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ON THE LONGITUDINAL VIBRATIONS OF TAPERED BARS

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In this paper an attempt is made to improve the elementary theory that describes the longitudinal vibrations of a tapered rod. The differential equation including both lateral inertia and shear is derived. In order to facilitate comparison with the known results, numerical values of the fundamental resonant frequencies and velocity gain factors are obtained for conical bars with linear taper.

1. Introduction

Rods of different shapes with varying cross-section and under longitudinal vibrations are widely used in ultrasonic installations for producing a gain in the particle velocity. An extensive bibliography and brief discussion of the various important works on this subject is available in Ref. [5]. The problem of longitudinal vibrations of tapered rods with a circular cross-section has been investigated by several authors. Most of these authors have assumed that the lateral conctractions and extensions of the cross-section of the rod are small and, therefore, may be ignored. The results obtained by these investigators appear reasonable provided that the length of the rod is large in comparison to its diameter.

Rayleigh [11] showed that, in prediction of the period of vibration, the error caused by ignoring the lateral motion is of the order of $(r/b)^2$ where r is the radius and b is the length of the rod. Thus, for sufficiently short rods the lateral motion of a particle is comparable in amplitude to its longitudinal motion and must not be ignored.

In the senior author's previous paper [1], the effect of Poisson's ratio on the longitudinal vibration was considered and it was found that it does have some effect on the desired parameters.

Martin [5] has derived a new wave equation for a conical rod which includes lateral inertia and shear. The numerical results presented by him show that Poisson's effect is appreciable and that it becomes more pronounced as the value of Poisson's parameter is increased. This author used the variational technique.

In this analysis the problem of longitudinal vibrations of a tapered rod is set up as a boundary value problem of the second kind. The displacement components as assumed in various papers are improved and then by satisfying the three equations of equilibrium, a differential equation is derived which seems to be more general as compared to those presented in the previous investigations.

2. Equations of motion

Let S be the cross-sectional area of the beam at any distance x. The origin of coordinates is taken at that end of the rod with a large cross-section. The rod being of varying cross-section thus, S is a function of x. Let u, v, w be the displacement component in the x, y and z directions, respectively. The equations of motion are

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}=\rho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yy}}{\partial z}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{2.1}$$

$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}=\rho\frac{\partial^2 w}{\partial t^2},$$

where ρ is the density.

For the first approximation let us assume that the stresses σ_{yy} and σ_{zz} are small as compared to σ_{xx} and therefore be neglected. The displacement components may be chosen as

$$u=u(x, t), \quad v=-\sigma y\frac{\partial u}{\partial x}, \quad w=-\sigma z\frac{\partial u}{\partial x}, \tag{2.2}$$

where σ is the Poisson's ratio.

The shear stresses are given by

$$\sigma_{xy}=-Gy\frac{\partial^2 u}{\partial x^2}, \quad \sigma_{xz}=-Gz\frac{\partial^2 u}{\partial x^2}, \quad \sigma_{yz}=0, \tag{2.3}$$

where G is the modulus of rigidity.

Using Eq. (2.3) we can integrate the last two equations of the set (2.1) and obtain the normal stresses σ_{yy} and σ_{zz},

$$\sigma_{yy}=\left(\rho\sigma\frac{\partial^3 u}{\partial x\,\partial t^2}-G\sigma\frac{\partial^3 u}{\partial x^3}\right)\left(\frac{y^{*2}-y^2}{2}\right),$$

$$\sigma_{zz}=\left(\rho\sigma\frac{\partial^3 u}{\partial x\,\partial t^2}-G\sigma\frac{\partial^3 u}{\partial x^3}\right)\left(\frac{z^{*2}-z^2}{2}\right), \tag{2.4}$$

where y^*, z^* refer to a point on the boundary of the cross-section. The value of σ_{xx}, is now obtained from the stress-strain relation and is given by

$$\sigma_{xx}=E\frac{\partial u}{\partial x}+\frac{\sigma^2}{2}\left[(y^{*2}+z^{*2})-(y^2+z^2)\right]\left[\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right]. \tag{2.5}$$

Substituting in the first equation of the set (2.1), the equilibrium equations are satisfied provided

$$(2.6)\qquad ES\frac{\partial^2 u}{\partial x^2}+E\frac{\partial u}{\partial x}\frac{\partial s}{\partial x}+\frac{\sigma^2}{2}\left[\left(\rho\frac{\partial^4 u}{\partial x^2\,\partial t^2}-G\frac{\partial^4 u}{\partial x^4}\right)I\right]+$$

$$+\frac{\sigma^2}{2}\left[\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right]\frac{\partial I}{\partial x}=\left(\rho\frac{\partial^2 u}{\partial t^2}+2G\sigma\frac{\partial^2 u}{\partial x^2}\right)S,$$

where E is the Young's modulus and

$$I=\int[(y^{*2}+z^{*2})-(y^2+z^2)]\,dS.$$

Under a suitable assumption this differential equation reduces to the ones reported earlier in Refs. [1, 4, 5, 7].

To improve the theory of longitudinal vibrations of the tapered bars, the expression for v and w as assumed in Eq. (2.2) should be improved. In the second approximation the values of σ_{yy} and σ_{zz} are chosen as in Eq. (2.4) and then the value of stress σ_{xx} and displacements v and w are obtained from the stress-strain relations. These are given by

$$(2.7)\qquad \sigma_{xx}=E\frac{\partial u}{\partial x}+\frac{\sigma^2}{2}\left(\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right)[(y^{*2}+z^{*2})-(y^2+z^2)]$$

and

$$(2.8)\qquad \begin{aligned} v&=-\sigma y\frac{\partial u}{\partial x}+\left(y^{*2}y-\frac{y^3}{3}\right)P+(z^{*2}-z^2)\,yQ,\\ w&=-\sigma z\frac{\partial u}{\partial x}+\left(z^{*2}z-\frac{z^3}{3}\right)P+(y^{*2}-y^2)\,zQ,\end{aligned}$$

where

$$P=\frac{\sigma(1-\sigma^2)}{2E/\rho}\left[\frac{\partial^3 u}{\partial x\,\partial t^2}-\frac{G}{\rho}\frac{\partial^3 u}{\partial x^3}\right],\qquad Q=\frac{\sigma}{1-\sigma}P.$$

From the expressions for displacement functions, the shear stresses are given by

$$(2.9)\qquad \begin{aligned} \sigma_{xy}&=G\left[-\sigma y\frac{\partial^2 u}{\partial x^2}+\left(y^{*2}y-\frac{y^3}{3}\right)\frac{\partial P}{\partial x}+(z^{*2}-z^2)\,y\frac{\partial Q}{\partial x}\right],\\ \sigma_{xz}&=G\left[-\sigma z\frac{\partial^2 u}{\partial x^2}+\left(z^{*2}z-\frac{z^3}{3}\right)\frac{\partial P}{\partial x}+(y^{*2}-y^2)\,z\frac{\partial Q}{\partial x}\right],\\ \sigma_{yz}&=G\,[-2zyQ-2yzQ]=-4GyzQ.\end{aligned}$$

With these values for displacements and shear components, the improved value of normal stresses σ_{yy} and σ_{zz} is now obtained from the last two of the equilibrium equations (2.1)

$$\sigma_{yy}=\frac{\sigma}{2}\left(\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right)(y^{*2}-y)-\frac{1}{12}\left(\rho\frac{\partial^2 P}{\partial t^2}-G\frac{\partial^2 P}{\partial x^2}\right)[5y^{*4}+y^4-6y^{*2}y^2]-$$
$$-\frac{1}{2}\left(\rho\frac{\partial^2 Q}{\partial t^2}-G\frac{\partial^2 Q}{\partial x^2}\right)[(z^{*2}-z^2)(y^{*2}-y^2)]-GQ(y^{*2}-y^2),$$

$$\sigma_{zz}=\frac{\sigma}{2}\left(\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right)(z^{*2}-z^2)-\frac{1}{12}\left(\rho\frac{\partial^2 P}{\partial t^2}-G\frac{\partial^2 P}{\partial x^2}\right)[5z^{*4}+z^4-6z^{*2}z^2]-$$
$$-\frac{1}{2}\left(\rho\frac{\partial^2 Q}{\partial t^2}-G\frac{\partial^2 Q}{\partial x^2}\right)[(z^{*2}-z^2)(y^{*2}-y^2)-2GQ(z^{*3}-z^2).$$

The value of the normal stress σ_{xx} is now obtained from the stress-strain relation

$$\sigma_{xx}=E\frac{\partial u}{\partial x}+\sigma(\sigma_{yy}+\sigma_{zz}),$$

$$\text{(2.10)}\quad \sigma_{xx}=E\frac{\partial u}{\partial x}+\frac{\sigma^2}{2}\left(\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right)[(y^{*2}+z^{*2})-(y^2+z^2)]-$$
$$-\frac{\sigma}{12}\left(\rho\frac{\partial^2 P}{\partial t^2}-G\frac{\partial^2 P}{\partial x^2}\right)[5(y^{*4}+z^{*4})+(y^4+z^4)-6(y^{*2}y^2+z^{*2}z^2)]-$$
$$-\sigma\left(\rho\frac{\partial^2 Q}{\partial t^2}-G\frac{\partial^2 Q}{\partial x^2}\right)[(z^{*2}-z^2)(y^{*2}-y^2)]-2GQ[(y^{*2}+z^{*2})-(y^2+z^2)].$$

The equation of motion in the x-direction requires that the following differential equation be satisfied:

$$\text{(2.11)}\quad ES\frac{\partial^2 u}{\partial x^2}+E\frac{\partial u}{\partial x}\frac{\partial s}{\partial x}+\frac{\sigma^2}{2}\left(\rho\frac{\partial^4 u}{\partial x^2\,\partial t^2}-G\frac{\partial^4 u}{\partial x^4}\right)I_1+$$
$$+\frac{\sigma^2}{2}\left(\rho\frac{\partial^3 u}{\partial x\,\partial t^2}-G\frac{\partial^3 u}{\partial x^3}\right)\frac{\partial I_1}{\partial x}-\frac{\sigma}{12}\left(\rho\frac{\partial^3 P}{\partial x\,\partial t^2}-G\frac{\partial^3 P}{\partial x^3}\right)I_2-$$
$$-\frac{\sigma}{12}\left(\rho\frac{\partial^2 P}{\partial t^2}-G\frac{\partial^2 P}{\partial x^2}\right)\frac{\partial I_2}{\partial x}-\sigma\left(\rho\frac{\partial^2 Q}{\partial t^2}-G\frac{\partial^2 Q}{\partial x^2}\right)\frac{\partial I_3}{\partial x}-$$
$$-\sigma\left(\rho\frac{\partial^3 Q}{\partial x\,\partial t^2}-G\frac{\partial^3 Q}{\partial x^3}\right)I_3-2G\frac{\partial Q}{\partial x}I-2GQ\frac{\partial I}{\partial x}=\rho s\frac{\partial^2 u}{\partial t^2}-I_4,$$

where

$$I_1=\int[(y^{*2}+z^{*2})-(y^2+z^2)]\,ds,$$
$$I_2=\int[5(y^{*4}+z^{*4})+(y^4+z^4)-6(y^{*2}y^2+z^{*2}z^2)]\,ds,$$
$$I_3=\int(z^{*2}-z^2)(y^{*2}-y^2)\,ds,$$
$$I_4=-\frac{\partial}{\partial y}\int\sigma_{xy}\,ds-\frac{\partial}{\partial z}\int\sigma_{xz}\,ds.$$

Equation (2.11) is a differential equation that describes the longitudinal vibrations of a bar with a variable cross-section and includes both the inertia and the shear effects.

3. SOLUTION

The purpose of this investigation is to provide an improved engineering theory and, in particular, to study the effect of lateral inertia on the longitudinal vibrations of a variable cross-section beam. Taking the shear component to be zero, the differential equation (2.11) is specialized to account for the effect of lateral inertia only. This is considered appropriate at this stage as results obtained can be meaningfully compared with those already known. Consider simple harmonic vibrations with angular velocity ω, that is, take $u(x, t)=u(x)\sin\omega t$, the differential equation (2.11) becomes

$$(3.1)\qquad \left[1-\frac{\lambda^2\sigma^2 s}{2\pi}+\frac{\lambda^4 s^2\sigma^2(2\sigma^2+\sigma-1)}{6\pi^2}\right]\frac{d^2u}{dx^2}+ \\ +\frac{s'}{s}\left[1-\frac{\lambda^2\sigma^2 s}{\pi}+\frac{\lambda^4 s^2\sigma^2(2\sigma^2+\sigma-1)}{2\pi^2}\right]\frac{du}{dx}+\lambda^2 u=0,$$

where $\lambda^2=\rho\omega^2/E$ and s in the area of cross-section and $s'=ds/dx$. For a half wave resonant system, if U denotes the amplitude of vibration at $x=0$, $u(x)$ may be chosen to satisfy the boundary conditions

$$(3.2)\qquad u(0)=U,\quad u'(0)=0,\quad u'(d)=0.$$

It must be noted that no attempt is made here to satisfy boundary conditions on the curved surface of the bar. If s' is sufficiently small as compared to s, the surface stress condition has little effect on the average displacement u.

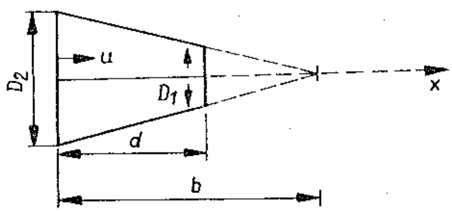

FIG. 1. Tapered rod of length b and area of cross-section $s(x)$.

In general it will be impossible to obtain solutions of Eq. (3.1) in a closed form. For a simpler equation [5], power series solutions have been obtained and are expressed in terms of Legendre functions.

Examination of Eq. (3.1) indicates that for a circular cross-section and Poisson ratio $\sigma=0$, the differential equation reduces to that given by MERKULOV [7]. The presence of high order terms in Eq. (3.1) is a consequence of considering σ not equal to zero. It follows that the solution of the differential equation (3.1) may be expressed as a power series of σ.

Let $u_0(x)$ denote the solution of Eq. (3.1) in the special instance when $\sigma=0$ and let $u(x)$ be expressed as

$$u(x)=\sum_{n=0}^{\infty} u_n(x)\,\sigma^n. \tag{3.3}$$

Substituting Eq. (3.3) in Eq. (3.1) and comparing terms with like powers of σ^n, we obtain

$$\begin{aligned}
&u_0''+\left(\frac{s'}{s}\right)u_0'+\lambda^2 u_0=0,\\
&u_2''+\frac{s'}{s}u_2'+\lambda^2 u_2=\left(\frac{\lambda^2 s}{2\pi}+\frac{\lambda^4 s^2}{6\pi^2}\right)u_0''+\frac{s'}{s}\left(\lambda^2\frac{s}{\pi}+\frac{\lambda^4 s^2}{2\pi^2}\right)u_0',\\
&u_3''+\frac{s'}{s}u_3'+\lambda^2 u_3=-\frac{\lambda^4 s^2}{6\pi^2}\left[u_0''+3\frac{s'}{s}u_0'\right],\\
&u_4''+\frac{s'}{s}u_4'+\lambda^2 u_4=\left[\left(\frac{\lambda^2 s}{2\pi}+\frac{\lambda^4 s^2}{6\pi^2}\right)u_2''+\frac{s'}{s}\left(\frac{\lambda^2 s}{\pi}+\frac{\lambda^4 s^2}{2\pi^2}\right)u_2'+\right.\\
&\qquad\left.-\left(\frac{\lambda^4 s^2}{3\pi^2}u_0''+\frac{s'}{s}\frac{\lambda^4 s^2}{\pi^2}u_0'\right)\right].
\end{aligned} \tag{3.4}$$

These equations are valid for any form of taper and for any form of cross-section. s and s' are functions of x and depend on the particular variation of the cross--section.

The value of u_0 is determined from Eq. $(3.4)_1$ subject to the boundary conditions $(3.2)_1$, whereas the solutions of Eqs. $(3.4)_{1,3,4}$ will be chosen to satisfy the boundary conditions

$$u_n(0)=0, \qquad u_n'(0)=0. \tag{3.5}$$

The solution given by the expansion Eq. (3.3) will then satisfy the boundary condition $(3.2)_2$, from which the value of (λd) will be determined. In order to carry numerical calculations, it is essential that Eqs. (3.4) be solved for a specific case.

The differential Eqs. (3.4) are specialized for a rod with a circular cross-section and having a linear taper. This case is chosen as a model since the numerical data for this case is available in literature.

Without getting into further explanations, the solutions of Eqs. (3.4) are presented and are given by

$$\begin{aligned}
&u_0(x)=\frac{U}{b-x}\left(b\cos\lambda x-\frac{1}{\lambda}\sin\lambda x\right),\\
&u_2(x)=\frac{U}{b-x}\,\frac{b(\lambda d_2)^2}{8}\left[\sum_{i=1}^{6} A_i\left(1-\frac{x}{b}\right)^{6-i}\right]\cos\lambda x+\\
&\qquad+\sum_{i=1}^{6} B\left[1-\frac{x}{b}\right]^{6-i}\sin\lambda x,
\end{aligned} \tag{3.6}$$

where

$$A_1=\frac{(\lambda d_2)^2}{12}\left(\frac{1}{10}\right), \qquad B_1=\frac{1}{12}\,\frac{(\lambda d_2)^2}{12}\,\frac{(b\lambda)}{10},$$

$$A_2=\frac{(\lambda d_2)^2}{12}\left(\frac{1}{4}\right), \qquad B_2=\frac{1}{12}\,\frac{(\lambda d_2)^2}{12}\left[-\frac{1}{4(\lambda b)}\right],$$

$$A_3=\frac{1}{6}+\frac{(\lambda d_2)^2}{12}\left[\frac{7}{6}\,\frac{1}{(\lambda b)^2}\right], \qquad B_3=\frac{\lambda b}{6}+\frac{1}{12}(\lambda d_2)^2\left[\frac{7}{6}\,\frac{1}{(\lambda b)}\right],$$

(3.7)
$$A_4=\frac{1}{4}+\frac{(\lambda d_2)^2}{12}\left[-\frac{7}{4}\,\frac{1}{(\lambda b)^2}\right], \qquad B_4=-\frac{1}{4}\,\frac{1}{(\lambda b)}+\frac{1}{12}(\lambda d_2)^2\left[\frac{7}{4}\,\frac{1}{(\lambda b)^3}\right],$$

$$A_5=\frac{5}{4}\,\frac{1}{(\lambda b)^2}+\frac{1}{12}(\lambda d_2)^2\left[-\frac{7}{4}\,\frac{1}{(\lambda b)^3}\right], \qquad B_6=-\sum_{i=1}^{5}B_i+\frac{1}{\lambda b}\sum_{i=1}^{5}(6-i)A_i,$$

$$A_6=-\sum_{i=1}^{5}A_i$$

and

(3.8)
$$u_3(x)=\frac{U}{b-x}\left(\frac{-b(\lambda d_2)^2}{8}\right)\left\{\left[\sum_{i=1}^{6}C_i\left(1-\frac{x}{b}\right)^{6-i}\right]\cos\lambda x+\sum_{i=1}^{6}\left[D_i\left(1-\frac{x}{b}\right)^{6-i}\right]\sin\lambda x\right\},$$

where

$$C_1=A_1, \qquad D_1=B_1,$$

$$C_2=A_2, \qquad D_2=B_2,$$

$$C_3=A_3-\frac{1}{6}, \qquad D_3=B_3-\frac{\lambda b}{6},$$

(3.9)
$$C_4=A_4-\frac{1}{4}, \qquad D_4=B_4+\frac{1}{4}\left(\frac{1}{\lambda b}\right),$$

$$C_5=A_5-\frac{5}{4}\,\frac{1}{(\lambda b)^2}, \qquad D_5=B_5-\frac{5}{4}\left(\frac{1}{\lambda b}\right),$$

$$C_6=-\sum_{i=1}^{5}C_i, \qquad D_6=-\sum_{i=1}^{5}D_i+\frac{1}{\lambda b}\sum_{i=1}^{5}(6-i)C_i;$$

$$U_4(x)=\frac{U}{b-x}\,\frac{b(\lambda d_2)^4}{64}\left\{\sum_{i=1}^{11}E_i\left(1-\frac{x}{b}\right)^{11-i}\cos\lambda x+\sum_{i=1}^{11}F_i\left[\left(1-\frac{x}{b}\right)^{11-i}\right]\sin\lambda x\right\},$$

where

$$E_1=\frac{1}{2(\lambda b)}\left(\frac{T_1}{10}\right),\quad F_1=\frac{1}{2(\lambda b)}\left[-\frac{G_1}{10}\right];$$

$$E_i=\frac{1}{2(\lambda b)}\left[\frac{T_i}{11-i}-(12-i)F_i-1\right],\quad F_i=\frac{1}{2(\lambda b)}\left[(12-{}_i i)E_i-1\frac{G_i}{11-i}\right]$$

$$(i=2,3,\dots,10),$$

$$E_{11}=-\sum_{i=1}^{10}E_i,\quad F_{11}=-\sum_{i=1}^{10}F_i+\frac{1}{(\lambda b)}\sum_{i=1}^{10}(11-i)E_i$$

and

$$G_1=\frac{(\lambda d_2)^2}{12}[-(\lambda b)^2A_1],$$

$$G_2=\frac{(\lambda d_2)^2}{12}[-14(\lambda b)B_1-(\lambda b)^2A_2],$$

$$G_3=\left\{-(\lambda b)^2A_1+\frac{(\lambda d_2)^2}{12}[36A_1-12(\lambda b)B_2-(\lambda b)^2A_3]\right\},$$

$$G_4=\left[\{-12(b\lambda)B_1-(\lambda b)^2A_2\}+\frac{(\lambda d_2)^2}{12}\{24A_2-10(\lambda b)B_3-(\lambda b)^2A_4\}\right],$$

$$G_5=\left[\{28A_1-10(\lambda b)B_2-(\lambda b)^2A_3\}+\frac{(\lambda d_2)^2}{12}\{14A_3-8(\lambda b)B_4-(\lambda b)^2A_5\}\right],$$

$$G_6=\left[\left\{18A_2-8(\lambda b)B_3-(\lambda b)^2A_4+\frac{4}{3}(\lambda b)^2\right\}+\frac{(\lambda d_2)^2}{12}\left\{6A_4-6(\lambda b)B_5+\right.\right.$$
$$\left.\left.+(\lambda b)^2\sum_{i=1}^{5}A_i\right\}\right],$$

$$G_7=\left[\left\{10A_3-6(\lambda b)B_4-(\lambda b)^2A_5-\frac{16}{3}\right\}+\frac{(\lambda d_2)^2}{12}\{-4(\lambda b)B_6\}\right],$$

$$G_8=\left[\left\{4A_4-4(\lambda b)B_5+(\lambda b)^2\sum_{i=1}^{5}A_i+\frac{16}{3}\right\}+\frac{(\lambda d_2)^2}{12}\left\{4\sum_{i=1}^{5}A_i\right\},\right.$$

$$G_9=[-2(\lambda b)B_6],$$

$$G_{10}=2\sum_{i=1}^{5}A_i$$

and

$$T_1=\frac{(\lambda d_2)^2}{12}[-(\lambda b)^2B_1],$$

$$T_2=\frac{(\lambda d_2)^2}{12}[14(\lambda b)A_1-(\lambda b)^2B_2],$$

$$T_3=\left[-(\lambda b)^2 B_1+\frac{(\lambda d_2)^2}{12}\{36B_1+12(\lambda b)A_2-(\lambda b)^2 B_3\}\right],$$

$$T_4=\left[\{12(\lambda b)A_1-(\lambda b)^2 B_2\}+\frac{(\lambda d_2)^2}{12}\{24B_2+10(\lambda b)A_3-(\lambda b)^2 B_4\}\right],$$

$$T_5=\left[\{28B_1+10(\lambda b)A_2-(\lambda b)^2 B_3\}+\frac{(\lambda d_2)^2}{12}\{14B_3+8(\lambda b)A_4-(\lambda b)^2 B_5\}\right],$$

$$T_6=\left[\left\{18B_2+8(\lambda b)A_3-(\lambda b)^2 B_4-\frac{8(\lambda b)}{6}\right\}+\frac{(\lambda d_2)^2}{12}\{6B_4+6(\lambda b)A_5-(\lambda b)^2 B_6\}\right],$$

$$T_7=\left[\left\{10B_3+6(\lambda b)A_4-(\lambda b)^2 B_5-\frac{16}{3}(\lambda b)\right\}+\frac{(\lambda d_2)^2}{12}\left\{-4(\lambda b)\sum_{i=1}^{5}A_i\right\}\right],$$

$$T_8=\left[\left\{4B_4+4(\lambda b)A_5-(\lambda b)^2 B_6-\frac{16}{3}\frac{1}{(\lambda b)}\right\}+\frac{(\lambda d_2)^2}{12}\{(-4)B_6\}\right],$$

$$T_9=-2(\lambda b)\sum_{i=1}^{5}A_i,\qquad T_{10}=-2[B_6].$$

Thus the solution of Eq. (3.1) may be represented as

$$u(x)=\frac{U}{b-x}[P(x)\cos\lambda x+Q(x)\sin\lambda x], \tag{3.10}$$

where

$$\begin{aligned}P(x)&=P_0+\sigma^2 P_2(x)+\sigma^3 P_3(x)+\sigma^4 P_4(x),\\ Q(x)&=Q_0+\sigma^2 Q_2(x)+\sigma^3 Q_3(x)+\sigma^4 Q_4(x).\end{aligned} \tag{3.11}$$

The value of λ must be chosen to satisfy Eq. $(3.2)_2$ which in turn requires the following equation to be satisfied:

$$\tan\lambda d=\frac{\dfrac{P(d)}{b}+\left(1-\dfrac{d}{b}\right)[P'(d)+\lambda Q(d)]}{\left(1-\dfrac{d}{b}\right)\left\{[\lambda P(d)-Q'(d)]-\dfrac{Q}{b}\right\}}. \tag{3.12}$$

Having found the value of λ from Eq. (3.12), the velocity gain ratio is then obtained from

$$\frac{u(d)}{u(0)}=\frac{1}{b-d}[P(d)\cos\lambda d+Q(d)\sin\lambda d]. \tag{3.13}$$

4. Conclusions

To demonstrate the effect of lateral inertia effects, numerical results were computed for various configurations. These are shown in Table 1. For each particular case, five numerical values are listed. The first and second values are taken from

Ref. [1]. The other three values have been computed from Eqs. (3.12) and (3.13), when the Poisson's parameter σ is chosen to be 0.3, 0.35 and 0.4, respectively. For comparison, the values corresponding to $\sigma=0$ have also been included. It must be noted that when $\sigma=0$, the value of λd depends only on the ratio b/d and hence on the ratio D_2/D_1. For computing numerical results the ratio d/D_2 is taken greater than 1. This is necessitated by the fact that series converge only when this ratio is maintained greater than 1.

Table 1. Values of gain for arious values of D_2/D_1 d/D_2 when $\sigma=0.0$, 0.3, 0.35 and 0.4.

σ	$\frac{D_2}{D_1}$	$\frac{d}{D_2}$	λd	GAIN
0.0	2.0	1.0	3.286	1.935
0.30	2.0	1.0	3.177	1.88
0.30	2.0	1.0	3.1856	1.8459
0.35	2.0	1.0	3.1510	1.8319
0.40	2.0	1.0	3.1128	1.8247
0.0	2.0	1.5	3.281	1.935
0.30	2.0	1.5	3.236	1.914
0.30	2.0	1.5	3.2495	1.9009
0.35	2.0	1.5	3.237	1.8912
0.40	2.0	1.5	3.2229	1.8817
0.0	3.0	1.5	3.473	2.647
0.30	3.0	1.5	3.426	2.605
0.30	3.0	1.5	3.4395	2.5811
0.35	3.0	1.5	3.4272	2.5626
0.40	3.0	1.5	3.4134	2.5441
0.0	3.0	2.0	3.473	2.647
0.30	3.0	2.0	3.447	2.605
0.30	3.0	2.0	3.4559	2.6137
0.35	3.0	2.0	3.4494	2.6031
0.40	3.0	2.0	3.4420	2.5918
0.0	4.0	1.0	3.629	3.148
0.30	4.0	1.0	3.517	3.009
0.30	4.0	1.0	3.5256	2.8967
0.35	4.0	1.0	3.4876	2.8573
0.40	4.0	1.0	3.4454	2.8364
0.0	8.0	10.0	3.979	4.048
0.30	8.0	10.0	3.978	4.044
0.30	8.0	10.0	3.978	4.045
0.35	8.0	10.0	3.978	4.0430
0.40	8.0	10.0	3.9775	4.0428

From Table 1 it is evident that the inertia terms are significant only for short rods and especially when the taper is large. In that instance the difference in the computed values differs by as much as 10%. For rods with sharp tapers and of considerable length, the lateral inertia effects are almost insignificant.

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Streszczenie

O PODŁUŻNYCH DRGANIACH ZWĘŻAJĄCYCH SIĘ PRĘTÓW

W pracy zaproponowano ulepszoną teorię, która opisuje podłużne drgania zwężających się prętów. Wyprowadzono równanie różniczkowe uwzględniające zarówno poprzeczne siły inercji, jak również inercję ścinania. W celu zapewnienia porównań ze znanymi rezultatami, wielkości numeryczne podstawowych częstości rezonansowych oraz współczynników wzrostu prędkości obliczono dla prętów stożkowych z liniowym zwężeniem.

Резюме

О ПРОДАЛЬНЫХ КОЛЕБАНИЯХ СУЖИВАЮЩИХСЯ СТЕРЖНЕЙ

В работе была предложена улучшенная теория, описывающая продольные колебания суживающихся стержней. Вводятся дифференциальные уравнения, учитывающие как поперечные силы инерции так и инерцию сдвига. С целью обеспечения сравнений с известными результатами, числовые значения основных резонансных частот и коэффициентов увеличения скорости были расчитаны для конусных стержней с линейным сужением.

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