ROZPRAWY INŻYNIERSKIE · ENGINEERING TRANSACTIONS · 30, 1, 67—75, 1982
Polska Akademia Nauk · Instytut Podstawowych Problemów Techniki

ON A CERTAIN METHOD FOR PARAMETER ESTIMATION OF A MECHANICAL SYSTEM WITH SENSITIVITY MODEL APPLICATION

J. WICHER (WARSZAWA)

The parameter estimation problem of a linear mechanical system with n degrees-of-freedom has been considered. The lumped masses of this system are connected by any combination of stiffness and viscous damping elements (called the rheological models). The sensitivity model of the system is used in a presented algorithm.

1. Introduction

Investigation of the dynamic properties of a real mechanical system is connected with a mathematical modelling problem. The process of determining a mathematical model (most frequently given as a set of differential equations) is called the identification problem.

The general identification problem, in its wider sense, includes all techniques devised to determine a mathematical model from input-output data. If the description of the system is totally unknown, then we have the "black box" identification problem. In the vast majority of cases some knowledge of the structure of the mechanical system is available and in this restricted case this is a parametric identification problem or parameter estimation problem.

The theory of lumped mechanical systems is commonly based on a mathematical model consisting of masses connected by a parallel combination of a linear stiffness and viscous damped elements. However, it is often found in practice that the rheological models joining the masses are very useful, e.g. in isolation systems with rubberlike materials.

The rheological model is the structure with the series combination of viscous damping and stiffness elements where the relations between the force P and displacement q are most generally represented by the linear operators Π and Γ, where $\Pi P = \Gamma q$.

In general case these operators can be represented by linear differential form:

$$\Pi = a_i \frac{\partial^i}{\partial t^i}, \qquad \Gamma = b_i \frac{\partial^i}{\partial t^i}, \tag{1.1}$$

where a_i, b_i are constant coefficients.

The ratio of force to displacement can be represented by the equivalent spring characteristic $k^z = P/q = \Gamma \Pi^{-1}$. The operators Γ and Π for three– and four–element rheological models are given in Table 1. These models are known as Bürger and Standard models. The Standard I and Standard II models are dynamically equivalent when the stiffness $k' = [n/(n+1)]\, k$ and the viscous damping coefficient $c' = [n/(n+1)]^2\, c$, where n is a dimensionless stiffness ratio [1].

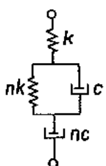
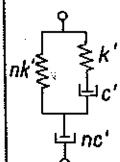
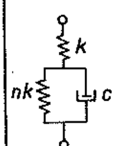
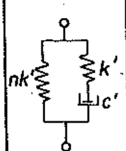
Consider the n-degrees-of-freedom system (Fig. 1) where the equations of motions are

$$(1.2)\qquad m_i \ddot{q}_i + k^z_{ii}\,(q_i - q_{i0}) + \sum_{\substack{j=1\\ j\neq i}}^{n} k^z_{ij}\,(q_i - q_j) = P_i, \qquad i = 1, \ldots, n.$$

Using the Γ and Π operators, the above equations can be written in the form

$$(1.3)\qquad m_i \Pi_{ii} \ddot{q}_i + \Gamma_{ii} q_i + \Pi_{ii} \sum_{\substack{j=1\\ j\neq i}}^{n} \Pi^{-1}_{ij}\, \Gamma_{ij} (q_i - q_j) = \Pi_{ii}\, P_i + \Gamma_{ii}\, q_{i0}, \qquad i = 1, \ldots, n.$$

Table 1.

Model	Γ	Π
 Bürger I	$nk\dfrac{d}{dt} + c\dfrac{d^2}{dt^2}$	$\dfrac{k}{c} + (1+n)\dfrac{d}{dt} + \dfrac{c}{k}\dfrac{d^2}{dt^2}$
 Bürger II	$nk'\dfrac{d}{dt} + c'(1-n)\dfrac{d^2}{dt^2}$	$\dfrac{k'}{c'} + (2+\dfrac{1}{n})\dfrac{d}{dt} + \dfrac{c'}{k'}\dfrac{d^2}{dt^2}$
 Standard I	$nk + c\dfrac{d}{dt}$	$(1+n) + \dfrac{c}{k}\dfrac{d}{dt}$
 Standard II.	$nk' + c'(1+n)\dfrac{d}{dt}$	$1 + \dfrac{c'}{k'}\dfrac{d}{dt}$

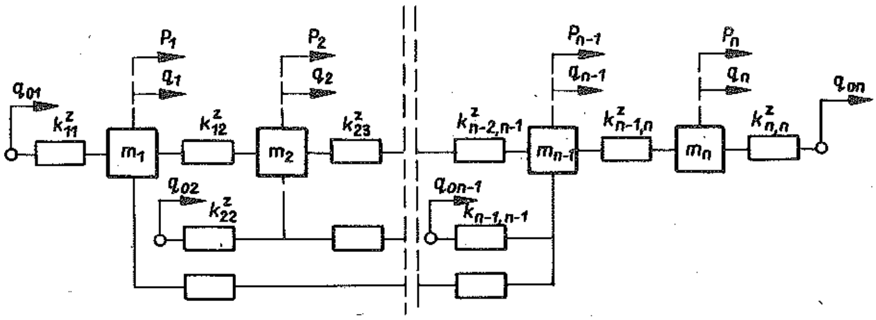


FIG. 1.

All operators Γ_{ij} and Π_{ij}, $ij = 1, \ldots, n$ depend on the unknown stiffness and damping coefficients.

Below the problem of how to find the values of these parameters will be considered.

2. Estimation of the unknown parameters

A general estimation scheme is the model and the real system. The model is connected to the system paralelly. The parameters of the model are changed according to the identification algorithm until the output error is zero or, in the case of additional noise, minimal.

In parameter estimation procedure the input-output schemes are very useful. In Eqs. (1.3) there are n-dimensional input vectors P and q_0 and the output vector q. In estimation procedure the experimental output vector of the real system is used. Let $\check{q} = \check{q}(t)$ be the n-dimensional output vector obtained from experimental data. Further the parameters of the considered system may be denoted as the r-dimensional parameter vector $a = a(\Gamma_{ij}, \Pi_{ij})$, $i, j = 1, \ldots, n$. Then the estimation problem can be performed through minimalization of the functional Φ of a function φ

$$\min_a Q(a) = \min_a \Phi \{\varphi [\check{q}(t) - q(t, a)]\}, \tag{2.1}$$

where the output error $e = \check{q}(t) - q(t, a)$ and Q is called the quality function. The minimum value of Q, which is a scalar function of the parameter vector a, can be obtained by adjusting the parameter a. Computationally, this can be accomplished by the steepest descent method, where the value of a at the $(i+1)$ iteration is given in terms of its value at the i-th iteration as

$$a^{(i+1)} = a^{(i)} + \Delta a, \tag{2.2}$$

where the superscript i indicates the iteration and Δa is given by

$$\Delta a = -k \operatorname{grad} Q, \quad k > 0. \tag{2.3}$$

The velocity of the iteration process can be controlled by the factor k. The value of the factor k has to be chosen carefully as in all gradient methods. If k is chosen too large, the iteration process diverges, if it is chosen too small, the process takes too much time. Commonly k is chosen empirically.

The squared output error is a suitable criterion for the estimation of unknown parameters. In this case

$$Q(a) = \Phi\,[e^T Ae], \tag{2.4}$$

where $e = \check{q}(t) - q(t, a)$ and A is symmetric positive definite matrix that weighs the individual components $e_j = \check{q}_j - q_j$. Commonly the matrix A is chosen empirically or is given as the unity matrix.

The gradient of the quality function can be determined by a different method. This is the biggest computational task in the parameter estimation problem.

An iteresting method is the procedure with a sensitivity model application.

3. Application of the sensitivity model

In the general case the grad Q can be obtained by differentiation:

$$\operatorname{grad} Q = \frac{\partial Q}{\partial a} = \Phi\left[\frac{\partial \varphi}{\partial e}\,\frac{\partial e}{\partial a}\right], \tag{3.1}$$

where

$$\frac{\partial e}{\partial a} = \frac{\partial[\check{q}(t) - q(t, a)]}{\partial a} = -\frac{\partial q(t, a)}{\partial a} = -u. \tag{3.2}$$

The partial derivative of q with respect to a is the sensitivity matrix $u = (u_{ij}) = (\partial q_i / \partial a_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, r$.

The elements u_{ij} of the matrix u are called the sensitivity function. These functions can be evaluated by solving the sensitivity equations. These equations can be obtained by differentiating Eqs. (1.3) with respect to all unknown parameters. Taking the partial derivatives with respect to one of them, say m_i, we obtain

$$m_i \Pi_{ii} \ddot{u}_{im_i} + \Gamma_{ii}(u_{im_i} - u_{iom_i}) + \Pi_{ii} \sum_{\substack{j=1 \\ j \neq i}}^{n} \Pi_{ij}^{-1} \Gamma_{ij}(u_{im_i} - u_{jm_i}) = \Pi_{ii}\ddot{q}_i, \qquad i = 1, \ldots, n, \tag{3.3}$$

where $u_{im_j} = \partial q_i / \partial m_j$, $\ddot{u}_{im_i} = d^2/dt^2 \; (u_{im_i})$.

The system described by the sensitivity equations is known as the sensitivity model.

The squared output error is often used. In this case $Q\, e^T Ae$ and the gradient

(3.4) $$\operatorname{grad} Q = \left(\frac{\partial e}{\partial a}\right)^T Ae = -u^T Ae$$

and Eq. (2.3) may be written in the form

(3.5) $$\Delta a = ku^T Ae.$$

A general configuration for the system of parameter estimation with the sensitivity model is shown in Fig. 2.

The algorithm described above can be expressed as follows:

1) Assign an initial value of a^0 of the parameter vector a.

2) With the current value of $a^{(i)}$, $i = 1, 2, \ldots$ solve the differential equations (1.2).

3) Compute the grad Q using Eqs. (3.1) or, in the special case, Eqs. (3.4).

4) Update the parameter vector according to Eq. (2.2) until $\Delta a < \varepsilon$, where ε is a tolerance defined by the user.

5) If this test is satisfied, take the values $a^{(i+1)}$ as the estimate of the unknown parameter; if not—go to step 2. This concept is illustrated by the flow chart of Fig. 3. The initial value a^0 can be either obtained from physical considerations or assumed arbitrarily.

4. Example

Let us consider the simple mechanical system where an element of mass is supported by a linear rubberlike material to a vibrated exciter. The system and the instrumentation setup are illustrated in Fig. 4. According to Eqs. (1.2) and (1.3) the equation of motion may be written as

(4.1) $$m\ddot{q}_1 + k^z (q_1 - q_0) = 0$$

or

(4.2) $$m\Pi\ddot{q}_1 + \Gamma(q_1 - q_0) = 0.$$

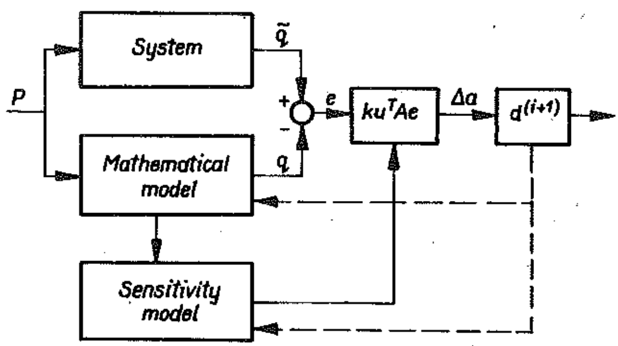


FIG. 2.

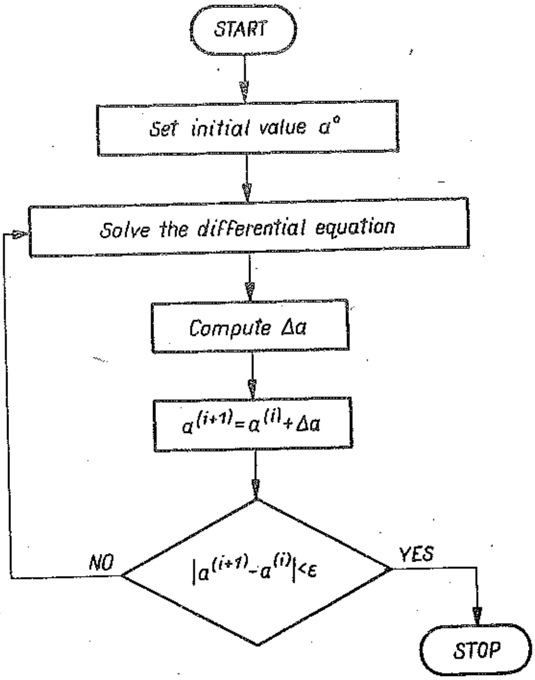


FIG. 3.

The rubberlike material may be described by the Standard I model (see Table 1). In this case Eq. (4.2) can be written in the form

$$m\left[(l+n)+\frac{c}{k}\frac{d}{dt}\right]\ddot{q}_1+\left[nk+c\frac{d}{dt}\right](q_1-q_0)=0 \tag{4.3}$$

or (after an easy transformation)

$$mc\dddot{q}_1+m(k+nk)\ddot{q}_1+ck\dot{q}_1+nkkq_1=ck\dot{q}_0+nkkq_0. \tag{4.4}$$

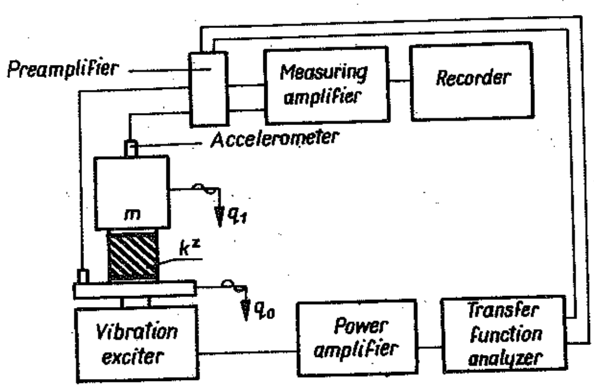


FIG. 4.

Using the substitutions

$$a_1 = nkk/mc, \quad a_2 = ck/mc, \quad a_3 = m(k+nk)/mc, \tag{4.5}$$
$$q_1 = y_1, \quad \dot{q}_1 = \dot{y}_1 = y_2, \quad \ddot{q}_1 = \dot{y}_2 = y_3.$$

Eq. (4.4) can be written in this canonical form:

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3,$$
$$\dot{y}_3 = -a_1 y_1 - a_2 y_2 - a_3 y_3 + a_1 q_0 + a_2 \dot{q}_0.$$

The output error may be defined as

$$e = (e_1, e_2, e_3)^T = (\check{q}_1 - y_1, \dot{\check{q}}_1 - y_2, \ddot{\check{q}}_1 - y_3)^T. \tag{4.6}$$

The rule for generating the parameter vector $a = (a_1, a_2, a_3)$ has the form (3.5) if the squared output error is used. The matrix A is chosen as the unity matrix and u is the (3×3) sensitivity matrix with elements $u_{ij} = \partial y_i/\partial a_j$, $i, j = 1, 2, 3$. The j-the component of Eq. (3.5) can be written in the form

$$\Delta a_j = k(u_{1j} e_1 + u_{2j} e_2 + u_{3j} e_3). \tag{4.7}$$

The sensitivity functions $u_{ij} = \partial y_i/\partial a_j$ can be obtained by differentiating Eqs. (4.5). Taking the partial derivatives with respect to one of the parameters, say a_j, the sensitivity equations are

$$\partial \dot{y}_1/\partial a_j = \partial y_2/\partial a_j,$$
$$\partial \dot{y}_2/\partial a_j = \partial y_3/\partial a_j, \tag{4.8}$$
$$\partial \dot{y}_3/\partial a_j = -a_1(\partial y_1/\partial a_j) - a_2(\partial y_2/\partial a_j) - a_3(\partial y_3/\partial a_j) + w_j, \quad j = 1,2,3$$

and

$$w_1 = -y_1 + q_0, \quad w_2 = -y_2 + \dot{q}_0 = -\dot{y}_1 + \dot{q}_0 = \dot{w}_1, \quad w_3 = -y_3. \tag{4.9}$$

According to Fig. 2 all sensitivity functions ought to be measured simultaneously. From this structure it can be seen tha if a system has r parameters, r sensitivity models are needed. In the considered example three sensitivity models (4.8) are needed. But it is possible to build a more simplified structure containing less than three sensitivity models.

The only difference between the equations (4.8) for $j = 1$ and $j = 2$ is the input signal which is $w_1 = y_1 + q_0$ and in the second case $w_2 = \dot{w}_1$. Then the signals of the latter sensitivity model are the derivatives of the corresponding signals of the former. Hence

$$\partial y_1/\partial a_2 = \partial \dot{y}_1/\partial a_1 = \partial y_2/\partial a_1,$$
$$\partial y_2/\partial a_2 = \partial \dot{y}_2/\partial a_1 = \partial y_3/\partial a_1, \tag{4.10}$$
$$\partial y_3/\partial a_2 = \partial \dot{y}_3/\partial a_1 = -a_1(\partial y_1/\partial a_1) - a_2(\partial y_2/\partial a_1) - a_3(\partial y_3/\partial a_1) - y_1 + q_0.$$

Thus the sensitivity functions with respect to a_2 may be obtained directly as the signals in the suitable nodes in the diagram shown in Fig. 5.

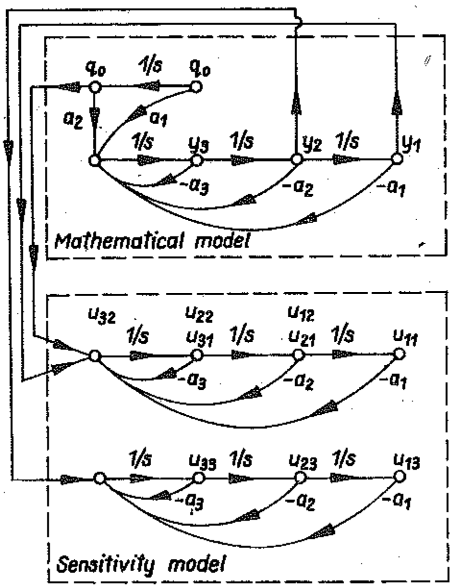


FIG. 5.

In the case of the sensitivity functions for the parameter a_3, the input signal $w_3 = -y_3$ is not the derivative (with respect to time) of the input signal of the sensitivity equation obtained for parameter a_2, which implies that the signals of this sensitivity model are not the derivatives of the corresponding signals of the model (4.8) for $j = 2$. Then the sensitivity functions for the parameter a_3 may be obtained from the new diagram connected with the one considered earlier (see Fig. 5).

The complete identification scheme for the considered example is illustrated in the diagram shown in Fig. 6.

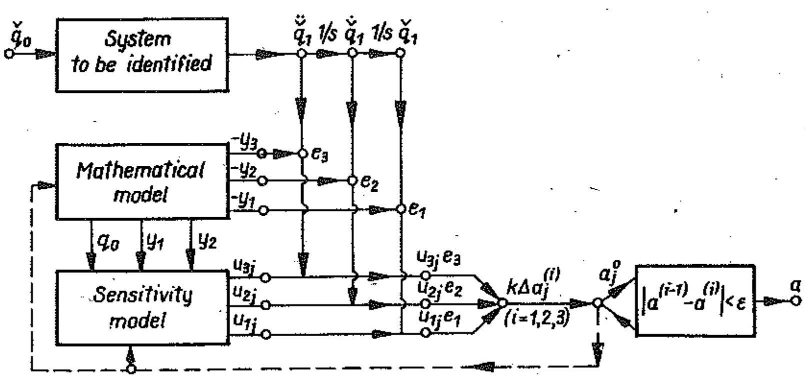


FIG. 6.

Giving the input signal as the impluse function, for $k = 10$ and the initial values $a_1^0 = 10^8$, $a_2^0 = 10^6$, $a_3^0 = 2 \times 10^2$ after the twentieth iterations were obtained $a_1 = 1.266561 \times 10^8$, $a_2 = 4.165732 \times 10^5$, $a_3 = 3.395733 \times 10^2$. Using the substitutions (4.5) the following values of the unknown parameters of the rheological model were obtained; $k = 461.6$, $nk = 1783$, $c = 6.61$ (for known $m = 1.105 \times 10^{-3}$).

References

1. J. E. Ruzicka, *Resonance characteristics of unidirectional viscous and Coulomb-damped vibration isolation systems*, J. of Eng. for Industry, pp. 729–740, Nov. 1967.
2. W. S. Sperry, *Rheological-model concept*, J. of the Acoust. Soc. of America, **36**, 2, pp. 376–385, Feb. 1964.
3. V. Vemuri, A. Raefsky, *On a new approach to parameter estimation by the method of sensitivity functions*, Int. J. Systems Sci., **10**, 4, pp. 395–407, 1979.
4. J. Wicher, *Sensitivity analysis in parametric identification of dynamic system*, Proc. Symp. "Stochastische Schwingungen und Zuverlässigkeit," Wartburg bei Eisenach, pp. 143—148, Oct. 1980.

Streszczenie

O PEWNEJ METODZIE ESTYMACJI PARAMETRÓW UKŁADU MECHANICZNEGO Z ZASTOSOWANIEM MODELU WRAŻLIWOŚCI

Rozpatrywany jest problem estymacji parametrów liniowego układu mechanicznego o n stopniach swobody, w którym masy skupione połączone są za pomocą dowolnej kombinacji elementów sprężystych i tłumieniowych (tzw. modeli reologicznych). W przedstawionym algorytmie wykorzystuje się model wrażliwości układu.

Резюме

О НЕКОТОРОМ МЕТОДЕ ОЦЕНОК ПАРАМЕТРОВ МЕХАНИЧЕСКОЙ СИСТЕМЫ С ПРИМЕНЕНИЕМ МОДЕЛИ ЧУВСТВИТЕЛЬНОСТИ

Рассматривается проблема оценок параметров линейной механической системы с n степенями свободы, в которой сосредоточенные массы соединены при помощи произвольной комбинации упругих и демпфирующих элементов (т. наз. реологических моделей). В представленном алгоритме используется модель чувствительности системы.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received March 12, 1981.