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ELECTROMECHANICAL ANALOGIES FOR THE THEORY OF CONSOLIDATION

R. UKLEJEWSKI (POZNAŃ) and M. KRAKOWSKI (ŁÓDŹ)

The electromechanical analogies between one-dimensional systems with distributed parameters are examined, applying Firestone's system of electromechanical analogies, by means of dimensional analysis. The existence of such analogies for the theore of consolidation is provede: it constitutes the basis to the application of some methods and concepts from electrotechnics in this theory. As an example of such application, an algorithm of the method of determination of elastic coefficients of a porous medium is presented.

1. Introduction

The existence of analogies, i.e. similarities of mathematical models of various physical phenomena, makes possible to transfer methods, concepts and completed formulas from one research branch to another. In this way electromechanical analogies contribute to the development of the general theory of vibration.

This paper deals with the electromechanical analogies for the M. A. Biot theory of consolidation [1, 6].

2. Electromechanical analogies for one-dimensional systems with distributed parameters

2.1. *Definitions of system of electromechanical analogies*

The concept of dimensional space Π over the reals was introduced by S. Drobot in [8]. The dimensions of all physical quantities are the elements of space Π. It is proved in [8] that each dimensional space Π is isomorphic with a certain vector space. For this reason the dimensions can be considered as vectors. The other concepts concerning space Π such as the system of fundamental dimensions. the dimensional independence of elements and the dimensional combination of the dimensions correspond respectively to the following concepts from the vector space theory, namely: the basis of the vector space, the linear independence of vectors and the linear combination of vectors [8].

Let Π_m be the set of dimensions of all the mechanic quantities, whereas Π_e—the set of all the electric quantities in the dimensional space Π. It is easy to show that the sets Π_m and Π_e are the subspaces of the dimensional space Π.

DEFINITION. *Each isomorphism of the dimensional space of the mechanic quantities Π_m in the dimensional space of the electric quantities Π_e will be called the system of electromechanical analogies.*

Likewise as in the vector space theory, the following theorem holds true [11]:

THEOREM. *The dimensional spaces Π_m and Π_e are isomorphic if and only if their systems of fundamental dimensions contain the same (finite) number of elements.*

2.2. Firestone's systems of electromechanical analogies for onedimensional systems with distributed parameters

In the case of systems with lumped parameters we have the following systems of fundamental dimensions:

in the dimensional space Π_m:

$$[m] \equiv [\text{mass}], \quad [l] \equiv [\text{displacement}], \quad [t] \equiv [\text{time}],$$

in space Π_e:

$$[u] \equiv [\text{voltage}], \quad [i] \equiv [\text{current}], \; [t] \equiv [\text{time}],$$

where the dimensions are denoted by the square brackets. By virtue of the Theorem, the dimensional spaces Π_m and Π_e are in this case isomorphic.

In the case of one-dimensional systems with distributed parameters, the following four dimensions constitute the system of fundamental dimensions in space Π_e (transmission lines):

$$[u], \quad [i], \quad [t], \quad [\mathscr{L}] \equiv [\text{distance}].$$

The dimensional space Π_m will be in this case isomorphic with the dimensional space Π_e if, for example, the dimensions [displacement] and [distance] will be considered conventionally to be independent in the space Π_m [16], [10]. Thus we assume the following system of fundamental dimensions in the dimensional space Π_m:

$$[m], \quad [l], \quad [t], \quad [\mathscr{L}] \equiv [\text{distance}].$$

Firestone's system of analogy [17] for one-dimensional systems with distributed parameters is obtained by the imposition of the following four correspondences:

$$\begin{aligned} &[\text{time}]_m \leftrightarrow [\text{time}]_e, \\ &[\text{distance}]_m \leftrightarrow [\text{distance}]_e, \\ &[\text{power}]_m \leftrightarrow [\text{power}]_e, \\ &[\text{force}]_m \leftrightarrow [\text{current}]_e. \end{aligned} \tag{2.1}$$

It is easy to show that the dimensions appearing on the same side in the correspondences (2.1) are independent. Therefore the correspondences (2.1) determine an isomorphism of the dimensional spaces Π_m and Π_e. The remaining correspondences between mechanic and electric dimensions (as well as quantities)—see Table 1—are the consequence of the correspondences (2.1) and they result from the solution of the suitable dimensional equations [13]. A likewise treatment is applied by J. WEHR in [16] to another system of electromechanical analogies.

Table 1.

Firestone's system of analogies for one-dimensional systems with distributed parameters (transmission lines)

Electric quantities and dimensions	Mechanic quantities and dimensions
voltage $[u]$	velocity of displacement $[l/t]$
current $[i]$	force $[ml/t^2]$
magnetic flux $[ut]$	displacement $[l]$
electric charge $[it]$	momentum $[ml/t]$
inductance per unit length $[ut/i\mathscr{L}]$	deformability per unit length $[t^2/m\mathscr{L}]$
capacitance per unit length $[it/u\mathscr{L}]$	mass per unit length $[m/\mathscr{L}]$
characteristic impedance $[u/i]$	$\dfrac{\text{velocity of displacement}}{\text{force}}$ $[t/m]$
current density $[i/\mathscr{L}^2]$	stress $[ml/t^2\mathscr{L}^2]$
magnetic flux per unit length $[ut/\mathscr{L}]$	strain $[l/\mathscr{L}]$
	etc.

2.3. One-dimensional state of strain of porous medium in comparison with electric transmission lines

On the left-hand side in Table 2, the equations of the consolidation theory describing the one-dimensional quasi-static state of strain of porous medium are presented. The corresponding equations due to the system of two electric transmission lines with the magnetic and conductance coupling (Fig. 1) are given on the right-hand side of the table.

Basic denotations used in the paper are:

mechanic quantities:

σ_{11} — normal component of the stress tensor of the skeleton of a porous medium in the direction x_1,

σ — stress in the fluid (perfect liquid),

w_1 — component of the displacement vector of the porous skeleton in the direction x_1,

W_1 — component of the displacement vector of the fluid in the direction x_1,

N, A, Q, R—Biot-Willis elastic coefficients,

C — permeability constant of the porous medium

$$M = A - \frac{Q^2}{R}, \quad H = Q + R, \quad K_1 = \frac{H}{R(M+2N)};$$

electric quantities:

i_1 (J_1)—current (current density) in the line 1,

Table 2.

Equation of consolidation theory (one-dimensional state of strain)	Equations of transmission lines
physical relations: $\sigma_{11}=(2N+A)\dfrac{\partial w_1}{\partial x_1}+Q\dfrac{\partial W_1}{\partial x_1}$ $\sigma=Q\dfrac{\partial w_1}{\partial x_1}+R\dfrac{\partial W_1}{\partial x_1}$	$i_1=J_1 S=\dfrac{L_2}{L_1L_2-M_e^2}\dfrac{\partial(-\Psi)}{\partial x}+\dfrac{M_e}{L_1L_2-M_e^2}\dfrac{\partial(-\Psi_2)}{\partial x}$ $i_2=J_2 S=\dfrac{M_e}{L_1L_2-M_e^2}\dfrac{\partial(-\Psi_1)}{\partial x}+\dfrac{L_1}{L_1L_2-M_e^2}\dfrac{\partial(-\Psi_2)}{\partial x}$
$\dfrac{\partial w_1}{\partial x_1}=\dfrac{1}{M+2N}\sigma_{11}-\dfrac{Q}{R(M+2N)}\sigma$ $\dfrac{\partial W_1}{\partial x_1}=-\dfrac{Q}{R(M+2N)}\sigma_{11}+\dfrac{2N+A}{R(M+2N)}\sigma$	$\dfrac{\partial(-\Psi_1)}{\partial x}=L_1(J_1S)-M_e(J_2S)$ $\dfrac{\partial(-\Psi_2)}{\partial x}=-M_e(J_1S)+L_2(J_2S)$
Darcy law $\dfrac{\partial w_1}{\partial t_1}-\dfrac{\partial w_1}{\partial t}=C\dfrac{\partial\sigma}{\partial x_1}$	$\dfrac{\partial(-\Psi_2)}{\partial t}-\dfrac{\partial(-\Psi_1)}{\partial t}=\dfrac{1}{G}S\dfrac{\partial J_2}{\partial x}$
equilibrium equation $\dfrac{\partial\sigma_{11}}{\partial x_1}+\dfrac{\partial\sigma}{\partial x_1}=0$	$\dfrac{\partial J_1}{\partial x}+\dfrac{\partial J_2}{\partial x}=0$
complete set of equations $\dfrac{\partial^2 w_1}{\partial x_1^2}=-K_1\dfrac{\partial\sigma}{\partial x_1}$ $C\dfrac{\partial^2\sigma}{\partial x_1^2}=\dfrac{1}{R}\dfrac{\partial\sigma}{\partial t}-\dfrac{H}{R}\dfrac{\partial^2 w_1}{\partial x_1\partial t}$	$\dfrac{\partial^2(-\Psi_1)}{\partial x^2}=-(L_1+M_e)S\dfrac{\partial J_2}{\partial x}$ $\dfrac{1}{G}S\dfrac{\partial^2 J_2}{\partial x^2}=\dfrac{L_1L_2-M_e^2}{L_1}S\dfrac{\partial J_2}{\partial t}-\dfrac{L_1+M_e}{L_1}\dfrac{\partial^2(-\Psi_1)}{\partial x\,\partial t}$

i_2 (J_2)—current (current density) in the line 2,
S—cross-section of the line 1 (line 2),
Ψ_1, Ψ_2—magnetic flux associated with the line 1 or 2,
u_1, u_2—line voltage of the line 1 or 2,
$u = \frac{\partial \Psi}{\partial t}$—relationship between voltage and magnetic flux,
L_1, L_2—self-inductance of the lines 1 and 2 per unit length.
M_e—mutual inductance between lines 1 and 2 per unit length
G—conductance between lines 1 or 2 per unit length.

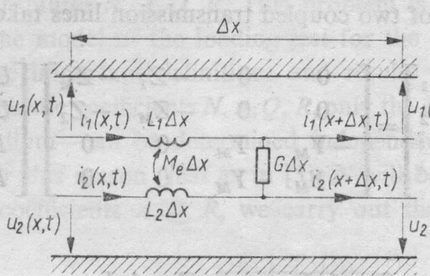

FIG. 1. The element of two electrical lines with magnetic and conductance coupling.

It follows from Tables 2 and 1 that an analogy between the one-dimensional quasi-static state of strain of a porous medium and the system of two coupled transmission lines exists. The analogy belongs to Firestone's system of electromechanical analogies.

It may be shown [13] that the analogy discussed can be extended to dynamic problems in the same system of electromechanical analogies. Then the forces of inertia [2.7] appear in the equations. Since in Forestone's system of analogies (Table 1) we have

$$[\text{mass}]_m \leftrightarrow [\text{capacitance}]_e$$

the electric scheme from Fig. 1 involving suitable capacitances constitutes the electric analogon for dynamical problems of the consolidation theory (Fig. 2).

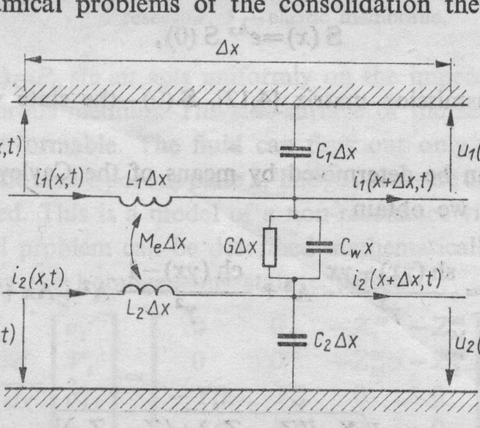

FIG. 2. The electric analogon for one-dimensional dynamic problems of the theory of consolidation.

3. Solution of the coupled transmission lines equations

Consider the system from Fig. 1 with the sinusoidal current excitation. The instantaneous values of the voltages and currents in both lines are given by

$$(3.1)\qquad \begin{aligned} u_k(x,t) &= \mathrm{Im}\,(U_k(x)\,e^{j\omega t}),\\ i_k(x,t) &= \mathrm{Im}\,(I_k(x)\,e^{j\omega t}), \qquad k=1,2, \end{aligned}$$

where $U_k(x)$ and $I_k(x)$ are the complex voltage and current respectively, $j=\sqrt{-1}$ and ω is the angular frequency.

The matrix equation of two coupled transmission lines takes the form [4]

$$(3.2)\qquad \frac{d}{dx}\begin{bmatrix} U_1\\ U_2\\ I_1\\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -Z_1 & -Z_M\\ 0 & 0 & -Z_M & -Z_2\\ -Y_M & Y_M & 0 & 0\\ Y_M & -Y_M & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1\\ U_2\\ I_1\\ I_2 \end{bmatrix},$$

where

$$(3.3)\qquad \begin{aligned} Z_1 &= j\omega L_1, & Z_2 &= j\omega L_2,\\ Z_M &= -j\omega M_e, & Y_M &= G \end{aligned}$$

are the impedances and admittance.

Equation (3.2) is called the homogeneous state equation and can be written in the form

$$(3.4)\qquad \frac{d\mathbf{S}(x)}{dx} = \mathbf{A}\mathbf{S}(x),$$

where $\mathbf{S}(x)$ is the state vector, $\mathbf{A}$—the matrix of the system. The state vector $\mathbf{S}(x)$ possesses $n=4$ components $[U_1, U_2, I_1, I_2]$ in the considered case. Thus the constant square matrix $\mathbf{A}$ is of the order $n=4$.

The solution of Eq. (3.4) is given by

$$(3.5)\qquad \mathbf{S}(x) = e^{\mathbf{A}x}\,\mathbf{S}(0),$$

where $e^{\mathbf{A}x}$ is the transmission matrix [4,12], $\mathbf{S}(0)$—the state vector on the input of the system.

The matrix $e^{\mathbf{A}x}$ can be determined by means of the Cayley-Hamilton theorem [4] and as the result we obtain

$$(3.6)\qquad e^{\mathbf{A}x} = \frac{\mathrm{sh}(\gamma x)-\gamma x}{\gamma^3}\mathbf{A}^3 + \frac{\mathrm{ch}(\gamma x)-1}{\gamma^2}\mathbf{A}^2 + \mathbf{A}x + \mathbf{1}.$$

The quantity

$$(3.7)\qquad \gamma = \sqrt{Y_M\,[(Z_1-Z_M)+(Z_2-Z_M)]}$$

is the propagation constant [4]; the characteristic impedance of the system is defined by the formula

$$Z_0 = \sqrt{\frac{(Z_1 - Z_M) + (Z_2 - Z_M)}{Y_M}}. \tag{3.8}$$

4. Some applications of the analogies

Using the analogies derived in Sects. 2, 3 we can, for example, propose the procedure for determining the elastic coefficients of the consolidation theory, i.e. the Biot-Willis constants: N, A, Q, R. This problem is discussed in detail in [14]. We present here the model of the loading test for the determination of the elastic coefficients and the important definitions and results only.

From the four elastic coefficients N, A, Q, R, only the coefficient N—shear modulus of the porous medium—can be determined independently of the remaining coefficients [3], and for this reason (just as in [3]) N will be taken as known. In order to determine the coefficients A, Q, R, we carry out the loading test as in Fig. 3.

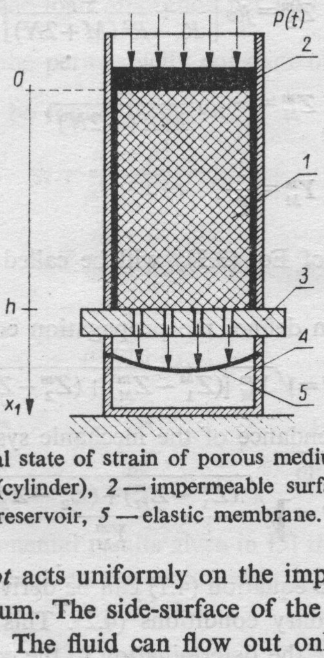

Fig. 3. One-dimensional state of strain of porous medium by $P(t) = P_0 \sin \omega t$
1 — sample of porous medium (cylinder), *2* — impermeable surface, *3* — porous plate, *4* — gas reservoir, *5* — elastic membrane.

The load $P(t) = P_0 \sin \omega t$ acts uniformly on the impermeable top-surface of the sample of the porous medium. The side-surface of the sample is impermeable for a fluid and underformable. The fluid can flow out only into the gas reservoir 4, through the underfomable porous plate 3. The gas pressure under the elastic membrane 5 can be regulated. This is a model of a non-resonance vibroisolator [5, 15].

The presented problem can be described mathematically as the one-dimensional state of strain by the homogeneous state equation

$$\frac{d}{dx}\begin{bmatrix} v_1 \\ V_1 \\ \sigma_{11} \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 & 0 & -Z_1^m & -Z_M^m \\ 0 & 0 & -Z_M^m & -Z_2^m \\ -Y_M^m & Y_M^m & 0 & 0 \\ Y_M^m & -Y_M^m & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ V_1 \\ \sigma_{11} \\ \sigma \end{bmatrix}, \tag{4.1}$$

where v_1 and V_1 are the complex velocities of the skeleton and the fluid, σ_{11} and σ—the complex stresses in the skeleton and the fluid respectively, and by the following boundary conditions:

$$\begin{aligned} &\sigma_{11}(x_1=h)=0, \\ &v_1(x_1=h)=0, \\ &v_1(x_1=0)=V_1(x_1=0), \\ &\sigma_{11}(x_1=0)+\sigma(x_1=0)=p_0, \end{aligned} \tag{4.2}$$

where

$$p_0=\frac{P_0}{S},$$

S—area of cross-section of the sample, h—height of the sample.

The quantities

$$\begin{aligned} Z_1^m &= j\omega\frac{1}{M+2N}, \\ Z_2^m &= j\omega\left[\frac{1}{R}+\frac{Q^2}{R^2(M+2N)}\right], \\ Z_M^m &= -j\omega\frac{Q}{MR(+2N)}, \\ Y_M^m &= \frac{1}{C}, \end{aligned} \tag{4.3}$$

according to the analogy of Eq. (3.3), will be called the mechanical impedances and admittance.

Using Eqs. (4.3) we can define: the propagation constant (cf. (3.7))

$$\gamma^m=\sqrt{Y_M^m\,[(Z_1^M-Z_M^m)+(Z_2^m-Z_M^m)]}, \tag{4.4}$$

and the characteristic impendance of the mechanic system (cf. Eq. (3.8))

$$Z_0^m=\sqrt{\frac{(Z_1^m-Z_M^m)+(Z_2^m-Z_M^m)}{Y_M^m}}. \tag{4.5}$$

The solution of the state equation (4.1) can be derived on the basis of Eqs. (3.5) and (3.6), using the boundary conditions (4.2). This solution permits to derive the following procedure for the determination to the coefficients A, Q, R [13]. One should then:

a) solve the nonlinear equation with respect to the real x

$$x=\frac{Y_M^m h|V_1(h)|}{\sqrt{2}|\sigma(0)|}\sqrt{\frac{\operatorname{ch}x-\cos x}{\operatorname{ch}x+\cos x}};$$

b) calculate $|\xi|$ from

$$|\xi|=\frac{2x^2}{Y_M^m h^2},$$

where

$$\xi = p^m + q^m, \qquad |\xi| = |p^m| + |q^m|;$$

c) calculate $|p^m|$ from the formula

$$|p^m| = \frac{|\xi|\,|\sigma(0)|}{p_0} \sqrt{\frac{\operatorname{ch}^2 x - \sin^2 x}{\operatorname{ch} x - \cos x}};$$

d) calculate

$$|q^m| = |\xi| - |p^m|$$

and

$$k^m = j\frac{|\xi|\, v_1(0) - |p^m|\, V_1(h)}{p_0 h}.$$

It should be noted that the following quantities must be measured:

$v_1(0) = |v_1(0)|\, e^{j\psi_1(0)}$ —the complex velocity of the skeleton for $x_1 = 0$;
$V_1(h) = |V_1(h)|\, e^{j\psi_2(h)}$ —the complex velocity of the fluid for $x_1 = h$;
$\sigma(0) = |\sigma(0)|\, e^{j\varphi_2(0)}$ —the complex fluid pressure for $x_1 = 0$;
$p = |p_0|\, e^{j0}$ —the load amplitude per unit area;
$Y_M^m = \frac{1}{C}$; C —the permeabicity constant of the porous medium.

The coefficients A, Q, R can be found from

$$A = \frac{\kappa^2 + \mu^2}{\mu^2(\delta + \kappa)} - 2N, \tag{4.6}$$

$$Q = \frac{\delta\kappa - \mu^2}{\mu^2(\delta + \kappa)}, \tag{4.7}$$

$$R = \frac{\delta^2 + \mu^2}{\mu^2(\delta + \kappa)}, \tag{4.8}$$

where

$$\delta = \frac{p^m}{\omega}, \qquad \kappa = \frac{q^m}{\omega}, \qquad \mu^2 = \frac{-k^m}{\omega^2}.$$

ω—angular frequency.

It follows from the experimental results given in [5] that the presented formulas for the coefficients A, Q, R include the measurable quantities.

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Streszczenie

ANALOGIE ELEKTROMECHANICZNE W TEORII KONSOLIDACJI

Rozpatrzono analogie elektromechaniczne pomiędzy jednowymiarowymi układami o parametrach rozłożonych posłużono się systemem analogii Firestone'a oraz analizą wymiarową. Wykazano istnienie takich analogii dla teorii konsolidacji, co stanowi podstawę do stosowania w tej teorii pewnych metod i pojęć z elektrotechniki. Jako przykład takiego zastosowania przedstawiono algorytm metody wyznaczania stałych materiałowych ośrodka porowatego.

Резюме

ЭЛЕКТРОМЕХАНИЧЕСКИЕ АНАЛОГИИ В ТЕОРИИ КОНСОЛИДАЦИИ

Рассмотрены электромеханические аналогии между одномерными системами с распределенными параметрами; послужились системой аналогий Фаестоуна и размерным анализом. Показано существование таких аналогий для теории консолидации, что составляет основу для применения в этой теории некоторых методов и понятий из электротехники. Как пример такого применения представлен алгоритм метода определения материальных постоянных пористой среды

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
and
TECHNICAL UNIVERSITY OF ŁÓDŹ

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