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LYAPUNOV FUNCTIONAL FOR LATERAL BUCKLING OF I BEAMS SUBJECTED TO PURE BENDING

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Lateral buckling of I beams subjected to pure bending is investigated by the direct method of Laypunov. A metric space and an energy type Lyapunov functional are proposed for the solution of the problem. These readily yield familiar expressions for the buckling loads of simply supported and fixed ended beams.

NOTATION

E Young's modulus of elasticity,
G shear modulus,
I_x, I_y second moments of area of cross section about x and y axes,
J torsion constant of cross section,
M_x bending moment about x axis,
l span of beam,
T, t time,
u lateral displacement in the x direction,
V Lyapunov functional,
v vertical displacement in the y direction,
x, y principal axes of cross section of beam,
z longitudinal centroidal axis of beam,
β rotation about z axis
Γ warping constant of cross section,
ξ, φ state variables, i.e. displacement, velocity, stress component or temperature,
ρ metric.

1. INTRODUCTION

For several decades engineers and mathematicians have successfully employed the energy criterion to investigate stability characteristics of both discrete and continuous systems. As an alternative to this approach, the direct method of Lyapunov constitutes a potentially useful tool to obtain stability information from even more complex problems. The main advantage of Lyapunov's method lies in the fact that it avoids explicit solutions of the governing differential equations. Sufficient conditions for stability or instability of a system are obtained by investigating the behaviour of certain constructed functions known as Lyapunov functions for discrete systems and Lyapunov functionals for continuous systems.

The approach has, hovever, a serious weakness; there is no known universal method for constructing the Lyapunov functions or functionals. It has been observed that the stability information obtained by a particular investigator confronted with a specific problem depends to a large extent on his sharpness and ingenuity. For this reason the method may be considered as an art.

Whilst some innovative techniques for the construction of Lyapunov functions are known to have been proposed, still fewer such methods exist for continuous dynamical systems. In this case the problem is more complicated since a state which may be stable with respect to one metric may be unstable with respect to another metric.

In this paper the stability of lateral buckling of I beams subjected to pure bending is investigated by means of Lypunov's direct method. It is considered pertinent to state the essential factors of Lyapunov's theorem before applying it to the problem of interest.

The functional used is fairly simple and has a direct relationship to the energy functionals. In spite of the simple nature of the functional, it yields valuable information on the stability characteristics of the problem.

2. Definition and theorem of Lyapunov's functional

This represents, with minor modifications, a brief discussion of the fundamental concepts of Lyapunov's method for the continuous system following the exposition of Zubov [1].

It is useful to begin by introducing some notation. The physical system under consideration will be specified by a set Φ such that $\xi_1 \in \Phi$. Also assume that the time $t \in T$, a finite or infinite time interval and consequently the state of the dynamical system is specified by (ξ_i, t) $i=1, 2, ..., n$. The variables ξ_i represent displacement or velocity components, stress components, temperature etc. It will also be necessary to assume that Φ is a metric space with a metric $\rho(\xi_1, \xi_2)$ satisfying the well-known properties:

1. $\rho(\xi_1, \xi_2) \geqslant 0$,
2. $\rho(\xi_1, \xi_2) = 0$ if and only if $\xi_1 = \xi_2$,
3. $\rho(\xi_1, \xi_2) = \rho(\xi_2, \xi_1)$,
4. $\rho(\xi_1, \xi_2) + \rho(\xi_2, \xi_3) > \rho(\xi_1, \xi_3)$.

With this the ground is set for stating a stability theorem due to Zubov [1] which is a generalization of Lyapunov's theorem for discrete systems. Zubov's theorem [2]

For the solution $\xi = 0$ of the boundary value problem to be stable with respect to ρ, it is necessary and sufficient that in a sufficiently small neighbourhood $S(0, d)$ of $\xi = 0$ there exists a functional V having the following properties when $\xi \in S(0, d)$:

1. V is positive definite with respect to ρ.
2. V admits an infinitely small upper bound with respect to ρ.
3. $V(\xi(t, \xi^0))$ is non-increasing for $t \geqslant 0$ whenever $\xi^0 \in S(0, d)$.

If in addition there exists a d^1, $0 \leqslant d^1 \leqslant d$, such that

4. $V(\xi(t, \xi^0)) \to 0$ as $t \to +\infty$, whenever $\xi \in S(0, d^1)$,

then $\xi = 0$ is asymptotically stable with respect to ρ.

Satisfaction of the conditions stipulated in the theorem can be shown through the following three steps:

1. $V(\xi) \geqslant \alpha\rho^2(\xi, 0)$ where α is a positive constant.
2. $|V(\xi)| \leqslant \gamma\rho^2\ \xi, 0)$ where γ is a positive constant.
3. $\dfrac{dV(\xi(t, \xi^0))}{dt} \leqslant 0$ for $t \geqslant 0$.

3. Governing equations

An I beam, subject to a uniform bending moment, is shown in Fig. 1. The x and y coordinates are taken along the principal axes of the cross section and the z axis coincides with the longitudinal centroidal axis of the member. Deformation of the member at any section can be broken down into three distinct components, a lateral displacement u in the x direction, a vertical displacement v in the y direction and a rotation β about the z axis.

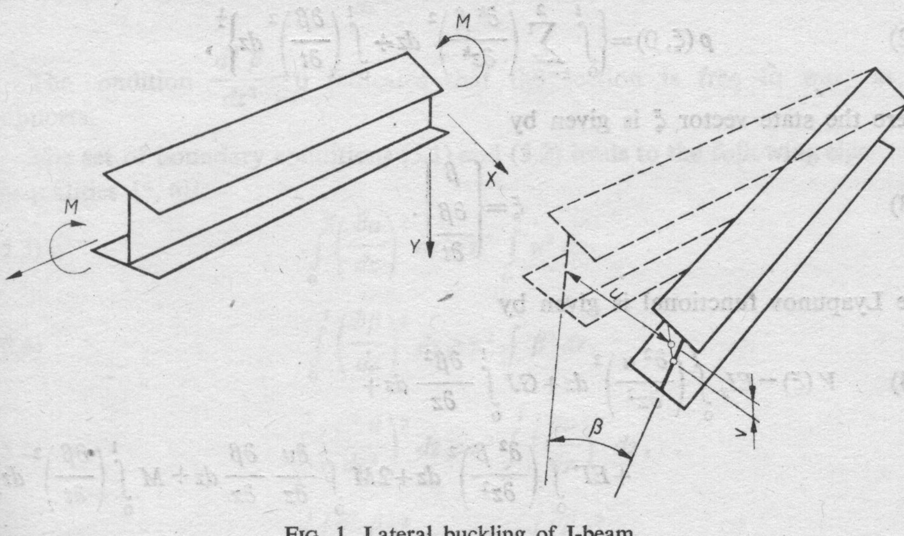

FIG. 1. Lateral buckling of I-beam.

The governing differential equation of the problem are [3, 4]

$$EI_x \frac{d^2 v}{dz^2} + M_x = 0, \tag{3.1}$$

$$EI_y \frac{d^2 u}{dz^2} + M_x \beta = 0, \tag{3.2}$$

$$GJ \frac{d\beta}{dz} - E\Gamma \frac{d^3 \beta}{dz^3} - M_x \frac{du}{dz} = 0. \tag{3.3}$$

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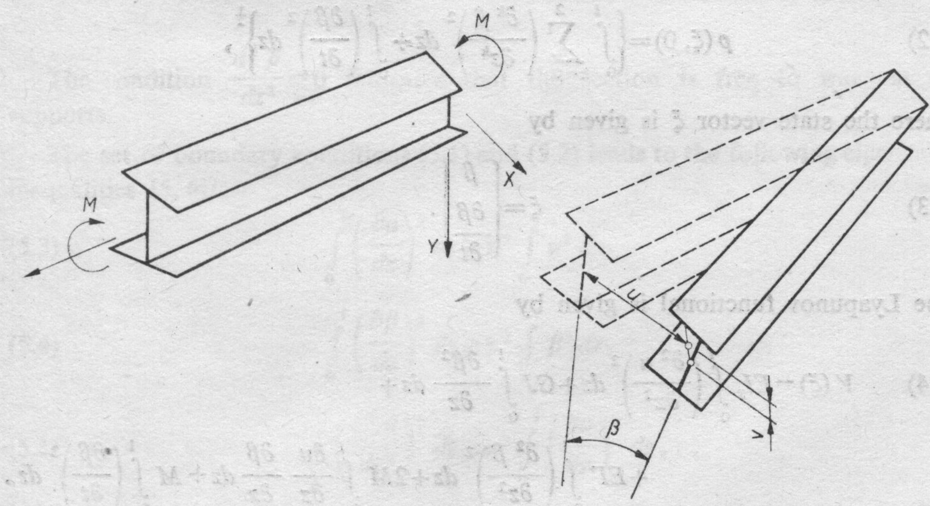

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The variable v which occurs in the first equation alone describe bending in the vertical plane and can therefore be solved independently of the other two. The second and third equations describe lateral bending and twisting and have to be solved simultaneously because they are coupled. In the third equation, the first term represents the resistance of the section to twist whilst the second term represents the resistance to warping.

4. Lyapunov functional

Since the use of metric space is indispensable in the Lyapunov stability approach for continuous systems, it is necessary to choose a metric $\rho(\xi_1, \xi_2)$ for measuring „distances" between any two states ξ_1 and ξ_2 in the form

$$\rho(\xi_1, \xi_2)=\left\{\int_0^l \sum_{k=0}^{2}\left[\frac{\partial^k \beta_1}{\partial z^k}-\frac{\partial \beta_2}{\partial z^k}\right]^2 dz+\int_0^l\left|\frac{\partial \beta_1}{\partial t}-\frac{\partial \beta_2}{\partial t}\right| dz\right\}^{\frac{1}{2}}. \tag{4.1}$$

Therefore, for the investigation of the equilibrium state $\xi=0$ and the deformed state $\xi=\xi$, this metric reduces to

$$\rho(\xi, 0)=\left\{\int_0^l \sum_{k=0}^{2}\left(\frac{\partial^k \beta}{\partial z^k}\right)^2 dz+\int_0^l\left(\frac{\partial \beta}{\partial t}\right)^2 dz\right\}^{\frac{1}{2}}, \tag{4.2}$$

where the state vector ξ is given by

$$\xi=\left\{\begin{matrix}\beta\\ \dfrac{\partial \beta}{\partial t}\end{matrix}\right\}. \tag{4.3}$$

The Lyapunov functional is given by

$$V(\xi)-EI_y\int_0^l\left(\frac{\partial^2 u}{\partial z^2}\right)^2 dz+GJ\int_0^l\frac{\partial \beta^2}{\partial z}\,dz+ +E\Gamma\int_0^l\left(\frac{\partial^2 \beta}{\partial z^2}\right)^2 dz+2M\int_0^l\frac{\partial u}{\partial z}\,\frac{\partial \beta}{\partial z}\,dz+M\int_0^l\left(\frac{\partial \beta}{\partial t}\right)^2 dz. \tag{4.4}$$

Equation (4.4) represents twice the total kinetic energy and the total potential energy minus four times the potential energy of the external load M.

By substituting for Eqs. (3.2) and (3.3) and considering the warping stiffness to be negligible, Eq. (4.4) reduces to

$$V(\xi)=-\frac{M^2}{EI_y}\int_0^l \beta^2\,dz+3GJ\int_0^l\left(\frac{\partial \beta}{\partial z}\right)^2 dz+ +E\Gamma\int_0^l\left(\frac{\partial^2 \beta}{\partial z^2}\right)^2 dz+M\int_0^l\left(\frac{\partial \beta}{\partial t}\right)^2 dz. \tag{4.5}$$

The chosen Lyapunov functional must also admit an infinitely small upper bound in the neighbourhood of the equilibrium state $\xi \equiv 0$. In qualitative terms, this implies

$$V(\xi) \leqslant \gamma \rho(\xi, 0), \tag{4.6}$$

where γ is a positive constant. By simple comprison of Eqs. (4.2) and (4.5), the inequality (4.6) is satisfied if

$$\gamma = \max\left\{1, \frac{M^2}{EI_y}\right\}. \tag{4.7}$$

5. Simple support

The flexural and torsional boundary conditions corresponding to simple supports are

$$u = \frac{d^2 u}{dz^2} = 0 \quad \text{at} \quad z = 0, l, \tag{5.1}$$

$$\beta = \frac{d^2 \beta}{dz^2} = 0 \quad \text{at} \quad z = 0, l. \tag{5.2}$$

The condition $\frac{d^2 \beta}{dz^2} = 0$ indicates that the section is free to warp at the supports.

The set of boundary conditions (5.1) and (5.2) leads to the following eigenvalue inequalities [5, 6]:

$$\int_0^l \left(\frac{\partial u}{dz}\right)^2 dz \geqslant \pi^2 \int_0^l u^2\, dz, \tag{5.3}$$

$$\int_0^l \left(\frac{\partial \beta}{dz}\right)^2 dz \geqslant \pi^2 \int_0^l \beta^2\, dz, \tag{5.4}$$

$$\int_0^l \left(\frac{\partial^2 u}{dz^2}\right)^2 dz \geqslant \pi^2 \int_0^l \left(\frac{\partial u}{\partial z}\right)^2 dz, \tag{5.5}$$

$$\int_0^l \left(\frac{\partial^2 \beta}{\partial z^2}\right)^2 dz \geqslant \pi^2 \int_0^l \left(\frac{\partial \beta}{\partial z}\right)^2 dz. \tag{5.6}$$

With the aid of the inequalities (5.4) and (5.6), Eq. (4.5) can be manipulated into the following form:

$$V(\xi) \geqslant \left[-\frac{M^2}{EI_y} + GJc\pi^2 + E\Gamma c^2 \pi^4\right] \int_0^l \beta^2\, dz + \\ + GJ(3-c) \int_0^l \left(\frac{\partial \beta}{\partial z}\right)^2 dz + E\Gamma(1-c^2) \int_0^l \left(\frac{\partial^2 \beta}{\partial z^2}\right)^2 dz + M \int_0^l \left(\frac{\partial \beta}{\partial t}\right)^2 dz. \tag{5.7}$$

Choosing $c=1/l^2$ the inequality (5.7) can be written as

(5.8) $$V(\xi)\geqslant \alpha\rho^2(\xi, 0),$$

where α is a positive constant given by

(5.9) $$\alpha=-\frac{M^2}{EI_y}+\frac{GJ\pi^2}{l^2}+\frac{E\Gamma\pi^4}{l^4}>0.$$

Equation (5.9) yields the familiar expression for the lateral buckling load of a simply supported beam under uniform bending as

(5.10) $$M_{cr}=\frac{\pi}{l}\sqrt{EI_y(GJ+E\Gamma\pi^2/l^2)}\,.$$

6. Fixe end

In the case of a member whose ends are free to rotate about the horizontal axis but fully restrained against all other displacements, the boundary conditions are as follows:

(6.1) $$v=\frac{d^2v}{dz^2}=0 \quad \text{at} \quad z=0, l,$$

(6.2) $$u=\frac{\partial u}{\partial z}=0 \quad \text{at} \quad z=0, l,$$

(6.3) $$\beta=\frac{\partial \beta}{\partial z}=0 \quad \text{at} \quad z=0, l,$$

The set of boundary conditions (6.2) and (6.3) lead to the following eigenvalue inequalities [5, 6]:

(6.4) $$\int_0^l\left(\frac{\partial u}{\partial z}\right)^2 dz\geqslant \pi^2\int_0^l u^2\,dz,$$

(6.5) $$\int_0^l\left(\frac{\partial \beta}{\partial z}\right)^2 dz\geqslant \pi^2\int_0^l \beta^2\,dz,$$

(6.6) $$\int_0^l\left(\frac{\partial^2 u}{\partial z^2}\right)^2 dz\geqslant 4\pi^2\int_0^l\left(\frac{\partial u}{\partial z}\right)^2 dz,$$

(6.7) $$\int_0^l\left(\frac{\partial^2 \beta}{\partial z^2}\right)^2 dz\geqslant 4\pi^2\int_0^l\left(\frac{\partial \beta}{\partial z}\right)^2 dz.$$

With the aid of the inequalities (6.5) and (6.7), Eq. (4.5) can be easily manipulated into the following form:

$$(6.8)\qquad V(\xi)\geqslant\left[GJc\pi^2+E\Gamma c^2\pi^4-\frac{M}{EI_y}\right]\int\limits_0^l \beta^2\,dz+$$

$$+GJ(3-c)\int\limits_0^l\left(\frac{\partial\beta}{\partial z}\right)^2 dz+E\Gamma(1-c^2/4)\int\limits_0^l\left(\frac{\partial^2\beta}{\partial z^2}\right)^2 dz+M\int\limits_0^l\left(\frac{\partial\beta}{\partial t}\right)^2 dz.$$

By choosing $c=4/l^2$ and using a method similar to the one employed for the simple supports, the positivity of $V(\xi)$ with respect to $\rho(\xi, 0)$ leads to the familiar expression for the critical moment as

$$(6.9)\qquad M_{cr}=2\pi/l\sqrt{EI_y\,(GJ+4E\Gamma\pi^2/l^2)}.$$

7. Conclusion

A simple Lyapunov functional has been proposed for the lateral buckling of *I* beams subjected to uniform bending. The functional has effectively produced familiar expressions for the critical buckling moments for simple supports and fixed ends. As it is well known, these two boundary conditions represent a set of limits which most practical restraints imply.

The solution presented has demonstrated once more that whilst no universally accepted suggestions for constructing Lyapunov functionals have yet been proposed, there is evidence that they bear some relations to the energy functionals.

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Streszczenie

FUNCJONAŁ LAPUNOWA DLA WYBOCZENIA POPRZECZNEGO BELEK DWUTEOWYCH PODDANYCH CZYSTEMU ZGINANIU.

Problem wyboczenia poprzecznego belek dwuteowych poddanych czystemu zginaniu zbadano bezpośrednią metodą Lapunova. Dla rozwiązania zagadnienia zaproponowano odpowiednią przestrzeń metryczną i energetyczny funkcjonal Lapunowa. W ten sposób otrzymuje się bezpośrednio znane wzory dla obciążeń krytycznych belek o końcach swobodnie podpartych lub utwierdzonych.

Резюме

ФУНКЦИОНАЛ ЛЯПУНОВА ДЛЯ ПОПЕРЕЧНОГО ИЗГИБА ДВУТАВРОВЫХ БАЛОК ПОДВЕРГНУТЫХ ЧИСТОМУ ИЗГИБУ

Проблема поперечного изгиба двутавровых балок, подвергнутых чистому изгибу, исследована непосредственным методом Ляпунова. Для решения проблемы предложены соответствующее метрическое пространство и энергетический функционал Ляпунова. Таким образом получаются непосредственно известные формулы для критических нагрузок балок со свободно подпертыми или закрепленными концами.

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