(m+M)(Mel - mge\cos\varphi) - me\cos\varphi(me\omega^2\sin\varphi - kx - bv \\ &\qquad - Mg - mg)\Big]\frac{\Psi_2 m^2e^2\cos\varphi\sin\varphi}{\Delta^2}.\end{aligned}$$

Then – on the grounds of extremalising the conditions of the Hamilton's function versus moment $M(t)_{el}$ – we formulate the Law of Control in the form (2.18):

$$M_{el} = M_0\mathrm{sgn}\left[\frac{(M+m)\Psi_2 - me\cos\varphi\Psi_1}{\Delta}\right], \tag{2.18}$$

where M_0 – given value of the moment of force.

Using the transversality conditions (2.8) for coupled functions, Hamiltonian zeroing along the optimal trajectory as well as the start conditions for the motion coordinates of the system, we are able to determine the set of start-end conditions (2.19), which allow to solve the system (2.15) and (2.17) in an explicit way.

$$\begin{aligned} &v(0) = 0, && x(0) = -\frac{(M+m)g}{k}, && H(t_k^*) = 0, \\ &\omega(0) = 0, && \varphi(0) = -\frac{\pi}{2}, && \\ &\varphi(t_k^*) = +\frac{\pi}{2}, && \Psi_1(t_k^*) = 0, && \\ &\Psi_2(t_k^*) = 0, && \Psi_3(t_k^*) = 0. && \end{aligned} \tag{2.19}$$

The problem has been solved numerically for $M_0 = 65.88$ [Nm], using the set of parameters given in Table 1. The results are presented in Fig. 5.

Table 1.

Vibrator mass	382.3 [kg]
Radius of vibrator unbalance	0.04 [m]
Moment of inertia reduced to the axis of rotation	1.3 [kgm^2]
Machine body mass	7263.2 [kg]
Coefficient of elasticity of the machine body support	$4.83{\cdot}10^6$ [N/m]
Viscous damping coefficient of the machine body support	$3.84{\cdot}10^4$ [Ns/m]

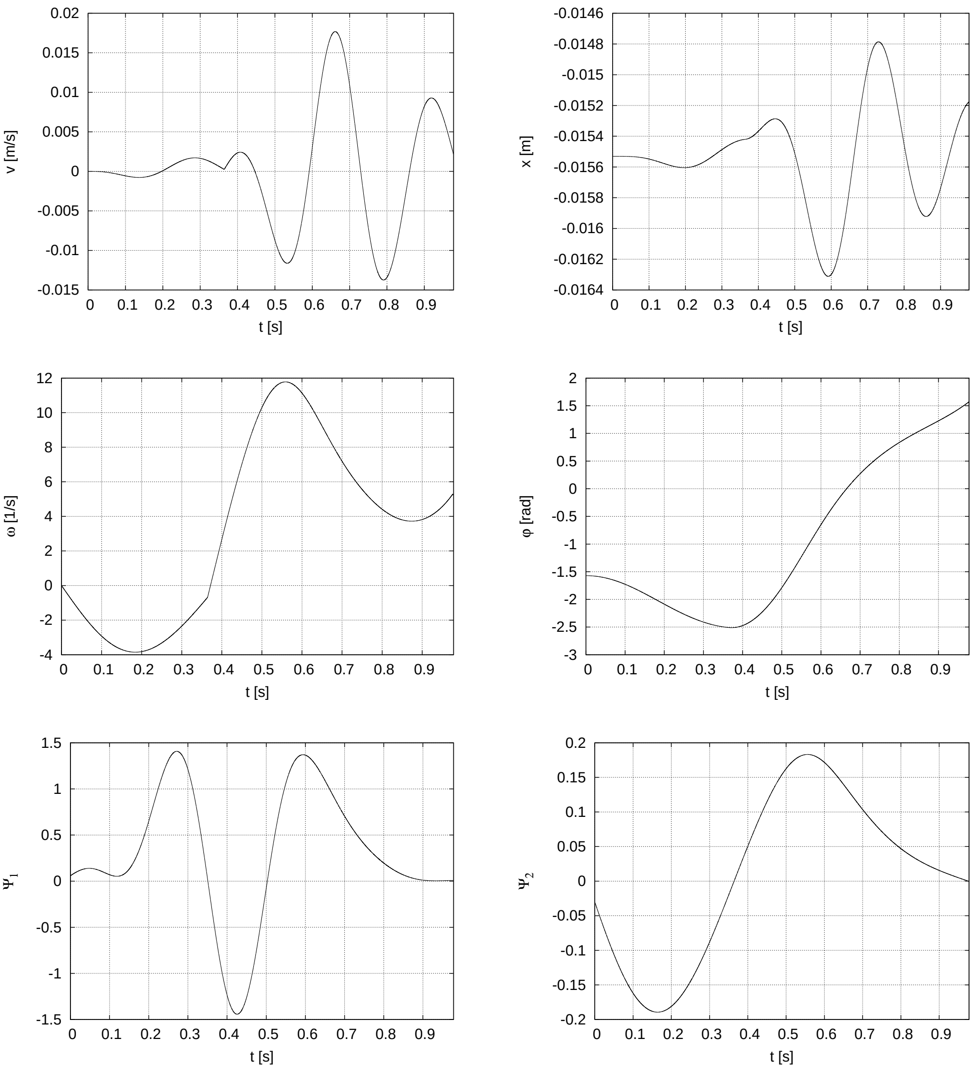


Fig. 5. The solutions of systems Eqs. (2.15) and (2.17).

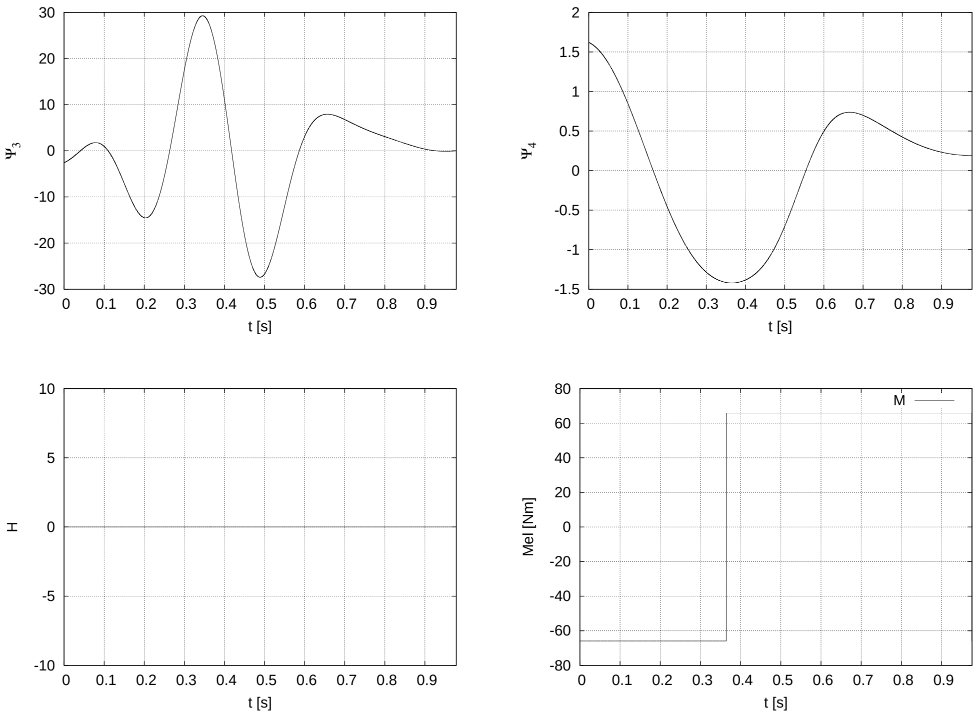


Fig. 6. The solutions of systems Eqs. (2.15) and (2.17) – continuation.

3. Influence of variability of the electric driving motor moment on the optimal solutions

The most popular source of the moment of force in vibrating machine drives is – in practice – an asynchronous motor. The instantaneous form of the moment of this motor – in the velocity range being near zero – is far from constant and exhibits very strong oscillations with the power network frequency. However, its mean value is – in this velocity range – near the value of the starting moment determined on the basis of the mechanical characteristic of the motor. Assuming this value as M_0 in the previously given models, the comparing simulations can be performed. They will allow estimate the influence of the electromagnetic moment variability on the optimal solutions determined for models with bi-stable moments. The interdependence of the electromagnetic moment of the driving motor and the vibrator shaft was determined in simulations based on the grounds of the asynchronous machine equations obtained as the result of two transformations applied separately for the stator circuit and for the rotor circuit. The first transformation, the Park's one described by the transformation matrix $[T]$ (3.1), changes the phase system of stator circuit equations into quasi-coordinates $0, d, q$. The second transformation, the Clark's transformation (3.2), transforms the phase system of rotor circuit equations to the coordinates $0, \alpha, \beta$.

$$[T] = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos(p_b\varphi) & \cos\left(p_b\varphi - \frac{2}{3}\pi\right) & \cos\left(p_b\varphi - \frac{4}{3}\pi\right) \\ -\sin(p_b\varphi) & -\sin\left(p_b\varphi - \frac{2}{3}\pi\right) & -\sin\left(p_b\varphi - \frac{4}{3}\pi\right) \end{bmatrix}, \tag{3.1}$$

$$[C] = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}. \tag{3.2}$$

The relations between currents and quasi-currents $i'_{\alpha r}$ $i'_{\beta r}$ i_{ds} i_{dq} and the transformed power voltages u_{ds}, u_{qs} can be presented in the following matrix notation:

$$\begin{bmatrix} u_{ds} \\ u_{qs} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_{\sigma s} + L_m & 0 & L_m & 0 \\ 0 & L_{\sigma s} + L_m & 0 & L_m \\ L_m & 0 & L'_{\sigma r} + L_m & 0 \\ 0 & L_m & 0 & L'_{\sigma r} + L_m \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i'_{\alpha r} \\ i'_{\beta r} \end{bmatrix} + \tag{3.3}$$

$$+ p_b\dot{\varphi} \begin{bmatrix} 0 & -(L_{\sigma s} + L_m) & 0 & -L_m \\ L_{\sigma s} + L_m & 0 & L_m & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i'_{\alpha r} \\ i'_{\beta r} \end{bmatrix} +$$

$$+ \begin{bmatrix} R_s & 0 & 0 & 0 \\ 0 & R_s & 0 & 0 \\ 0 & 0 & R'_r & 0 \\ 0 & 0 & 0 & R'_r \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i'_{\alpha r} \\ i'_{\beta r} \end{bmatrix},$$

while the electric moment is formulated as:

$$M_{el} = p_b\left(\Psi_{ds} i_{qs} - \Psi_{qs} i_{ds}\right), \tag{3.4}$$

where Ψ_{ds}, Ψ_{qs} – components of the electromagnetic flow associated with the stator and rotor, which can be presented in the matrix notation (3.5):

$$
\begin{bmatrix} \Psi_{ds} \\ \Psi_{qs} \\ \Psi_{\alpha r} \\ \Psi_{\beta r} \end{bmatrix} = \begin{bmatrix} L_{\sigma s}+L_m & 0 & L_m & 0 \\ 0 & L_{\sigma s}+L_m & 0 & L_m \\ L_m & 0 & L'_{\sigma r}+L_m & 0 \\ 0 & L_m & 0 & L'_{\sigma r}+L_m \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \\ i'_{\alpha r} \\ i'_{\beta r} \end{bmatrix}. \tag{3.5}
$$

The commutation process related to switching-over the phase sequence of the stator circuit, being done in order to change the sense of the moment to the opposite one, was determined on the basis of the Continuity Principle of the electromagnetic flow associated with the rotor circuit. Taking into account Eq. (3.5) – the components of the relevant currents – we can state that: $\Psi_{\alpha r}^{-} = \Psi_{\alpha r}^{+}$ and $\Psi_{\beta r}^{-} = \Psi_{\beta r}^{+}$, where indices „+" and „−" mark the time just before and just after the switch-over, respectively.

The continuity of the flow associated with the rotor leads to the formulae for the current components values just after the switch-over, in the form:

$$
\begin{aligned}
i_{ds}^{+} &= 0, \\
i_{qs}^{+} &= 0, \\
i_{\alpha r}^{\prime +} &= \frac{L_m i_{ds}^{-} + (L'_{\sigma r}+L_m) i_{\alpha r}^{\prime -}}{L'_{\sigma r}+L_m}, \\
i_{\beta r}^{\prime +} &= \frac{L_m i_{qs}^{-} + (L'_{\sigma r}+L_m) i_{\beta r}^{\prime -}}{L'_{\sigma r}+L_m}.
\end{aligned} \tag{3.6}
$$

Table 2. Parameters of the asynchronous motor.

Motor power rating P_n	8.1 [kW]
Motor rated velocity n_n	1420 [rot/min]
Stator leakage inductance $L_{\sigma s}$	4.77 [mH]
Rotor leakage inductance in stator terms $L'_{\sigma w}$	4.77 [mH]
Magnetic induction L_μ	82.4 [mH]
Stator resistance R_s	0.62 [Ω]
Rotor resistance in stator terms R'_w	0.84 [Ω]

The simulation results presented in Fig. 7 indicate that the switch-over performed at the times determined for models presented in Secs. 2.1 and 2.2, does not cause any significant differences, neither in the curve of the angular coordinate nor in the curve of the angular velocity mean value – regardless of the visible oscillatory influence on the driving moment.

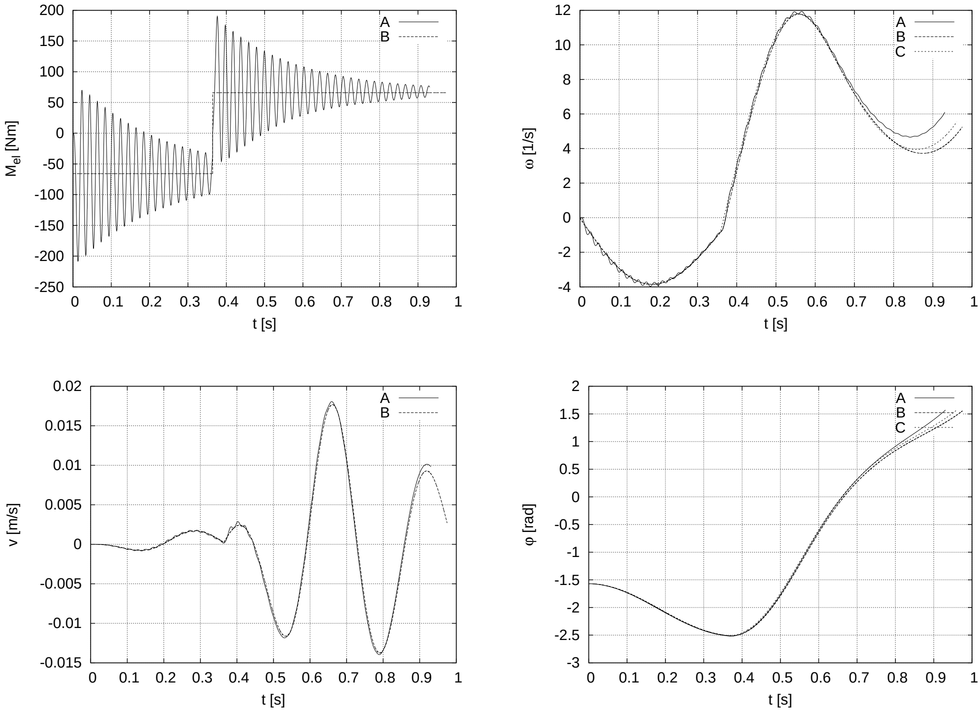


Fig. 7. Graphs of the motion coordinates for: A – model with an electric moment originated from an asynchronous motor, B – full variant with a bi-stable moment of force, C – simplified variant with a bi-stable moment of force.

4. Number of switch-overs of the driving moment

The need to determine the number of the moment switch-overs realizing the first half-turn of the vibrator for the given M_0 value, might be essential in the preliminary designing process. For the simplified model this problem can be solved analytically, without the necessity of performing an explicit solution of the pendulum equation of motion. Thus, on the basis of the condition of the mass lifting – in the gravitational field of force [8] – subjected to the influence of the constant value M_0 moment, for the case in which only one switch-over occurs, we are entitled to state the following:

1. During lifting of the vibrator mass centre, the motion will take place until the work performed by the moment of force equals the potential energy stored in the vibrator. Denoting by φ_1 the angular distance at which this equalisation occurs, we obtain the formula:

$$M_0\varphi_1 = mge(1 - \cos\varphi_1). \tag{4.1}$$

2. During the reverse in the intermediate point, related to the lowest position of the vibrator mass centre, the vibrator kinetic energy constitutes the sum of the recovered potential energy connected with the height determined by angle φ_1 and the work performed at returning at the angular distance φ_1. This energy – increased by the work performed on the further part of the distance φ and decreased by the potential energy of the repeated lifting of the mass centre - should retain its positive value to avoid the stopping of motion and changing its direction. Thus, we can write:

$$E = mge(1 - \cos\varphi_1) + M_0\varphi_1 + M_0\varphi - mge(1 - \cos\varphi) > 0. \tag{4.2}$$

3. To quarantee the fulfilment of inequality (4.2), which after rearrangement assumes the form:

$$\frac{M_0}{mge}(2\varphi_1 + \varphi) > 1 - \cos\varphi, \tag{4.3}$$

   the care should be taken that the straight line of the left-hand side of the inequality should be above the function value of the right-hand side. The smallest M_0 value corresponds to the situation when tangents to the function on its both sides are equal at certain critical point φ_{kr} and the function values are also equal at this point. Thus, respectively:

$$\frac{M_{0\,\min}}{mge} = \sin\varphi_{kr}, \tag{4.4}$$

$$\frac{M_{0\,\min}}{mge}(2\varphi_1 + \varphi_{kr}) = 1 - \cos\varphi_{kr}. \tag{4.5}$$

4. Solution of the set of Eqs. (4.1), (4.4), (4.5):

$$\begin{aligned} M_{0\,\min} &= 0.422mge, \\ \varphi_1 &= 0.905 \text{ [rad]}, \\ \varphi_{kr} &= 2.706 \text{ [rad]} \end{aligned} \tag{4.6}$$

   determines the smallest value of the driving moment $M_{0\,\min}$, which allows to perform the first half-turn with only one switch-over of the moment. It also determines two angles: φ_1 and φ_{kr}, for which the angular velocity of the vibrator equals zero.

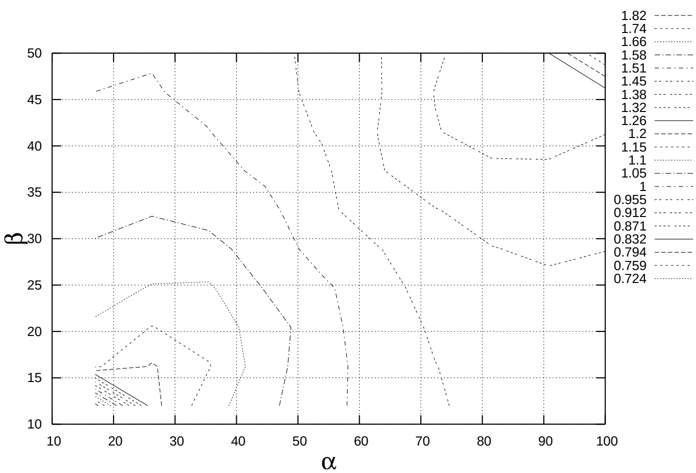
By means of a similar reasoning we can determine the switch-over times and the minimum moment M_0 for a larger number of switch-overs. Table 3 presents the determined dependences, with notations: n – number of switch-overs, $k = \dfrac{M_{0\,\min}}{mge}$, φ_i – successive angles corresponding to switch-overs of the moment, φ_{kr} – critical value of the angle of rotation.

Table 3.

n	k	φ_i [rad]	φ_{kr} [rad]
0	0.724	–	+2.31
1	0.422	−0.905	+2.706
2	0.293	+0.605, −1.306	+2.844
3	0.224	−0.456, +0.948, −1.546	+2.916

Knowing the switch-over number of the moment, it is possible to estimate the least upper bound of the time needed for performing the first half-turn. Fig. 8 presents the nomogram, obtained by solving the pendulum equation of

a)



b)

alpha	beta	t [s]
...	...	...
75.650	34.000	0.924
75.700	33.000	0.993
75.700	33.050	0.988
75.700	33.100	0.983
75.700	33.150	0.979
75.700	33.200	0.975
*75.700	33.250	0.971
75.700	33.300	0.967
75.700	33.350	0.963
...	...	...

FIG. 8. Time of the first half-turn as a function of α and β parameters for the case of one switch-over of the moment. a) Nomogram, b) Table of data (* – indicates data for the example discussed in the paper).

motion, relating the time – which is looked for – to two parameters $\alpha = \frac{mge}{J_{zr}}$ and $\beta = \frac{M_0}{J_{zr}}$ for the case of one switch-over of the moment.

5. Conclusions

Summarising the problem of the first half-turn, we can state that:

1. It is possible to find the driving moment of the variable sense, which will allow the first half-turn to be performed by vibrators, without unnecessery overmotoring.
   (a) The driving moment is contained within the boundary of this moment variability, and due to this feature – for the values being in the range $[-M_0, +M_0]$ – it becomes the bi-stable moment of the value $\pm M_0$.
   (b) Times of the switch-over of the driving moment can be estimated on the basis of Eq. (2.18) and dependences (2.15), (2.17), (2.19) necessary for solving the Hamiltonian extreme (2.16).
2. The start, in which the first half-turn is put into practice by switch-overs of the sense of the moment, allows to decrease radically the starting moment of the motor, which corresponds to decreasing the driving unit power. As a result, the system efficiency at the nominal operations improves significantly.
3. Simplifications – following from the assumption that the driving moment is of a constant value and from the omission of body and vibrator interactions – do not introduce any essential errors, concerning the determination of the switch-over time of the moment of force, for typical driving systems of vibratory machines. It means for such systems, in which an asynchronous motor is the source of the moment of force and the vibrator is fixed to the elastically supported machine body. The value of the motor starting moment found directly from the catalogue data or determined on the basis of the Kloss mechanical characteristics – with taking into consideration the displacement of current for deep-groove motors (like in B NEMA [National Electrical Manufacturers Association] design) and for multi-cage motors (like in C NEMA design) – should be applied in calculations as the M_0 value.
4. When determining the M_0 value for over-resonance machines, special attention should be directed towards the situation when the vibrator is passing through the resonance zone. When the selected value of the moment (although allowing to perform the first half-turn and to overcome the motion resistance at nominal operations) is too low, it might be not sufficient to

overcome the increased resistance during passing through the resonance zone. In consequence, the stall of a motor in this zone occurs.

5. If the start is realised at low values of the M_0 moment, which prolongs the time when the motor stays in the zone of short-circuit currents, the problem of motor thermal load and the selection of safety measures should be considered. For estimation of the lowest upper bound of the time of the first half-turn and for determination of the minimum number of switch-overs of the moment, one can use dependences presented in Sec. 4.

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