ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 29, 2, 295—310, 1981
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

VIBRATION AND BENDING OF A CRACKED PLATE

B. D. AGGARWALA and P. D. ARIEL (CALGARY)

The vibration and bending of a square plate with four cracks emanating from the centre of the plate or the midpoints of the edges is considered. Dual series equations which result from the mixed boundary conditions along the lines of the cracks are derived. By isolating the singular part of the solution, the problem is reduced to a Fredholm integral equation which is solved numerically. For the vibrating plate the frequencies of the first two symmetric modes are obtained and the displacement and the strain energy are determined for the case of bending of the plate. Also in both cases, stress intensity factors are calculated.

Nomenclature

a length of side of the plate,
A_m, B_m constants occurring in the solution of Eqs. (1.3) and (1.29)
c crack-length,
c' $1/2\,\pi-c$,
D $Eh^3/[12(1-\nu^2)]$,
E Young's Modulus,
E_m constants defined by Eqs. (1.9) and (2.6),
F_m, G_m, H_m constants occurring in the integral equations,
f computed frequency,
f_0 true value of the frequency,
$f_1, f_2, \ldots$ appropriate constants given by Eq. (1.36),
E'_m, G'_m, H'_m, R'_m defined by Eqs. (1.29) and (2.17),
h plate thickness,
$h(\rho)$ defined by Eqs. (2.12) and (2.21),
$K(\rho, r)$ Kernel of integral equations,
k $(\mu/D)^{1/2}$,
M_x, M_y edge moment,
P $qa^4/D\pi^4$,
q load applieu to the plate in the z-direction,
R_m defined by Eqs. (2.8),
r, ρ parameters in the interval [0, 1),
r_1, r_2 defined by Eqs. (1.6),
U strain energy,
U_1 strain energy when cracks emanate from the edges,
U_2 strain energy when cracks emanate from the centre,
V_x, V_y transverse shear force,
w displacement of the plate in the z-direction,
w_{01} central deflection when cracks emanate from the edges,
w_{02} central deflection when cracks emanate from the centre,

- x, y coordinates in the plane of the undisturbed plate,
- z coordinate in the transverse direction,
- β $1/2\, m\pi$,
- γ $(1-\nu)/(3+\nu)$,
- Δ_m defined by Eqs. (1.11),
- $\Phi(t)$ unknown function related to E_m,
- ν stress intensity facter,
- μ mass per unit area,
- ν Poisson's ratio,
- Ω frequency of vibration,

INTRODUCTION

Using the classical equations of plate theory [1], a number of research workers have analysed the bending and/or vibration of plates with internal cracks or internal support. YANG [2] considered the bending of a finite plate with an internal support symmetrically placed with respect to parallel boundaries. The author made use of Green's function approach to formulate the problem in terms of singular integral equations. Related problems occur when there are cracks in the plate. A large number of such problems have been considered by SNEDDON [3].

In order to solve such problems, it is necessary to make the proper assumptions regarding the nature of singularity at the crack-tip. WILLIAMS [4] considered the bending of an infinite plate containing a semi-infinite crack and be deduced that there is a moment singularity at the crack tip which varies as the inverse square root of the distance from the tip of the crack.

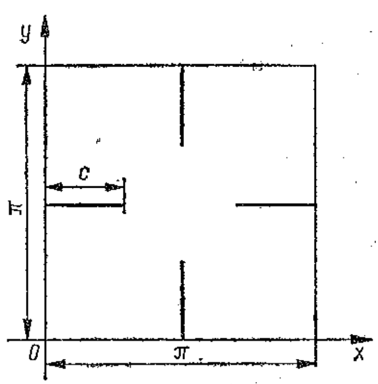

FIG. 1. Geometry of the plate with cracks emanating from the edges.

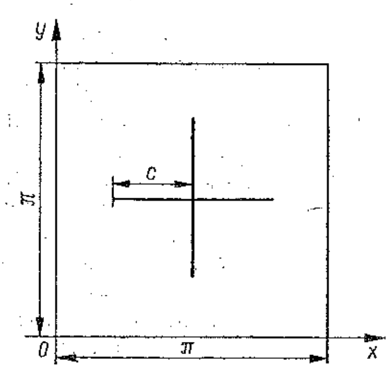

FIG. 2. Geometry of the plate with cracks emanating from the centre.

Using Green's function approach, the vibration of a plate was considered by LYNN and KUMBASAR [5] who obtained a Fredholm integral equation of the first kind. There have also been attempts at numerical solutions, a survey of which has been made by LEISSA *et al.* [6] who concluded that without taking into account the appropriate singularities the solutions were poorly approximated.

The bending of a rectangular plate with one or two cracks was investigated by KEER and SVE [7] who obtained various physical quantities such as the strain energy and stress intensity factor. The investigation of the vibrations of a rectangular plate with one crack emanating from the edge or through the centre was undertaken by STAHL and KEER [8] who gave the frequencies of vibrations.

In the present problem we have considered the vibrations and bending of a rectangular plate which has four symmetrical cracks parallel to edges either emanating from the edges or from the centre, cf. Figs. 1 and 2. It has been shown that the solution is reduced to solving a pair of dual series equations which in turn leads to a Fredholm integral equation of the second kind. The solution of this integral equation is found numerically and various physical quantities relevant to the problem are determined.

1. VIBRATIONS OF A SQUARE PLATE

We shall follow TIMOSHENKO and WOINOWSKY-KRIEGER [1] who, using the classical plate theory, obtained the following partial differential equation for $w_0(x, y, t)$, the free transverse displacement at the point (x, y) at time t[1]

$$\nabla^4 w_0 + \frac{\mu \partial^2 w_0}{D\, \partial t^2} = 0. \tag{1.1}$$

Assuming

$$w_0(x, y, t) = w(x, y) \exp(i\Omega t), \tag{1.2}$$

we eliminate the time dependence to obtain

$$\nabla^4 w - \frac{\mu \Omega^2}{D} w = 0. \tag{1.3}$$

In the follwing we shall restrict ourselves to symmetric-symmetric vibrations of a plate which is simply supported at the edges $x=0$, $y=0$, $x=\pi$ and $y=\pi$. Consequently, in view of the boundary conditions

$$w=0, \quad \frac{\partial^2 w}{\partial x^2}=0, \quad x=0, \quad x=\pi, \quad w=0, \quad \frac{\partial^2 w}{\partial y^2}=0, \quad y=0, \quad y=\pi \tag{1.4}$$

we choose

$$w(x, y) = \sum_{m=1,3}^{\infty} (A_m \sinh r_1 y + B_m \sinh r_2 y) \sin mx + \sum_{m=1,3}^{\infty} (A_m \sinh r_1 x + B_m \sinh r_2 x) \sin my, \tag{1.5}$$

[1] In Eqs. (1.1) and (2.1) x and y have been non-dimensionalised as $x = x\pi/a$, $y = y'\pi/a$ where x' and y' are the dimensional coordinate.

where

$$r_1=(m^2+k\Omega)^{1/2}, \quad r_2=(m^2-k\Omega)^{1/2}, \quad k=(\mu/D)^{1/2}. \tag{1.6}$$

Equation (1.5) gives the value of w only in the region $\left(0<x<\frac{\pi}{2}\right)\cup\left(0<y<\frac{\pi}{2}\right)$. In the rest of the plate w must be calculated by symmetry.

1.1. Cracks from outside

In this section we consider a cracked plate in which there are four symmetrical cracks along the central lines emanating from the midpoints of the edges. Let c be the length of each crack.

The other boundary conditions to be satisfied in the present case are

$$(1.7)\quad \begin{aligned}
&V_y\equiv D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2}+(2-\nu)\frac{\partial^2 w}{\partial x^2}\right]=0, \quad y=\frac{\pi}{2}, \quad 0\leqslant x\leqslant\frac{\pi}{2};\\
&V_x\equiv D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2}+(2-\nu)\frac{\partial^2 w}{\partial y^2}\right]=0, \quad x=\frac{\pi}{2}, \quad 0\leqslant y\leqslant\frac{\pi}{2};\\
&M_y\equiv D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)=0, \quad y=\frac{\pi}{2}, \quad 0\leqslant x\leqslant c;\\
&M_x\equiv D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)=0, \quad x=\frac{\pi}{2}, \quad 0<y\leqslant c;\\
&\frac{\partial w}{\partial y}=0, \quad y=\frac{\pi}{2}, \quad c<x\leqslant\frac{\pi}{2};\\
&\frac{\partial w}{\partial x}=0, \quad x=\frac{\pi}{2}, \quad c<y\leqslant\frac{\pi}{2}.
\end{aligned}$$

Using the boundary conditions $(1.7)_1$ or $(1.7)_{2,3}$ or $(1.7)_4$ and $(1.7)_5$ or $(1.7)_6$, we arrive at the following relations between A_m and B_m:

$$(1.8)\quad \begin{aligned}
&r_1(r_2^2-\nu m^2)A_m\cosh\frac{1}{2}r_1\pi+r_2(r-\nu m^2)B_m\cosh\frac{1}{2}r_2\pi=0,\\
&\sum_{m=1,3}^{\infty}\left(A_m r_1\cosh\frac{1}{2}r_1\pi+B_m r_2\cosh\frac{1}{2}r_2\pi\right)\sin mx=0, \quad c\leqslant x<\frac{\pi}{2};\\
&\sum_{m=1,3}^{\infty}\left\{\left[(r_1^2-\nu m^2)A_m\sinh\frac{1}{2}r_1\pi+(r_2^2-\nu m^2)B_m\sinh\frac{1}{2}r_2\pi\right]\sin mx+\right.\\
&\left.+[(\nu r_1^2-m^2)A_m\sinh r_1 x+(\nu r_2^2-m^2)B_m\sinh r_2 x]\sin\frac{m\pi}{2}\right\}=0, \quad 0\leqslant x<c.
\end{aligned}$$

From the relation $(1.8)_1$ we have

$$-\frac{r_1 A_m\cosh\frac{1}{2}r_1\pi}{r_1^2-\nu m^2}=\frac{r_2 B_m\cosh\frac{1}{2}r\pi}{r_2^2-\nu m^2}=\frac{E_m}{2k\Omega}. \tag{1.9}$$

Substituting for A_m and B_m from Eq. (1.9) in Eqs. $(1.8)_{2,3}$, we obtain the following pair of dual series equations:

$$\sum_{m=1,3}^{\infty} E_m \sin mx = 0, \qquad c < x \leqslant \frac{\pi}{2}, \tag{1.10}$$

$$\sum_{m=1,3}^{\infty} E_m [m(1+F_m)\sin mx + r_1 G_m \sinh r_1 x + r_2 H_m \sinh x_2 x] = 0, \qquad 0 \leqslant x < c,$$

where

$$F_m = \left[\frac{1}{r_1}(vm^2 - r_1^2)^2 \tanh\frac{r_1\pi}{2} - \frac{1}{r_2}(vm^2 - r_2^2)^2 \tanh\frac{r_2\pi}{2}\right] \Big/ \Delta_m - 1,$$

$$G_m = -\frac{1}{r_1^2}(vr_1^2 - m^2)(vm^2 - r_1^2)\,\mathrm{sech}\frac{r_1\pi}{2} \sin\frac{m\pi}{2} \Big/ \Delta_m, \tag{1.11}$$

$$H_m = \frac{1}{r_2^2}(vr_2^2 - m^2)(vm^2 - r_2^2)\,\mathrm{sech}\frac{r_2\pi}{2} \sin\frac{m\pi}{2} \Big/ \Delta_m,$$

$$\Delta_m = mk\Omega(3+v)(1-v).$$

In order to solve Eqs. (1.10), we assume

$$E_m = \int_0^c t\Phi(t) J_1(mt)\, dt, \tag{1.12}$$

where the function $\Phi(t)$ is to be determined.

With the choice of E_m given by Eq. (1.12) it can be seen in view of the identity

$$\sum_{m=1,3}^{\infty} J_1(mt) \sin mx = \frac{x}{2t}(t^2 - x^2)^{-1/2} H(t-x), \qquad x+t<\pi; \tag{1.13}$$

that Eq. $(1.10)_1$ is automatically satisfied.

Integrating Eq $(1.10)_2$ with respect to x, we obtain

$$\sum_{m=1,3}^{\infty} E_m [(1+F_m)\cos mx - G_m \cosh r_1 x - H_m \cosh r_2 x] = \alpha \qquad 0 \leqslant x < c, \tag{1.14}$$

where α is an appropriate constant.

Substituting for E_m from Eq. (1.12) in Eq. (1.13) and making use of the identity

$$\sum_{m=1,3}^{\infty} J_1(m_t) \cos mx = \frac{1}{2t} - \frac{2}{2t}(x^2 - t^2)^{-1/2} H(x-t) + \int_0^{\infty} \frac{I_1(ts)\cosh sx}{1+e^{\pi s}}\, ds, \qquad x+t<\pi \tag{1.15}$$

we can express Eq. (1.13) in the form of Abel's equation

$$\int_0^x \frac{\Phi(t)}{\sqrt{x^2 - t^2}} dt = h(x), \tag{1.16}$$

where

$$(1.17)\qquad h(x)=\frac{2}{x}\left\{\int_0^c t\Phi(t)\left[\frac{1}{2t}+\sum_{m=1,3}^{\infty}(F_m\cos mx-G_m\cosh r_1 x-H_m\cosh r_2 x)\times\right.\right.$$
$$\left.\left.\times J_1(mt)+\int_0^\infty\left[\frac{I_1(ts)\cosh sx}{1+e^{\pi s}}ds\right]dt-\alpha\right\}.$$

The solution for $\Phi(t)$ is

$$(1.18)\qquad \Phi(\eta)=\frac{2}{\pi}\frac{d}{d\eta}\int_0^\eta\frac{xh(x)}{\sqrt{\eta^2-x^2}}dx.$$

Evaluating the integrals and making use of some well-known identities (GRADESHTYN and RYZHYK [9]), we finally arrive at the following homogeneous integral equation:

$$(1.19)\qquad \theta(\rho)+\int_0^1 K(\rho,r)\,\theta(r)\,dr=0,\qquad 0\leqslant\rho<1,$$

where

$$(1.20)\qquad K(\rho,r)=2c^2 r\left\{\sum_{m=1,3}^{\infty}[mF_m J_1(m\rho c)+r_1 G_m I_1(r_1\rho c)+r_2 H_m I_1(r_2\rho c)]\times\right.$$
$$\left.\times J_1(mrc)-\int_0^\infty\frac{sI_1(crs)\,I_1(c\rho s)}{1+e^{\pi s}}ds\right\}$$

and

$$(1.21)\qquad \theta(\rho)=\Phi(\rho c).$$

Stress intensity factor. By substituting for A_m and B_m from Eq. (1.9) in Eq. $(1.7)_3$, it can be shown that the behaviour of the moment at the crack-tip ($x\to c+$) is given by

$$(1.22)\qquad M_y=-\frac{1}{4}D(3+\nu)(1-\nu)\,\theta(1)\,(x^2-c^2)^{-1/2},$$

which shows that near the crack-tip the moment varies inversely as the square root of the distance from the crack-tip which is consistent with the results derived by WILLIAMS [4] and STAHL and KEER [8].

One of the physical quantities of importance for workers in fracture mechanics is the stress intensity factor. In the problem of vibration, one can define the stress intensity factor κ as follows:

$$(1.23)\qquad \frac{\kappa}{\sqrt{s}}=\frac{(M)_{x\to c+}}{w_0 D},$$

where s is the non-diemensional distance of a point on the central line from the crack-tip and w_0 is the central deflection. With this definition we have

$$(1.24)\qquad \kappa = -\frac{1}{4}(3+\nu)(1-\nu)k\Omega \Big/ \sum_{m=1,3}^{\infty} b_m \left\{(2c^3)^{1/2} \int_0^1 r \frac{\theta(r)}{\theta(1)} J_1(mrc)\,dr\right\},$$

where

$$(1.25)\qquad b_m = \left[\frac{r_2^2-\nu m^2}{r_2}\tanh\frac{r_2\pi}{2} - \frac{r_1^2-\nu m^2}{r_1}\tanh\frac{r_1\pi}{2}\right]\sin\frac{m\pi}{2}.$$

1.2. Cracks from inside

We now consider a cracked plate in which there are four symmetrical cracks along the central lines starting from the centre of the plate.

Let

$$(1.26)\qquad c' = \frac{\pi}{2} - c.$$

For this case the boundary conditions $(1.7)_{1,2}$ still hold. The other boundary conditions, however, change to

$$(1.27)\qquad \begin{aligned} &M_y \equiv D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = 0, \quad y = \frac{\pi}{2}, \quad c' < x \leqslant \frac{\pi}{2};\\ &M_x \equiv D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0, \quad x = \frac{\pi}{2}, \quad c' < y \leqslant \frac{\pi}{2};\\ &\frac{\partial w}{\partial y} = 0, \quad y = \frac{\pi}{2}, \quad 0 \leqslant x < c';\\ &\frac{\partial w}{\partial x} = 0, \quad x = \frac{\pi}{2}, \quad 0 \leqslant y < c'. \end{aligned}$$

For the present case the dual series equations are

$$(1.28)\qquad \begin{aligned} &\sum_{m=1,3}^{\infty} E'_m \cos m\xi = 0, \quad c < \xi \leqslant \frac{\pi}{2};\\ &\sum_{m=1,3}^{\infty} E'_m \left[m(1+E_m)\cos m\xi + r_1 G'_m \sinh r_1\left(\frac{\pi}{2}-\xi\right) + \right.\\ &\qquad\qquad \left. + r_2 H'_m \sinh r_2\left(\frac{\pi}{2}-\xi\right)\right] = 0, \quad 0 \leqslant \xi < c, \end{aligned}$$

where

$$(1.29)\qquad E'_m = E_m \operatorname{cosec}\frac{1}{2}m\pi, \quad G' = G_m \operatorname{cosec}\frac{1}{2}m\pi, \quad H'_m = H_m \operatorname{cosec}\frac{1}{2}m\pi$$

Integrating Eq. $(1.28)_2$ with respect to ξ, we obtain

$$\sum_{m=1,3}^{\infty} E'_m \left\{(1+F_m)\sin m\xi + G'_m\left[\cosh\frac{1}{2}r_1\pi - \cosh r_1\left(\frac{1}{2}\pi-\xi\right)\right] + \right.$$
$$\left. + H'_m\left[\cosh\frac{1}{2}r_2\pi - \cosh r_2\left(\frac{1}{2}\pi-\xi\right)\right]\right\} = 0, \quad 0\leqslant\xi<c. \tag{1.30}$$

In order to solve Eqs. (1.28), we assume

$$E'_m = \int_0^c \Phi(t)\, J_0(mt)\, dt \tag{1.31}$$

and make use of the identities

$$\sum_{m=1,3}^{\infty} J_0(mt)\cos m\xi = \frac{1}{2}(t^2-\xi^2)^{-1/2} H(t-\xi), \quad \xi+t<\pi;$$
$$\sum_{m=1,3}^{\infty} J_0(mt)\sin m\xi = \frac{1}{2}(\xi^2-t^2)^{-1/2} H(\xi-t) + \int_0^{\infty}\frac{I_0(st)\sinh\xi s}{1+e^{\pi s}}\,ds, \tag{1.32}$$
$$\xi+t<\pi.$$

Proceeding as in the preceding section we arrive at the following integral equation:

$$\theta(\rho) + \int_0^1 K(\rho, r)\,\theta(r)\,dr = 0, \quad 0\leqslant\rho<1, \tag{1.33}$$

where

$$K(\rho, r) = 2c^2 r\left\{\Sigma\left[mF_m J_0(m\rho c) + r_1 G'_m\left(\sinh\frac{1}{2}r_1\pi I_0(r_1\rho c) - \right.\right.\right.$$
$$\left. -\cosh\frac{1}{2}r_1\pi L_0(r_1\rho c)\right) + r_2 H'_m\left(\sinh\frac{1}{2}r_2\pi I_0(r_2\rho c) + \right.$$
$$\left.\left.\left. -\cosh\frac{1}{2}r_2\pi L_0(r_2\rho c)\right)\right] + \int_0^{\infty}\frac{sI_0(crs)\,I_0(c\rho s)}{1+e^{\pi s}}\,ds\right\}, \tag{1.34}$$

$$\theta(\rho) = \Phi(\rho c)/(\rho c).$$

Stress intensity factor. It can again be demonstrated that the moment near the crack-tip varies inversely as the square root of the distance. The stress intensity factor as defined by Eq. (1.23) can be shown to be

$$\kappa = -\frac{1}{4}(3+\nu)(1-\nu)\,k\Omega \Big/ \sum_{m=1,3}^{\infty} b_m\left\{(2c^3)^{1/2}\int_0^1 \frac{r\theta(r)}{\theta(1)} J_0(mrc)\,dr\right\}, \tag{1.35}$$

where b_m is given by Eq. (1.25).

Numerical results and discussion. Equations (1.19) and (1.33) cannot be solved by a numerical procedure. By discretizing, these equations are reduced to a system of linear algebraic equations. The Gauss quadrature formula has been used, which, as a rule, is more accurate than the Simpson's quadrature formula. From these equations the first two frequencies have been computed for different crack-lengths and they are presented in Table 1

Table 1. Illustrating the variation of frequency ($= k\Omega$) of vibration of a square plate with the length of the crack

$\frac{2c}{\pi}$	Cracks from outside		Cracks from inside	
	first mode	second mode	first mode	second mode
0.0	2.0	18.0	2.0	18.0
0.1	1.99992	17.996	1.9771	17.871
0.2	1.99888	17.946	1.9129	17.603
0.3	1.99448	17.773	1.8205	17.421
0.4	1.98312	17.450	1.7172	17.379
0.5	1.96069	17.071	1.6164	17.369
0.6	1.92338	16.783	1.5281	17.277
0.7	1.86846	16.661	1.4570	17.023
0.8	1.79414	16.647	1.4055	16.690
0.9	1.69611	16.612	1.3578	16.395
1.0	1.51876	16.421	1.3649	16.279

The most interesting observation one can make from Table 1 is that for full crack-length the frequencies of vibration are different for the two cases, i.e. when the cracks are from outside and when the cracks are from inside. The frequency when the cracks start from inside turns out to be exactly the same as when the problem is treated as the vibrations of a square plate of length of size $1/2\pi$ with two adjacent sides simply supported and the other two free (WARBURTUN [10]). Whereas when the cracks start from outside, the frequency appears to be higher. The reason for this apparent discrepancy is as follows: when the cracks start from outside, then in the limiting case when they reach the centre, on account of symmetry, the condition $\partial w/\partial x=0$ must be satisfied at the centre, which is not applicable when the cracks start from the centre. This additional constraint, therefore, pushes up the fraquency. For the verification of this number, using the point matching technique under the constraint $\partial w/\partial x=0$ at the centre, frequency was computed for a square plate with two adjacent edges simply supported and other two free. The numbers turned out to be 1.6075, and 1.5695 for $n=10, 15, 20$, respectively, where n is the number of points matched on each side. Assuming the error to be of the order of $1/n$, we write

$$k\Omega \equiv f = f_0 + \frac{f_1}{n} + \frac{f_2}{n^2} + \frac{f_3}{n^3} + \dots \tag{1.36}$$

The true value of frequency f_0 is calculated by substituting the values of computed frequencies and it was found that it was 1.5177 which is in substantial agreement with that calculated in the other manner.

In Table 2 the stress intensity factors are given for the first mode of vibration which are calculated according to Eqs. (1.24) and (1.35). It is interesting to note that the stress intensity factor keeps on increasing with the crack-length when the cracks are from outside; however, when the cracks are from inside it has a maximum value when c is approximately 0.15π.

Table 2. Illustrating the variation of stress intensity factor for first mode of vibration with the crack-length

$\frac{2c}{\pi}$	Cracks from outside	Cracks from inside
0.1	0.02875	0.35569
0.2	0.07988	0.45794
0.3	0.14421	0.48426
0.4	0.21572	0.44740
0 5	0.28874	0.40421
0.6	0.35696	0.34978
0.7	0.41455	0.29337
0.8	0.45861	0.23507
0.9	0.49917	0.13953

2. Bending of a square plate

We consider a square plate simply supported at the edges $x=0$, $y=0$, $x=\pi$ and $y=\pi$, which is unstressed in its plane. A constant load q is applied in the z-direction which produces a displacement $w(x, y)$. The differential equation for w is

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}=P. \tag{2.1}$$

We choose w in the form

$$w=w_1+w_2, \tag{2.2}$$

where w_2 is given by (SZILARD [11])

$$w_2=\frac{4P}{\pi}\sum_{m=1,3}^{\infty}\frac{1}{m^5}\left\{1-\operatorname{sech}\frac{m\pi}{2}\left[\cosh m\left(\frac{\pi}{2}-y\right)+\frac{my}{2}\sinh m\left(\frac{\pi}{2}-y\right)-\right.\right.$$
$$\left.\left.-\frac{m\pi}{4}\operatorname{sech}^2\frac{m\pi}{2}\sinh my\right]\right\}\sin mx \tag{2.3}$$

and w_1 is to be determined with the help of appropriate boundary conditions. It may be remarked here that w_2 satisfies the following boundary conditions at the edges:

$$\begin{aligned} &w=0, \quad \frac{\partial^2 w}{\partial x^2}=0, \quad x=0,\ x=\pi, \quad 0\leqslant y\leqslant\pi;\\ &w=0, \quad \frac{\partial^2 w}{\partial y^2}=0, \quad y=0,\ y=\pi, \quad 0\leqslant x\leqslant\pi. \end{aligned} \tag{2.4}$$

We now assume that in the region $0 \leqslant x \leqslant \pi/2$, $0 \leqslant y \leqslant \pi/2$, w_1 is given by

$$(2.5) \qquad w_1 = \sum_{m=1,3}^{\infty} (A_m \sinh my + B_m y \cos hmy) \sin mx + \\ + \sum_{m=1,3}^{\infty} (A_m \sin hmx + B_m x \cos hmx) \sin my .$$

In the rest of the plate, w must be obtained by symmetry considerations.

2.1. *Cracks from outside*

In this section we consider the bending of a cracked plate in which there are four symmetrical cracks along the central line starting from the edges. The length of each crack is c.

The additional boundary conditions to be satisfied are Eqs. (1.7).

The boundary condition $(1.7)_1$ or $(1.7)_2$ leads to the following relation between A_m and B_m:

$$(2.6) \qquad \frac{m}{1+\nu}\left(A_m \cosh \frac{1}{2} m\pi + \frac{1}{2}\pi B_m \sinh \frac{1}{2} m\pi\right) = \frac{B_m}{1-\nu} \cosh \frac{1}{2} m\pi = E_m (\text{say}) \ldots .$$

Making use of the other two pairs of boundary conditions and substituting for A_m and B_m, we obtain the pair of dual series equations

$$\Sigma E_m \sin mx = 0, \qquad c < x \leqslant \frac{\pi}{2};$$

$$(2.7) \qquad \Sigma m E_m \left\{(1+F_m) \sin mx + G_m \frac{d}{dx}(x \sinh mx) + H_m \sinh mx\right\} = \Sigma R_m \sin mx, \\ 0 \leqslant x < c,$$

where

$$F_m = \tanh \beta + \gamma\beta \operatorname{sech}^2 \beta - 1,$$

$$G_m = -\gamma \sin \beta \operatorname{sech} \beta,$$

$$H_m = \gamma\beta \tanh \beta \operatorname{sech} \beta \sin \beta,$$

$$(2.8) \qquad R_m = \frac{4P}{\pi m^3 (3+\nu)(1-\nu)} [2\nu(1-\operatorname{sech}\beta) + (1-\nu)\beta \tanh \beta \operatorname{sech} \beta],$$

$$\beta = \frac{1}{2} m\pi .$$

Integrating Eq. $(2.7)_2$ with respect to x, we obtain

$$(2.9) \qquad \Sigma E_m \{(1+F_m) \cos mx - mG_m x \sinh mx - H_m \cosh mx\} = \\ = \Sigma \frac{R_m}{m} \cos mx + \alpha, \qquad 0 \leqslant x < c,$$

where α is a suitable constant.

Consistent with the desired nature of moment singularity, we choose

$$E_m = \int_0^c t\Phi(t) J_1(mt)\, dt. \tag{2.10}$$

Substituting for E_m in (2.9) and proceeding in a manner similar to that in Sect. 1.1, we arrive at the following integral equation:

$$\theta(\rho) + \int_0^1 K(\rho, r)\, \theta(r)\, dr = h(\rho), \quad 0 \leqslant \rho < 1, \tag{2.11}$$

where

$$K(\rho, r) = 2c^2 r \left\{ \Sigma \left[F_m J_1(m\rho c) + mcG_m I_0(m\rho c) + H_m I_1(m\rho c) \right] J_1(mrc) + \right. \\ \left. - \int_0^\infty \frac{sI_1(s\rho c)\, I_1(src)}{1+e^{\pi s}}\, ds \right\}, \quad h(\rho) = 2\Sigma R_m J_1(m\rho c), \\ \theta(\rho) = \Phi(\rho c). \tag{2.12}$$

Physical quantities. The strain energy U is the work done by the applied load through the plate displacement and is given by

$$U = \frac{1}{2} \int_A qw\, dA. \tag{2.13}$$

The moment at the crack-tip has a singularity varying inversely as the square root of the distance. The stress intensity factor κ is defined by the relation

$$\frac{M_y}{qa^4} \quad \frac{\kappa}{\sqrt{s}}, \tag{2.14}$$

Substituting for M_y we obtain the expression for κ as

$$\kappa = \frac{1}{4\pi^4 P} (3+\nu)(1-\nu)(c/2)^{1/2}\, \theta(1). \tag{2.15}$$

2.2. Cracks from inside

We now consider the bending of a cracked plate in which there are four symmetrical cracks along the central lines starting from the centre. The length of each crack is c.

The appropriate boundary conditions are Eqs. (1.27). With the help of these boundary conditions we obtain the dual series equations

$$\sum_{m=1,3}^{\infty} E'_m \cos m\xi = 0, \quad 0 < \xi \leqslant \frac{\pi}{2},$$

$$\sum_{m=1,3}^{\infty} m E'_m \left\{ (1+F_m) \cos m\xi + G'_m \frac{d}{d\xi} \left[\left(\frac{1}{2}\pi - \xi \right) \sinh m \left(\frac{1}{2}\pi - \xi \right) \right] + \right. \\ \left. - H'_m \sinh m \left(\frac{1}{2}\pi - \xi \right) \right\} = \Sigma R'_m \cos m\xi, \quad 0 \leqslant \xi < c, \tag{2.16}$$

where

$$E'_m = E_m \operatorname{cosec} \frac{1}{2} m\pi,$$

(2.17)

$$G'_m = G_m \operatorname{cosec} \frac{1}{2} m\pi, \quad H'_m = H_m \operatorname{cosec} \frac{1}{2} m\pi, \quad R'_m = R_m \operatorname{cosec} \frac{1}{2} m\pi.$$

Integrating Eq. $(2.16)_2$ from 0 to ξ, we obtain

(2.18)
$$\sum_{m=1,3}^{\infty} E'_m \Big\{ (1+F_m) \sin m\xi - mG'_m \Big[\frac{1}{2} \pi \sinh \frac{1}{2} m\pi - \left(\frac{1}{2}\pi - \xi\right) \sinh m\left(\frac{1}{2}\pi - \xi\right)\Big] - H'_m \Big[\cosh \frac{1}{2} m\pi - \cosh m\left(\frac{1}{2}\pi - \xi\right)\Big]\Big\} = \Sigma \frac{R'_m}{m} \sin m\xi, \quad 0 \leqslant \xi < c.$$

As in case II, we choose

(2.19)
$$E'_m = \int_0^c \Phi(t) J_0(mt)\, dt,$$

which gives the right type of singularity at the crack-tip.

Substituting for E_m in Eq. (2.18) and proceeding along standard lines, we obtain the following integral equations:

(2.20)
$$\theta(\rho) + \int_0^1 K(\rho, r)\, \theta(r)\, dr = h(\rho), \quad 0 \leqslant \rho < 1,$$

where

(2.21)
$$K(\rho, r) = 2c^2 r \Big(\Sigma m \{F_m J_0(m\rho c) + G'_m [\beta(\cosh\beta I_0(m\rho c) - \sinh\beta L_0(m\rho c)) + \sinh\beta(I_0(m\rho c) + m\rho c\, I_1(m\rho c)) - \cosh\beta(L_0(m\rho c) + m\rho c\,(2/\pi + L_1(m\rho c)))] + H'_m [\sinh\beta\, I_0(m\rho c) - \cosh\beta\, L_0(m\rho c)]\} J_0(mrc) + \int_0^\infty \frac{s I_0(s\rho c)\, I_0(src)}{1+e^{\pi s}}\, ds \Big),$$

$$h(\rho) = 2\Sigma R_m J_0(m\rho c), \quad \theta(\rho) = \Phi(\rho c)/(\rho c).$$

Physical quantities. The strain energy and the stress intensity factor—two quantities of importance—are given by Eqs. (2.13) and (2.11), respectively.

Numerical results and discussion. It is not possible to solve Eqs. (2.13) and (2.20) by theoretical methods. Therefore, $\theta(\rho)$ has been computed numerically by discretizing the integrals using the Gauss quadrature formula. E_m and w were then

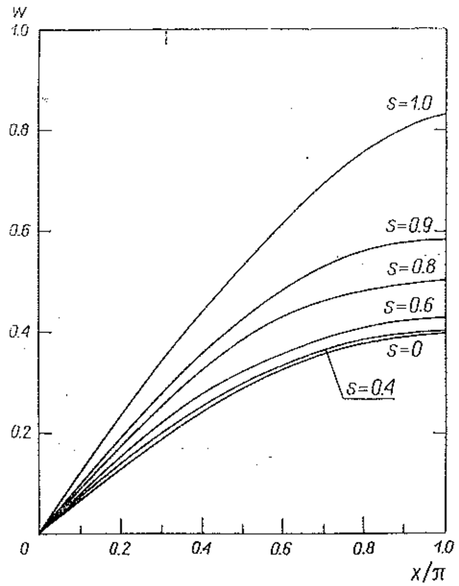

FIG. 3. Variation of ω with x/π along the line $y=\pi/2$ with $s(=2c/\pi)$ when the cracks are from the edges.

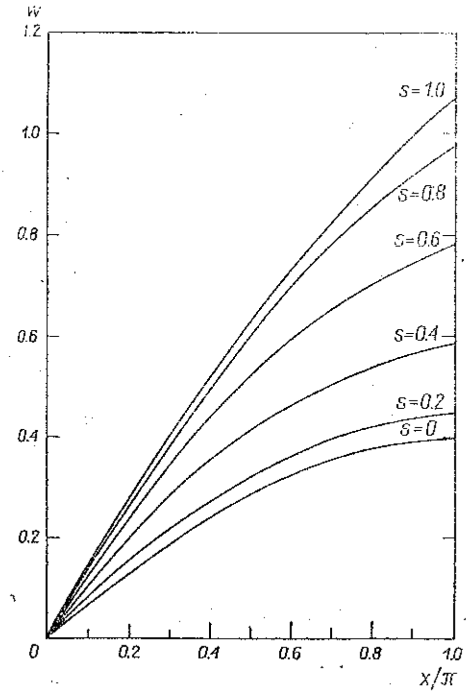

FIG. 4. Variation of ω with x/π along the line $y=\pi/2$ with $s(=2c/\pi)$ when the craks are from the centre.

Table 3. Illustrating the variation of strain energy and the central deflection with the crack-length when $P = 1$

$\frac{2c}{\pi}$	0.0	0.2	0.4	0.6	0.8	1.0
ω°_1	0.39571	0.39612	0.40208	0.42759	0.50207	0.83373
ω°_2	0.39571	0.44578	0.58500	0.78188	0.97804	1.08715
$2U_1/q$	0.32138	0.32187	0.32832	0.35233	0.40997	0.59219
$2U_2/q$	0.32138	0.35728	0.45120	0.56933	0.66815	0.70695

Table 4. Illustrating the variation of stress intensity factor for $P = 1$ with the crack-length

$\frac{2c}{\pi}$	Cracks from outside	Cracks from inside
0.1	0.000126	0.001502
0.2	0.000343	0.002065
0.3	0.000605	0.002422
0.4	0.000894	0.002635
0.5	0.001201	0.002719
0.6	0.001522	0.002675
0.7	0.001865	0.002488
0.8	0.002260	0.002139
0.9	0.002846	0.001560

evaluated. In Figs. (3) and (4), w has been exhibited against x for $y=\pi/2$ for different values of the crack-lengths. It can be again noted that there is a discrepancy in the value of w for full crack in the two cases, namely, when cracks start outside and when cracks start from inside. In the former case the value of w is less on account of the restraint $\partial w/\partial x=0$ at the centre.

In Table 3 strain energy and central deflection are shown against the crack length for both cases.

Finally, in Table 4 the stress intensity factors are given for different values of the crack-length for both cases. It may be again noted that the stress intensity factor keeps on increasing with the crack-length when the cracks emanate from the edges, however, it has a maximum value at about half the crack-length when the cracks start from inside.

Acknowledgements

Financial support of the Natural Sciences and Engineering Rsearch Council of Canada is gratefully acknowledged.

References

1. S. Timoshenko and S. Woinowsky-Krieger, *Theory of plates and shells*, McGraw Hill, 1959
2. W. H. Yang, *On an integral equation solution for a plate with internal support*, Quart. J. Mech Appl. Math., **21**, 503, 1968.
3. I. N. Sneddon, *Nixed boundary value problems in potential theory*, North Holland, 1966.
4. M. L. Williams, *The bending stress distribution at the base of a stationary crack*, J. Appl. Mech., **28**, 78, 1961.
5. P. P. Lynn and N. Kumbasar, *Free vibrations of thin rectangular plates having narrow cracks with simply supported edges*, Developments in Mechanics, **4**, 911–928, 1967.
6. A. W. Leissa, W. E. Clausen, L. E. Hulbert and A. T. Hopper, *A comparison of approximate methods for the solution of plate bending problems*, AIAA Int., **7**, 920–928, 1967.
7. L. M. Kerr and C. Sve, *On the bending of cracked plates*, Int. J. Solids Struct., **6**, 1545, 1970.
8. B. Stahl and L. M. Keer, *Vibration and stability of cracked rectangular plates*, Int. J. Solids Struct., **8**, 69, 1972.
9. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, 4th edition, 427, Academic Press, 1965.
10. G. B. Warburton, *The vibration of rectangular plates*, Proc. Inst. Mech. Engng., **168**, 371, London 1954.
11. R. Szilard, *Theory and analysis of plates*, p. 70, Prentice Hall Inc., 1974.

Streszczenie

DRGANIA I ZGINANIE PŁYTY ZE SZCZELINAMI

Rozważono zginanie i drgania płyty kwadratowej zawierającej cztery szczeliny wychodzące ze środka płyty lub ze środków jej krawędzi. Wyprowadzono dualne równania szeregowe wynikające z nieciągłych warunków brzegowych spełnionych wzdłuż linii, na których leżą szczeliny. Wyodrębnienie z rozwiązań członów osobliwych pozwala sprawdzić zagadnienie do równania całko wego Fredholma, które rozwiązuje się numerycznie. W przypadku płyty drgającej wyznaczono częstości odpowiadające dwóm pierwszym postaciom drgań własnych, a w przypadku zginania obliczono przemieszczenia oraz energię odkształcenia płyty. W obu przypadkach wyznaczono również współczynniki intensywności naprężenia.

Резюме

КОЛЕБАНИЯ И ИЗГИБ ПЛИТЫ СО ЩЕЛЯМИ

Рассмотрены изгиб и колебания квадратной плиты, содержавщей четыре щели выходящие из центра или из середины ее граней. Выведены дуальные последовательные уравнения, вытекающие из разрывных граничных условий удовлетворенных вдоль линий, на которых лежит щели. Выделение из решений особых членов позволяет свести проблему к интегральному уравнению Фредгольма, которое решается численным образом. В случае колеблющейся плиты определены частоты отвечающие двум первым видам собственных колебаний, а в случае изгиба вычислены перемещения и энергия деформации плиты. В обоих случаях определены тоже коэффициенты напряжения.

DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF CALGARY

Received October 25, 1979