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ON THE APPROXIMATE EVALUATION OF INTERACTION OF CRACKS IN ELASTIC MEDIA

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A method of approximate analysis is presented concerning the state of stress, and the stress intensity factors in particular, in elastic media subject to a plane state of strain and containing arbitrary arrays of cracks. In the case when the crack distribution is not too dense, the method proposed makes possible the determination of the required stress parameters in a manner resembling that used in solving the statically indeterminate systems of structural mechanics.

1. Introduction

In a recent paper [1] the problem of interaction of cracks in elastic bodies was outlined and discussed, the consideration being based on the approach due to which the cracks were considered as special case of material defects. Following the paper by H. Zorski [2] and several earlier papers by J. D. Eshelby [3, 4], the forces of interaction between individual cracks were expressed in terms of the corresponding stress intensity factors. Effective analysis of the phenomenon was demonstrated in the simplest case of bodies subject to the antiplane state of strain leading to the so-called Mode III crack deformation. Approximate results of the analysis were obtained by means of the procedure in which the cracks were replaced with suitably selected, concentrated force couples.

This procedure is now further developed and generalized to the cases of elastic media containing several, arbitrarily distributed cracks and subject to the plane state of strain. The cracks deform then according to Modes I and II, the opening and sliding modes. Analysis of fracture of bodies containing cracks is based on the knowledge of stress intensity factors at the crack tips. The usual mathematical approach leading to the determination of the intensity factors consists in constructing and solving rather complicated sets of integral equations; in most cases they must be solved numerically, and the problem has been extensively discussed in an unpublished dissertation by T. A. Pučik [5]. In the present paper, another approximate but relatively simple approach to the problem is proposed, resembling the well-known procedure of analyzing the statically indeterminate systems of structural mechanics.

2. Single crack analysis

Let us consider a crack in the form of a strip $|x| < a$, $y = 0$ extending in the direction of the z-axis from plus to minus infinity (Fig. 1) in an infinite elastic medium. The medium is loaded by forces T_i acting outside the crack and deforms according to the plane state of strain within the plane x, y. As a result, the crack itself is deformed according to the opening and sliding modes (Mode I and II, respectively), Figs. 1a and 1b.

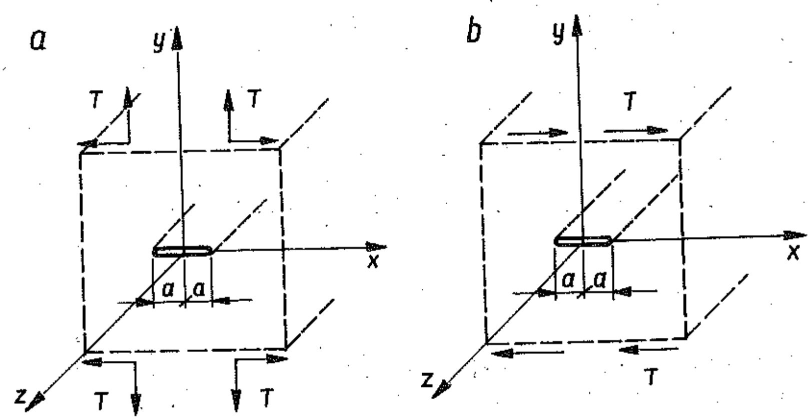

Fig. 1.

It was shown [6] that in such cases the states of stress and deformation outside the crack may be determined by analyzing the same solid body loaded by external forces T in which the crack is replaced with a suitably selected force dipole or force couple distributions depending on T and the crack size 2a. Let us introduce the following notations. The densities of horizontal and vertical forces distributed along the segment $-a < x < a$ of a solid body are denoted by ϱ^1 and ϱ^2, respectively.

A force couple is now constructed in the usual manner: for instance, two horizontal forces $\pm P_1$ are applied at points $(0, \delta)$, $(0, -\delta)$; the limiting procedure in which δ tends to zero and the product $2P_1 \delta = P_{12}$ remains constant yields the horizontal force couple (Fig. 2c); the density of its distribution is denoted by ϱ^{12}. In an analogous manner the other force couple (Fig. 2d) is introduced, and the two force dipoles (Figs. 2e, f); higher order couples and multipoles may also be constructed, and notations of the corresponding densities are shown in Fig. 2. The same indices may also be used in denoting the stresses produced in an infinite body by a single concentrated force, couple or dipole acting at the origin (0, 0) of the coordinate system x, y. Denote, for instance, by $\bar{\sigma}^1_{ij}(x, y)$ the stress distribution due to a concentrated horizontal force $P_1 = 1$ (Fig. 2a)

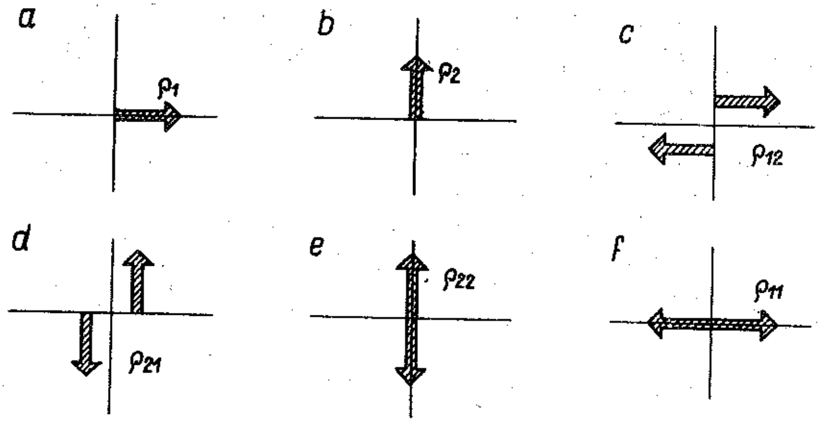

FIG. 2.

acting in the plane x, y. Then the stress due to the force couple (Fig. 2c) of unit intensity applied at the same point is equal to

$$\bar{\sigma}_{ij}^{12}(x, y) = -\frac{\partial}{\partial y}\bar{\sigma}_{ij}^{1}(x, y).$$

Similarly,

$$\bar{\sigma}^{22} = -\frac{\partial\bar{\sigma}^2}{\partial y}, \quad \bar{\sigma}^{21} = -\frac{\partial\bar{\sigma}^2}{\partial x}, \quad \bar{\sigma}^{11} = -\frac{\partial\bar{\sigma}^1}{\partial x} \quad \text{etc.} \tag{2.1}$$

are then the stresses produced by other higher order couples and dipoles.

In the case considered in [6], second order force distributions were sufficient to replace the action of cracks. It was shown there that the single and double forces distributed on the plane $y = 0$ lead to the following stress discontinuities across the plane:

$$\begin{aligned} &[\![\sigma_{xy}^{1}]\!] = -\varrho^1, \quad [\![\sigma_{xy}^{11}]\!] = (\varrho^{11})', \quad [\![\sigma_{xy}^{22}]\!] = -\frac{\nu}{1-\nu}(\varrho^{22})', \\ &[\![\sigma_{yy}^{2}]\!] = -\varrho^2, \quad [\![\sigma_{yy}^{12}]\!] = -(\varrho^{12})', \quad [\![\sigma_{yy}^{21}]\!] = (\varrho^{21})'. \end{aligned} \tag{2.2}$$

Here the symbol $[\![f]\!]$ denotes the jump of function $f(x, y)$ across the plane $y = 0$,

$$[\![f(x, y)]\!] = \lim_{\delta\to 0}[f(x, \delta) - f(x, -\delta)], \tag{2.3}$$

and the primes denote the derivatives with respect to x.

It was proved that the stresses produced in an infinite plane x, y by external loads and containing a crack according to Fig. 1 are the same as the stresses produced in a solid, uncracked plane by the same external loads, and the forces and double forces distributed along the segment $-a < x < a, y = 0$.

In the case of Mode I crack deformation the necessary forces to be applied along the segment are ϱ^{22} and ϱ^{11} (or ϱ^{22} and ϱ^{1}); in the case of Mode II crack deformation, the forces are ϱ^{12} and ϱ^{21} (or ϱ^{12} and ϱ^{2}). It is easily proved that both force systems are strictly equivalent. The double force intensities are calculated as follows [6].

Consider the elastic medium loaded by external forces according to Figs. 1a and 1b. The two-dimensional problem of an infinite medium loaded by external forces T may easily be solved (the plane does not contain any crack as yet) and yields the stress distribution $\sigma_{ij}^{0}(x, y)$. The distribution of stresses $\sigma_{ij}^{0}(x, y)$ along the segment $-a < x < a$, $y = 0$ is denoted by $p_2^0(x)$, while the stresses $\sigma_{yx}^{0}(x, y)$ along the same segment are denoted by(1) $p_1^0(x)$, i.e.

$$\text{(2.4)}\qquad \begin{aligned} \sigma_{yy}^{0}(x, 0) &= p_2^0(x), \quad \text{Mode I;}\\ \sigma_{yx}^{0}(x, 0) &= p_1^0(x), \quad \text{Mode II.} \end{aligned}$$

The double force intensities are now calculated from the integral formulae

$$\text{(2.5)}\qquad \varrho^{22}(x) = \frac{4(1-\nu)^2}{1-2\nu}\int_{-a}^{x}\frac{P_2^0(\xi)\,d\xi}{\sqrt{a^2-\xi^2}}, \qquad \varrho^{11} = \frac{\nu}{1-\nu}\varrho^{22},$$

where

$$P_2^0(\xi) = \frac{1}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-\xi}\,p_2^0(t)\,dt,$$

the last integral being considered in the sense of the Cauchy principal value.

The set of two double force distributions (2.5) solves the Mode I crack deformation problem. The other case of Mode II necessitates the application of two other double force distributions:

$$\text{(2.6)}\qquad \varrho^{12}(x) = \varrho^{21}(x) = 2(1-\nu)\int_{-a}^{x}\frac{P_1^0(\xi)\,d\xi}{\sqrt{a^2-\xi^2}},$$

where

$$P_1^0(\xi) = \frac{1}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2-t^2}}{t-\xi}\,p_1^0(t)\,dt,$$

the last integral being also considered in the CPV sense.

For instance, in the simplest case of an elastic medium loaded at $y = \pm\infty$ by uniformly distributed tension $T_2 = \sigma_{yy}(x, \pm\infty)$ and shear

(1) Functions $p_1^0(x)$, $p_2^0(x)$ are assumed to satisfy the Hölder condition.

$T_1 = \sigma_{yx}(x, \pm\infty)$, the integrals (2.5) and (2.6) are easily calculated to yield the "elliptical" force distributions

$$\varrho^{22}(x) = \frac{4(1-\nu)^2}{1-2\nu} T_2 \sqrt{a^2-x^2},$$

(2.7)
$$\varrho^{11}(x) = \frac{4\nu(1-\nu)}{1-2\nu} T_2 \sqrt{a^2-x^2},$$

$$\varrho^{12}(x) = \varrho^{21}(x) = 2(1-\nu) T_1 \sqrt{a^2-x^2}.$$

The final stress distribution in an infinite elastic medium containing the crack and loaded by external forces is now obtained by summing up the stresses $\sigma_{ij}^0(x, y)$ and the stresses produced by the double force distributions, Eqs. (2.5) and (2.6)

(2.8)
$$\sigma_{ij}(x, y) = \sigma_{ij}^0(x, y) + \sum_{m,n} \int_{-a}^{a} \varrho^{mn}(\xi)\, \bar{\sigma}_{ij}^{mn}(x-\xi)\, d\xi.$$

The sum has o be taken over all ecessary pairs of indices, 22 and 11 in the case of Mode I, and 12 and 21 in the case of Mode II. The functions $\bar{\sigma}_{ij}^{mn}$ play here the role of the Green functions and follow from differentiation of the known solutions concerning the plane loaded by horizontal or vertical unit forces according to the formulae (2.1). The corresponding stress distributions are given below.

Horizontal force (Fig. 2a)

$$\bar{\sigma}_{xx}^1 = -\frac{1}{4\pi(1-\nu)}\left[(3-2\nu)\frac{x}{r^2} - \frac{2xy^2}{r^4}\right],$$

(2.9)
$$\bar{\sigma}_{yy}^1 = \frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{x}{r^2} - \frac{2xy^2}{r^4}\right],$$

$$\bar{\sigma}_{xy}^1 = -\frac{1}{4\pi(1-\nu)}\left[2(1-\nu)\frac{y}{r^2} + \frac{y(x^2-y^2)}{r^4}\right].$$

Vertical force (Fig. 2b)

$$\bar{\sigma}_{xx}^2 = -\frac{1}{4\pi(1-\nu)}\left[2\nu\frac{y}{r^2} + \frac{y(x^2-y^2)}{r^4}\right],$$

(2.10)
$$\bar{\sigma}_{yy}^2 = -\frac{1}{4\pi(1-\nu)}\left[2(1-\nu)\frac{y}{r^2} - \frac{y(x^2-y^2)}{r^4}\right],$$

$$\bar{\sigma}_{xy}^2 = -\frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{x}{r^2} + \frac{2xy^2}{r^4}\right].$$

Horizontal force couple (Fig. 2c)

$$\bar{\sigma}_{xx}^{12} = -\frac{1}{4\pi(1-\nu)}\left[(3-2\nu)\frac{2xy}{r^4}+4\frac{xy(x^2-y^2)}{r^6}\right],$$

(2.11)
$$\bar{\sigma}_{yy}^{12} = \frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{2xy}{r^4}+4\frac{xy(x^2-y^2)}{r^6}\right],$$

$$\bar{\sigma}_{xy}^{12} = \frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{x^2-y^2}{r^4}+2\frac{x^2(x^2-3y^2)}{r^6}\right].$$

Vertical force couple (Fig. 2d)

$$\bar{\sigma}_{xx}^{21} = \frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{2xy}{r^4}-4\frac{xy(x^2-y^2)}{r^6}\right],$$

(2.12)
$$\bar{\sigma}_{yy}^{21} = -\frac{1}{4\pi(1-\nu)}\left[(3-2\nu)\frac{2xy}{r^4}-4\frac{xy(x^2-y^2)}{r^6}\right],$$

$$\bar{\sigma}_{xy}^{21} = -\frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{x^2-y^2}{r^4}+2\frac{y^2(3x^2-y^2)}{r^6}\right].$$

Horizontal force dipole (Fig. 2e)

$$\bar{\sigma}_{xx}^{11} = \frac{1}{4\pi(1-\nu)}\left[-(3-2\nu)\frac{x^2-y^2}{r^4}+2y^2\frac{3x^2-y^2}{r^6}\right],$$

(2.13)
$$\bar{\sigma}_{yy}^{11} = \frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{x^2-y^2}{r^4}-2y^2\frac{3x^2-y^2}{r^6}\right],$$

$$\bar{\sigma}_{xy}^{11} = \frac{1}{4\pi(1-\nu)}\left[-2(1-2\nu)\frac{xy}{r^4}-4xy\frac{x^2-y^2}{r^6}\right].$$

Vertical force dipole (Fig. 2f)

$$\bar{\sigma}_{xx}^{22} = -\frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{x^2-y^2}{r^4}-2x^2\frac{x^2-3y^2}{r^6}\right],$$

(2.14)
$$\bar{\sigma}_{yy}^{22} = \frac{1}{4\pi(1-\nu)}\left[(3-2\nu)\frac{x^2-y^2}{r^4}-2x^2\frac{x^2-3y^2}{r^6}\right],$$

$$\bar{\sigma}_{xy}^{22} = -\frac{1}{4\pi(1-\nu)}\left[(1-2\nu)\frac{2xy}{r^4}-4xy\frac{x^2-y^2}{r^6}\right].$$

It is known that the singular behaviour of stresses in the neighbourhood of crack tips as also the crack propagation conditions, energy relase rates etc. are easily expressed in terms of the stress intensity factors $K_{\rm I}^R$, $K_{\rm II}^R$, $K_{\rm I}^L$, $K_{\rm II}^L$, upper indices R, L reffering to the right-hand and left-hand crack tips, and subscripts denoting the corresponding deformation mode.

The stress intensity factors may be expressed by the functions P_1^0 and P_2^0 introduced in Eqs. (2.5) and (2.6):

$$K_I^R = -P_2^0(a)\sqrt{\frac{\pi}{a}}, \quad K_I^L = P_2^0(-a)\sqrt{\frac{\pi}{a}}, \tag{2.15}$$

$$K_{II}^R = -P_1^0(a)\sqrt{\frac{\pi}{a}}, \quad K_{II}^L = P_1^0(-a)\sqrt{\frac{\pi}{a}}. \tag{2.16}$$

Once the stress intensity factors are known, the stresses in the immediate vicinity of crack tips are determined by means of the well-known approximate formulae. Using the notations shown in Fig. 3, i.e. $r^2 = (x-a)^2 + y^2$, and

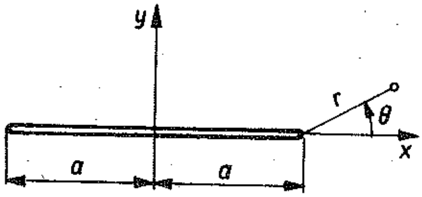

FIG. 3.

$\theta = \sin^{-1}[(x-a)/r]$, the stresses in the neighbourhood of the right-hand crack tip are written in the forms (cf., e.g. [8])

$$\begin{aligned}
\sigma_{xx} &= \frac{K_I^R}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right],\\
\sigma_{yy} &= \frac{K_I^R}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right],\\
\sigma_{xy} &= \frac{K_I^R}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\sigma_{xx} &= \frac{-K_{II}^R}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left[2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right],\\
\sigma_{yy} &= \frac{K_{II}^R}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2},\\
\sigma_{xy} &= \frac{K_{II}^R}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right].
\end{aligned} \tag{2.18}$$

The approximate formulae to be derived in the following sections of this paper will prove to be fairly accurate at a certain distance from the cracks. In order to determine the aproximate values of stresses close to the crack tips, the formulae (2.17) and (2.18) must be used, the values of the stress intensity factors being determined by the approximate analysis presented.

3. Approximate representation of a single crack

In the case of an elastic medium containing several cracks, evaluation of the stress distributions $\sigma^0_{yy}(x, y)$, $\sigma^0_{yx}(x, y)$ used in Eq. (2.4) may prove to be not so simple and leads to the necessity of solving a set of integral equations. This problem was discussed in the unpublished paper by T. A. Pućik [5] who reduced it to numerical analysis. However, as it was shown in [7], in certain cases another relatively simple and fairly accurate and effective procedure is possible. The procedure is applicable to such cases in which the crack distribution is not too dense, and namely the distances between the cracks are larger than their lengths.

The procedure is based upon a certain generalization of the Saint Venant principle. To illustrate the approach consider the simple case of the Mode I crack deformation under tension $T_2(x)$ applied at $\pm\infty$, Fig. 1a. Assuming the double force density to be known, the resulting stress distribution is written according to Eq. (2.8):

$$\sigma_{ij} = \sigma^0_{ij}(x, y) + \int_{-a}^{a} \varrho^{22}(\xi)\,\bar{\sigma}^{22}_{ij}(x-\xi, y)\,d\xi + \int_{-a}^{a} \varrho^{11}(\xi)\,\bar{\sigma}^{11}_{ij}(x-\xi, y)\,d\xi. \tag{3.1}$$

Since

$$\varrho^{11}(x) = \frac{\nu}{1-\nu}\,\varrho^{22}(x),$$

Equation (3.1) may be rewritten in a simplified form:

$$\sigma_{ij} = \sigma^0_{ij}(x, y) + \int_{-a}^{a} \varrho(\xi)\,\hat{\sigma}_{ij}(x-\xi, y)\,d\xi \tag{3.2}$$

with the notations

$$\varrho(\xi) = \varrho^{22}(\xi), \qquad \hat{\sigma}_{ij} = \bar{\sigma}^{22}_{ij} + \frac{\nu}{1-\nu}\,\bar{\sigma}^{11}_{ij}.$$

The kernel $\hat{\sigma}_{ij}(x-\xi, y)$ in Eq. (3.2) is now expanded (under the usual conditions) into the power series of ξ:

$$\hat{\sigma}_{ij}(x-\xi, y) = \hat{\sigma}_{ij}(x, y) - \frac{\xi}{1!}\frac{\partial \hat{\sigma}_{ij}(x, y)}{\partial x} + \frac{\xi^2}{2!}\frac{\partial^2 \hat{\sigma}_{ij}(x, y)}{\partial x^2} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{\partial^n \hat{\sigma}_{ij}(x, y)}{\partial x^n}. \tag{3.3}$$

Substitution of Eq. (3.3) into Eq. (3.2) yields

$$(3.4)\qquad \sigma_{ij}(x,y)=\sigma^0_{ij}(x,y)+\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\int_{-a}^{a}\xi^n\varrho(\xi)\,d\xi\right]\frac{\partial^n\hat{\sigma}_{ij}(x,y)}{\partial x^n}.$$

The above formula represents a particular case of a more general formula describing the stress field produced in an infinite medium by arbitrarily distributed forces $p_1(x)$, $p_2(x)$ applied along the segment $-a<x<a$, $y=0$. If $\sigma^0_{ij}\equiv 0$, then

$$\sigma_{ij}(x,y)=\int_{-a}^{a}p_1(\xi)\,\bar{\sigma}^1_{ij}(x-\xi,y)\,d\xi,+\int_{-a}^{a}p_2(\xi)\,\sigma^2_{ij}(x-\xi,y)\,d\xi$$

and, using the expansion (3.3), the stress asumes the form

$$(3.5)\qquad \sigma_{ij}(x,y)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\int_{-a}^{a}p_1(\xi)\,\xi^n\,d\xi\right]\frac{\partial^n\bar{\sigma}^1_{ij}(x,y)}{\partial x^n}+$$
$$+\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\int_{-a}^{a}p_2(\xi)\,\xi^n\,d\xi\right]\frac{\partial^n\bar{\sigma}^2_{ij}(x,y)}{\partial x^n}.$$

However, according to Eq. (2.1),

$$(3.6)\qquad \begin{aligned} -\frac{\partial\bar{\sigma}^1_{ij}}{\partial x}&=\bar{\sigma}^{11}_{ij}, & -\frac{\partial\bar{\sigma}^{11}_{ij}}{\partial x}&=\bar{\sigma}^{111}_{ij},\\ -\frac{\partial\bar{\sigma}^2_{ij}}{\partial x}&=\bar{\sigma}^{21}_{ij}, & -\frac{\partial\bar{\sigma}^{21}_{ij}}{\partial x}&=\bar{\sigma}^{211}_{ij}. \end{aligned}$$

Let us use the first six terms of the expansion (3.5)

$$(3.7)\qquad \sigma_{ij}(x,y)=[\int_{-a}^{a}p_1(\xi)\,d\xi]\,\bar{\sigma}^1_{ij}+[\int_{-a}^{a}\xi p_1(\xi)\,d\xi]\,\bar{\sigma}^{11}_{ij}+$$
$$+\frac{1}{2}[\int_{-a}^{a}\xi^2 p_1(\xi)\,d\xi]\,\sigma^{111}_{ij}+[\int_{-a}^{a}p_2(\xi)\,d\xi]\,\bar{\sigma}^2_{ij}+[\int_{-a}^{a}\xi p_2(\xi)\,d\xi)\,\bar{\sigma}^{21}_{ij}+$$
$$+\frac{1}{2}[\int_{-a}^{a}\xi^2 p_2(\xi)\,d\xi]\,\sigma^{211}_{ij}.$$

In the right-hand expansion of Eq. (3.7), the first term of the upper line and two terms of the lower line may also be written as

$$\mathscr{P}_1\,\bar{\sigma}^1_{ij}(x,y)$$

and

$$\mathscr{P}_2\,\bar{\sigma}^2_{ij}(x,y)+\mathscr{M}_2\,\bar{\sigma}^{21}_{ij}(x,y).$$

It is easily seen that they represent the action of resultant forces $\mathscr{P}_1$, $\mathscr{P}_2$ and couple $\mathscr{M}_2$ of the loads $p_1(x)$, $p_2(x)$ applied to the segment $2a$. In a rigid body they are statically equivalent to the distributed loads and suffice for the determination of static equilibrium of the system. The remaining terms of the expansion (3.7) are the necessary corrections which must be taken into account in discussing the behaviour of a deformable medium.

Let us return to Eq. (3.4) and write explicitly the first three terms of the expansion

$$\sigma_{ij}(x, y) \approx \sigma^0_{ij}(x, y) + R_0\, \hat{\sigma}_{ij}(x, y) - \frac{1}{1!} R_1 \frac{\partial \hat{\sigma}_{ij}(x, y)}{\partial x} + \frac{1}{2!} R_2 \frac{\partial \hat{\sigma}_{ij}(x, y)}{\partial x^2}. \tag{3.8}$$

Here

$$R_0 = \int_{-a}^{a} \varrho(\xi)\, d\xi, \quad R_1 = \int_{-a}^{a} \xi \varrho(\xi)\, d\xi, \quad R_2 = \int_{-a}^{a} \xi^2 \varrho(\xi)\, d\xi. \tag{3.9}$$

In order to stimate the accuracy of the representations (3.8) and (3.9), let us consider the well-known particular case of uniform tension $T_2 = \text{const}$ applied to an infinite elastic medium containing a single crack. The accurate, closed form solution to the problem is known [9]. Since in such a case $\sigma^0_{yy}(x, 0) = p^0(x) = T_2$ (cf. Eq. (2.4)), from Eq. (2.5) it follows that

$$\begin{aligned} P^0_2(\xi) &= \frac{T_2}{\pi} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}}{t - \xi}\, dt = -T_2\, \xi, \\ \varrho^{22}(x) &= -\frac{4(1-\nu)}{1-2\nu} T_2 \int_{-a}^{x} \frac{\xi d\xi}{\sqrt{a^2 - \xi^2}} = \frac{4(1-\nu)^2}{1-2\nu} T_2 \sqrt{a^2 - x^2}. \end{aligned} \tag{3.10}$$

Consequently, the corresponding values of R_0, R_1, R_2 of Eq. (3.9) are calculated,

$$\begin{aligned} R_0 &= 2\pi T_2\, a^2 \frac{(1-\nu)^2}{1-2\nu}, \qquad R_1 = 0, \\ R_2 &= \frac{1}{2} \pi T_2\, a^4 \frac{(1-\nu)^2}{1-2\nu}, \end{aligned} \tag{3.11}$$

and $\hat{\sigma}_{ij}$ is determined by means of Eqs. (2.13) and (2.14):

$$\bar{\sigma}_{yy} = \bar{\sigma}^{22}_{yy} + \frac{\nu}{1-\nu} \bar{\sigma}^{11}_{yy} = \frac{1-2\nu}{4\pi(1-\nu)^2} \frac{x^4 + 6x^2 y^2 - 3y^4}{(x^2+y^2)^3}. \tag{3.12}$$

Differentiating Eq. (3.12) twice with respect to x, the approximate formula (3.8) takes the form

$$\sigma_{yy}(x, y) \approx T_2 \left(1+\frac{a^2}{2} \frac{x^4+6x^2 y^2-3y^4}{(x^2+y^2)^3} - \frac{3}{8} a^4 \frac{x^6+15x^4 y^2-45x^2 y^4+5y^6}{(x^2+y^2)^5}\right). \tag{3.13}$$

The formula (3.13) representing the truncated series (3.4) holds approximately true for points (x, y) lying outside the region of divergence of that series, i.e. outside the circle $x^2+y^2 = a^2$. The accuracy of the formula (3.13) is also low in the vicinity of crack tips $x = \pm a, y = 0$ where the near-tip expansions (2.17) should be used. Let us, for instance, analyze the points lying along the x-axis, i.e. the points $x > a, y = 0$. The known accurate ormula yields the σ_{yy} stresses

$$\sigma_{yy}(x, 0) = T_2 \frac{x}{\sqrt{x^2-a^2}}, \tag{3.14}$$

while the two consecutive approximations following from Eq. (3.13) are

$$\sigma_{yy} \approx T_2 \left(1+\frac{1}{2} \frac{a^2}{x^2}\right), \tag{3.15}$$

$$\bar{\sigma}_{yy} = T_2 \left(1+\frac{1}{2} \frac{a^2}{x^2}+\frac{3}{8} \frac{a^4}{x^4}\right). \tag{3.16}$$

The near-tip solution (2.17) has for $\theta = 0, r = x$, the form

$$\sigma_{yy} \approx \frac{T_2}{\sqrt{2(x-a)}}. \tag{3.17}$$

The ranges of applicability of various approximations may be estimated by comparing the values given in Table 1 under the assumption that $T_2 = 1$.

The near-tip expansion (3.17) is seen to yield satisfactory results in the close vicinity of the crack tip $0 < x-a < 0.05a$, while the asymptotic expansions exhibit the accuracy of 99% (or better) for $x-a > 1.5a$ and $x-a > 0.8a$, respectively.

4. Two parallel cracks, Mode I crack deformation

Consider now an infinite medium containing two parallel cracks of lengths $2a_1$, $2a_2$ located symmetrically with respect to the y-axis (Fig. 4) at the distance of h from each other. Let us assume that the external loads are such that both cracks are deformed according to Mode I (normal loads are symmetric, and shearing loads — antisymmetric with

Table 1. Various approximations of stress σ_{yy} along the x-axis (Fig. 1a) in a plane subject to uniform tension $T_2 = 1$ and containing a crack of length $2a$.

x/a	Eq. (3.14)	Eq. (3.17)	Eq. (3.15)	Eq. (3.16)
1.001	22.377	22.361		
1.01	7.124	7.071		
1.05	3.280	3.162		
1.10	2.400	2.236		
1.20	1.809	1.581		
1.30	1.565			
1.40	1.429		1.255	1.353
1.50	1.342		1.222	1.296
2.00	1.155		1.125	1.148
3.00	1.061		1.056	1.061
5.00	1.021		1.020	1.021
10.00	1.005		1.005	1.005

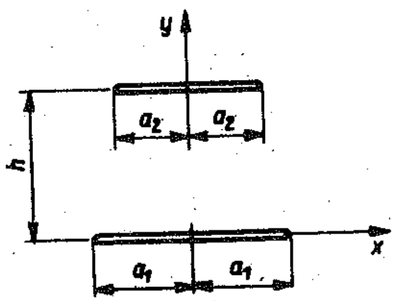

FIG. 4.

respect to the axis). In order to calculate the double force intensities $\varrho^{22}(x)$, $\varrho^{11}(x)$, Eq. (2.5), and the stress intensity factors, Eq. (2.16), for the crack $-a_1 < x < a_1, y = 0$ (Fig. 4), let us first determine the function $P_2^0(x)$. Due to the existence of the other crack, however, the function $p_2^0(x)$ defined in Eq. $(2.4)_1$ appearing in the integrand of Eq. (2.5) represents the stresses measured along the segment $-a_1 < x < a_1, y = 0$ of the plane, produced not only by the external tractions T (Fig. 1a) but also by the other crack $2a_2$; the latter crack is now represented by suitably distributed double forces ϱ^{22}, ϱ^{11} along the segment $-a_2 < x < a_2$, $y = h$. The forces are unknown as yet but let us make the assumption that the distance between both cracks is large enough (as compared with the crack lengths $2a_1$, $2a_2$) to enable us to write the function $p_2^0(x)$ in the form

$$p_2^0(x) = p_2^{0T}(x) + q_2^0(x),$$

$$q_2^0(x) = q_0 + q_1 \frac{x}{a_1} + q_2 + q_2\left(1 - \frac{x^2}{a_1^2}\right). \tag{4.1}$$

Here $p_2^{0T}(x)$ are the stresses $\sigma_{yy}(x, 0)$ produced by the external tractions T, and q_0, q_1, q_2 — unknown parameters of expansion of the stresses produced by the other crack $2a_2$ into a polynomial series. It is obvious that (cf. Fig. 5)

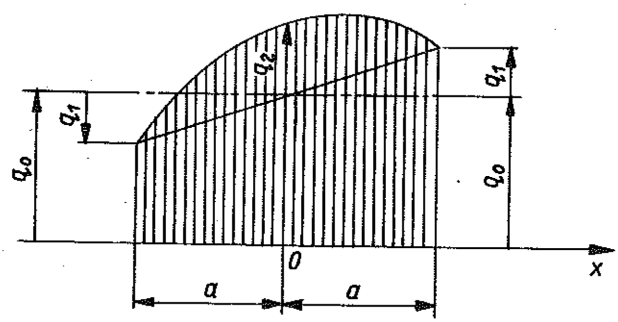

FIG. 5.

$$q_2^0(-a) = q_0 - q_1, \quad q_2^0(0) = q_0 + q_2, \quad q_2^0(a) = q_0 + q_1.$$

Substituting the expression (4.1) into the formulae (2.5) and (2.15), we obtain the values of $P_2(x)$, $\varrho^{22}(x)$, K_I^R and K_I^L produced by the unknown double forces distributed along the second segment $2a_2$. The following results are obtained:

$$P_2^0(x) = -q_0 a_1 \frac{x}{a_1} + q_1 a_1 \left(\frac{1}{2} - \frac{x^2}{a_1^2}\right) - \frac{q_2 a_1}{2}\left(3\frac{x}{a_1} - 2\frac{x^3}{a_1^3}\right); \tag{4.2}$$

$$\varrho^{22}(x) = \frac{4(1-\nu)^2}{1-2\nu}\left[q_0 a_1 \sqrt{1-\frac{x^2}{a_1^2}} + \frac{1}{2} q_1 a_1 \frac{x}{a_1}\sqrt{1-\frac{x^2}{a_1^2}} + \frac{1}{6} q_2 a_1 \left(5 - 2\frac{x^2}{a_1^2}\right)\sqrt{1-\frac{x^2}{a_1^2}}\right]. \tag{4.3}$$

The stress intensity factors are expressed in terms of q_0, q_1, q_2 in the following simple form (Eqs. (2.15) and (2.16)):

$$(4.4) \qquad \begin{aligned} K_I^R &= \left(q_0 + \frac{1}{2} q_1 + \frac{1}{2} q_2\right)\sqrt{\pi a_1}, \\ K_I^L &= \left(q_0 - \frac{1}{2} q_1 + \frac{1}{2} q_2\right)\sqrt{\pi a_1}. \end{aligned}$$

The complete values of P_2^0, ϱ^{22}, should also contain the contributions of external loads T applied to the infinite medium (the first right-hand term of Eq. $(4.1)_1$). Similar fomulae must also be written for the second crack.

Let us now return to the formulae (3.8) and (3.9). The stress field produced in an infinite plane by the loads $q_2^0(x)$ (4.1) distributed along the segment $-a_1 < x < a_1, y = 0$ may be written in the approximate form (3.8) resulting from the expansion (3.3). Normal stress $\sigma_{yy}(x, y)$ is found to have the form (for simplicity the sign $\approx$ is replaced with $=$)

$$(4.5) \qquad \sigma_{yy}(x, y) = \sigma_{yy}^0(x, y) + R_0\, \hat{\sigma}_{yy}(x, y) - R_1 \frac{\partial \hat{\sigma}_{yy}(x, y)}{\partial x} + \frac{1}{2} R_2 \frac{\partial^2 \hat{\sigma}_{yy}(x, y)}{\partial x^2}.$$

Here $\sigma_{yy}^0(x, y)$ is the stress produced in a solid body (without the cracks) by external tractions $\hat{\sigma}_{yy}(x, y)$ is given by Eq. (2.14), and R_0, R_1, R_2 follow from the formulae (3.9) into which $\varrho^{22}(x)$ (given by Eq. (4.3)) should be substituted for $\varrho(x)$. The necessary substitutions yield

$$(4.6) \qquad \sigma_{yy}(x, y) = \sigma_{yy}^0(x, y) + \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right)(\xi^4 + 6\xi^2\eta^2 - 3\eta^4)\varrho^{-6} + \frac{1}{8} q_1 (\xi^5 + 10\xi^3\eta^2 - 15\xi\eta^4)\varrho^{-8} + \left(\frac{3}{8} q_0 + \frac{1}{4} q_2\right)(\xi^6 + 15\xi^4\eta^2 - 45\xi^2\eta^4 + 5\eta^6)\varrho^{-10}.$$

In this formula ξ, η, ϱ are dimensionless coordinates,

$$\xi = x/a_1, \qquad \eta = y/a_1, \qquad \varrho^2 = \xi^2 + \eta^2.$$

The order of consecutive terms of the expansion (4.6) decreases and in practical applications the first two terms of the series lead to satisfactory results; under this assumption the formula (4.6) may be written in a simplified form. Together with the remaining two stresses, the following results are obtained:

$$(4.7) \qquad \sigma_{yy} = \sigma_{yy}^0 + \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right)(\xi^4 + 6\xi^2\eta^2 - 3\eta^4)\varrho^{-6} + \frac{1}{8} q_1 (\xi^5 + 10\xi^3\eta^2 - 15\xi\eta^4)\varrho^{-8},$$

$$(4.8) \qquad \sigma_{xx} = \sigma_{xx}^0 + \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right)(\xi^4 - 6\xi^2\eta^2 + \eta^4)\varrho^{-6} + \frac{1}{8} q_1 (\xi^5 - 14\xi^3\eta^2 + 9\xi\eta^4)\varrho^{-8},$$

(4.9) $$\sigma_{xy} = \sigma^0_{xy} + \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right) (2\xi^3 \eta - 6\xi\eta^3) \varrho^{-6} + \frac{1}{8} q_1 (3\xi^4 \eta - 18\xi^2 \eta^3 + 3\eta^5) \varrho^{-8}.$$

These formulae may be used in determining the interaction of cracks deforming according to Mode I.

In the case shown in Fig. 4 let us assume, for the sake of simplicity, the external loads to fulfill the symmetry conditions

$$\sigma^0_{yy}(x, y) = \sigma^0_{yy}(-x, y), \quad \sigma^0_{xy}(x, y) = -\sigma^0_{xy}(-x, y).$$

This assumption leads to the result $q_1 = 0$ as it is seen from the inspection of Eqs. (4.3) and (3.9).

Let us now assume that the distribution of normal stresses $\sigma_{yy}(x, 0)$ along the segment $-a_1 < x < a_1$ is written in the form of two terms of order zero and two:

(4.10) $$p^0_2(x) = q_0 + q_2\left(1 - \frac{x^2}{a_1^2}\right).$$

Then the combined action of external loads T and of the (unknown) distribution of double forces along the crack $2a_1$ produces the following stress field:

(4.11) $$\sigma_{yy}(x, h) = \sigma^0_{yy}(x, h) + \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right) a_1^2 \frac{x^4 + 6x^2 h^2 - 3h^4}{(x^2 + h^2)^3}.$$

This formula holds approximately true at a certain distance from the origin of the coordinate system, outside the circle $x^2 + y^2 = a^2$.

At the center of the other crack, $x = 0$, $y = h$, the stress (4.11) yields

(4.12) $$\sigma_{yy}(0, h) = \sigma^0_{yy}(0, h) - \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right) \frac{3a_1^2}{h^2},$$

and at its ends, $x = \pm a_2, y = h$,

(4.13) $$\sigma_{yy}(\pm a_2, h) = \sigma^0_{yy}(\pm a_2, h) - \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right) \frac{3h^4 - 6h^2 a_2^2 - a_2^4}{(a_2^2 + h^2)^3}.$$

These values may be denoted by $q'_0 + q'_2$ and q'_0, respectively, what is evident from the inspection of Fig. 6. Equations (4.12) and (4.13) make it possible to write the first set of wo equations with four unknowns q_0, q_2, q'_0, q'_2:

(4.14) $$\sigma^0_{yy}(0, h) - \frac{3}{2} q_0 \alpha_1^2 - \frac{9}{8} q_2 \alpha_1^2 = q'_0 + q'_2,$$

(4.15) $$\sigma^0_{yy}(0, h) - \frac{3}{2} q_0 F(\alpha_2) \alpha_1^2 - \frac{9}{8} q_2 F(\alpha_2) \alpha_1^2 = q'_0.$$

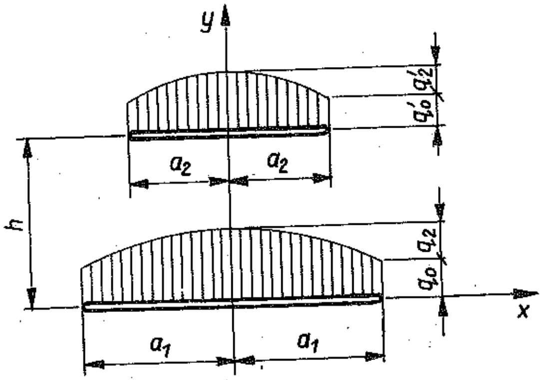

FIG. 6.

Here

$$\alpha_i = a_i/h, \qquad F(\alpha) = \frac{1-2\alpha^2-\alpha^4/3}{(1+\alpha^2)^3}. \tag{4.16}$$

The remaining two equations are obtained by expressing the values of $p_2^0(x)$ along the first crack $-a_1 < x < a_1$, $y = 0$ in terms of T and the double forces distributed along the second crack:

$$\sigma_{yy}^0(0,0) - \frac{3}{2} q_0' \alpha_2^2 - \frac{9}{8} q_2' \alpha_2^2 = q_0 + q_2, \tag{4.17}$$

$$\sigma_{yy}^0(a_1,0) - \frac{3}{2} q_0' F(\alpha_1) \alpha_2^2 - \frac{9}{8} q_2' F(\alpha_1) \alpha_2^2 = q_0. \tag{4.18}$$

This procedure may easily be generalized to the case of an arbitrary number of N cracks parallel to each other and symmetric with respect to the y-axis (Fig. 7). The lengths of the cracks are denoted by $2a_k$, $k = 1, 2, \ldots, N$, and their mutual distances — by h_{kn}. The set of $2N$ equations is written in the form

$$\sigma_{yy}^0(0, h_{1m}) - \sum_{\substack{k=1 \\ k \neq m}}^{N} \left(\frac{3}{2} q_0^k + \frac{9}{8} q_2^k \right) \alpha_{km}^2 = q_0^m + q_2^m, \tag{4.19}$$

$$\sigma_{yy}^0(a_m, h_{1m}) - \sum_{\substack{k=1 \\ k \neq m}}^{N} \left(\frac{3}{2} q_0^k + \frac{9}{8} q_2^k \right) \alpha_{km}^2 F(\alpha_{km}) = q_0^m. \tag{4.20}$$

Here $\alpha_{kn} = a_k/h_{kn}$. The solution of the system of Eqs. (4.19) and (4.20) yields the values of the parameters q_0^k, q_2^k which in turn enable the determination of all stress intensity factors and of the approximate stress fields outside the cracks.

In order to estimate the accuracy of the procedure leading to Eqs. (4.19), (4.20), (4.14)—(4.18), let us consider the simple case of two equal and

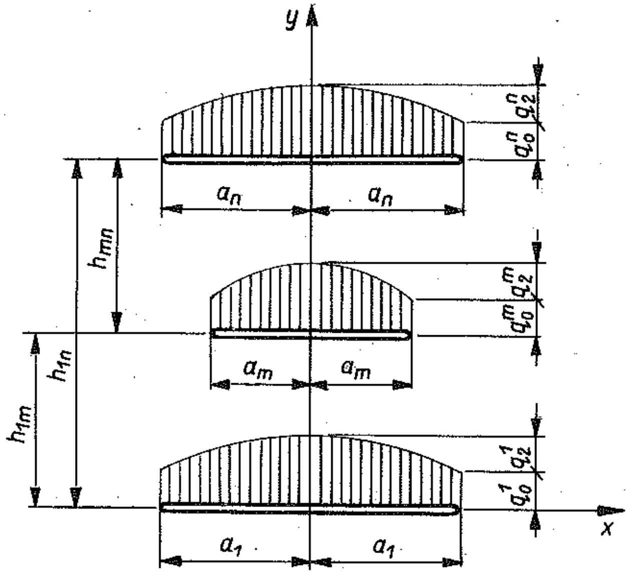

FIG. 7.

parallel cracks of lengths $2a$, located at the distance of h from each other. Let the infinite plane be loaded by uniform tension $T_2 = \text{const}$ at infinity (Fig. 8). The set of Eqs. (4.14) — (4.18) is then reduced to two equations with two unknowns ($q_0' = q_0$, $q_2' = q_2$):

(4.21)
$$q_0\left[1+\frac{3}{2}\alpha^2 F(\alpha)\right]+\frac{9}{8} q_2 \alpha^2 F(\alpha) = T_2,$$
$$q_0\left(1+\frac{3}{2}\alpha^2\right)+q_2\left(1+\frac{9}{8}\alpha^2\right) = T_2.$$

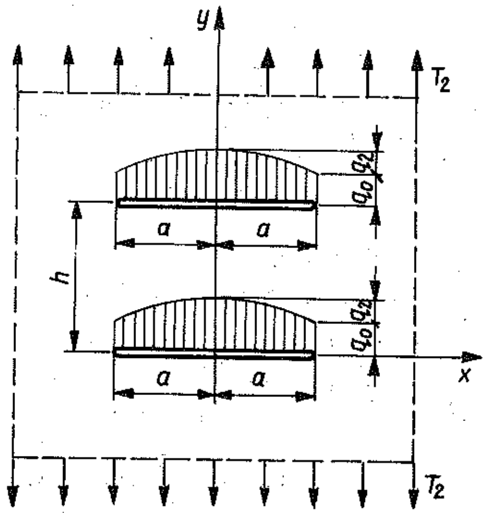

FIG. 8.

The solutions

$$q_0 = \frac{1+\frac{9}{8}\alpha^2[1-F(\alpha)]}{1+\frac{9}{8}\alpha^2+\frac{3}{8}\alpha^2 F(\alpha)} T_2, \quad q_2 = -\frac{\frac{3}{2}\alpha^2[1-F(\alpha)]}{1+\frac{9}{8}\alpha^2+\frac{3}{8}\alpha^2 F(\alpha)} T_2$$

are now used to determine the stress intensity factors (all the factors are identical due to full symmetry) from Eqs. (4.4),

$$K_{\mathrm{I}} = \frac{1+\frac{3}{8}\alpha^2[1-F(\alpha)]}{1+\frac{9}{8}\alpha^2+\frac{3}{8}\alpha^2 F(\alpha)} T_2\sqrt{\pi a}. \tag{4.22}$$

The formulae derived may be compared with the results obtained by means of two other, slightly modified procedures a) and b), one of them more accurate, and one based on certain additional simplifications.

a) Let us return to Eqs. (3.8) and (4.6)—(4.9). In the formula (4.7) for σ_{yy} the last terms of expansions (3.8) and (4.6) were disregarded. Taking now the term multiplied by R_2 into account and using the complete formula (4.6) (with $q_1 = 0$), the modified set of Eqs. (4.14) and (4.15) assumes the form

$$q_0'+q_2' = \sigma_{yy}^0(0,h) - q_0\left(\frac{3}{2}\alpha_1^2 - \frac{15}{8}\alpha_1^4\right) - q_2\left(\frac{9}{8}\alpha_1^2 - \frac{5}{4}\alpha_1^4\right); \tag{4.23}$$

$$q_0' = \sigma_{yy}^0(a_2,h) - q_0\left[\frac{3}{2}\alpha_1^2 F(\alpha_2) - \frac{15}{8}\alpha_1^4 H(\alpha_2)\right] - q_2\left[\frac{9}{8}\alpha_1^2 F(\alpha_2) - \frac{5}{4}\alpha_1^4 H(\alpha_2)\right]. \tag{4.24}$$

Here, in addition to Eq. (4.16), another notation is introduced:

$$H(\alpha) = \frac{1-9\alpha^2+3\alpha^4+\alpha^6/5}{(1+\alpha^2)^5}.$$

In the case of two equal cracks $a_1 = a_2 = a = h$, Eqs. (4.23) and (4.24) yield the solutions for $q_0 = q_0'$ and $q_2 = q_2'$, $\sigma_{yy}^0 = T$

$$q_0 = \frac{1+\frac{9}{8}\alpha^2(1-F)-\frac{5}{4}\alpha^4(1-H)}{\Delta} T,$$

(4.25)

$$q_2 = \frac{\frac{3}{2}\alpha^2(1-F)-\frac{15}{8}\alpha^4(1-H)}{\Delta} T,$$

with

$$F = F(\alpha), \quad H = H(\alpha),$$
$$\Delta = 1+\frac{3}{2}\alpha^2\frac{3+F}{4}-\frac{15}{8}\alpha^4\frac{2+H}{3}+\frac{15}{64}\alpha^6(F-H).$$

The corresponding value of the stress intensity factor $K_I^L = K_I^R = K_I$ is

$$K_I = T\sqrt{\pi a}\,\frac{1+\frac{3}{8}\alpha^2(1-F)-\frac{5}{16}\alpha^4(1-H)}{\Delta}. \tag{4.26}$$

b) A certain simplification of the procedures outlined above may be achieved by assuming the function $q_2^0(x)$ in Eq. (4.1) to be constant, $q_2^0(x) = \bar{q}_0$, the other parameter q_2 being usually much smaller than q_0. Equation (4.11) yields, with $\sigma_{yy}^0(x, y) = T_2$, the formula

$$\sigma_{yy}(x, h) = T_2 - \frac{3}{2}\bar{q}_0\,\alpha_1^2\,\frac{1-2x^2/h^2-x^4/3h^4}{(1+x^2/h^2)^3}. \tag{4.27}$$

The mean value σ_{yy}^M of $\sigma_{yy}(x, h)$ over the segment $-a_2 < x < a_2$ is given by the integral formula

$$\sigma_{yy}^M = \frac{1}{2a_2}\int_{-a_2}^{a_2}\sigma_{yy}(x, h)\,dx. \tag{4.28}$$

By substituting the expression (4.27) for the integrand in Eq. (4.28) and taking into account that

$$\frac{1}{2a_2}\int_{-a_2}^{a_2}\frac{1-2x^2/h^2-x^4/3h^4}{(1+x^2/h^2)^3}\,dx = \frac{1+\alpha_2^2/3}{(1+\alpha_2^2)^2},$$

the required mean value is obtained:

$$\sigma_{yy}^M = T_2 - \frac{1}{2}\bar{q}_0\,\alpha_1^2\,\frac{3+\alpha_2^2}{(1+\alpha_2^2)^2}. \tag{4.29}$$

Since, on the other hand, this value may be denoted by $\bar{q}_0'$, and a similar reasoning may be repeated with respect to the other crack, a simple set of two equations is obtained:

$$\begin{aligned} \bar{q}_0' &= T_2 - \frac{1}{2}\bar{q}_0\,\alpha_1^2\,\frac{3+\alpha_2^2}{(1+\alpha_2^2)^2}, \\ \bar{q}_0 &= T_2 - \frac{1}{2}\bar{q}_0'\,\alpha_2^2\,\frac{3+\alpha_1^2}{(1+\alpha_1^2)^2}, \end{aligned} \tag{4.30}$$

enabling the determination of the corresponding approximate values of the stress intensity factors.

In the case of equal cracks $a_1 = a_2 = a$, $\bar{q}_0$ is found from a single equation:

$$\bar{q}_0 = \frac{T_2}{1+\frac{3}{2}\alpha^2 \frac{1+\alpha^2/3}{(1+\alpha^2)^2}}, \tag{4.31}$$

and the stress intensity factors $K_I^L = K_I^R = K_I$ are

$$K_I = \frac{T_2\sqrt{\pi a}}{1+\frac{3}{2}\alpha^2 \frac{1+\alpha^2/3}{(1+\alpha^2)^2}}. \tag{4.32}$$

Numerical values of the approximate formulae (4.32), (4.22) and (4.26) for $0.05 < \alpha < 0.30$ are given in Table 2. The results confirm the applicability of the simplified approach b), Eq. (4.32).

Table 2. Various approximations of the stress intensity factors $K_I/T_2\sqrt{\pi a}$ for two equal cracks (Fig. 8) in a plane subject to uniform tension T_2.

α	(4.32)	(4.22)	(4.26)
0.05	0.9963	0.9963	0.9963
0.10	0.9855	0.9856	0.9857
0.15	0.9685	0.9691	0.9697
0.20	0.9469	0.9484	0.9506
0.25	0.9219	0.9255	0.9300
0.30	0.8952	0.9019	0.9098

5. Two collinear cracks, Mode I

Let us now consider the case of an infinite elastic body containing two collinear cracks $|x| < a_1$, $|x-l| < a_2$ shown in Fig. 9. The resulting stress field cannot be symmetric with respect to the centers of the cracks, and hence the load $q_2^0(x)$ in Eq. (4.1) must be assumed in the form containing the parameter q_1. In order to limit the number of equations, let us assume that $q_2 = 0$ and write the function $p_2^0(x)$ in the linear form:

$$p_2^0(x) = p_2^{0T}(x) + q_0 + q_1 \frac{x}{a_1}. \tag{5.1}$$

The stress σ_{yy} (4.7) is written in the form

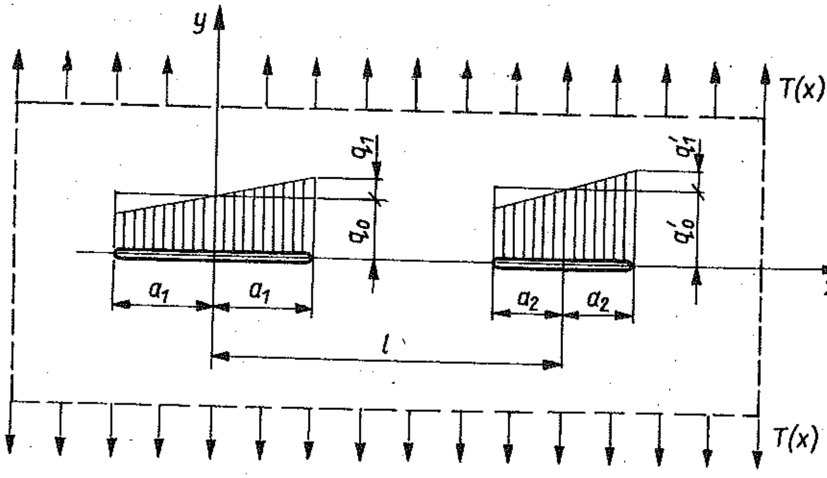

FIG. 9.

(5.2) $$\sigma_{yy}(\xi_1,\eta_1)=\sigma^0_{yy}(\xi_1,\eta_1)+\frac{1}{2}q_0\frac{\xi_1^4+6\xi_1^2\eta_1^2-3\eta_1^4}{\varrho_1^6}+\frac{1}{8}q_1\frac{\xi_1^5+10\xi_1^3\eta_1^2-15\xi_1\eta_1^4}{\varrho_1^8}.$$

Here

$$\xi_1=x/a_1,\quad \eta_1=y/a_1,\quad \varrho_1^2=\xi_1^2+\eta_1^2.$$

Equation (5.2) represents the stress field produced by external loads T and a central crack $2a_1$ for sufficiently large distances from the center of the crack, $\varrho_1>1$. The parameters q_0, q_1 depend on the external load T and upon the other crack at which their counterparts are q_0' and q_1', Fig. 9. It is seen that the stress (5.2) at the center of the other crack, i.e. at $x=l$, $y=0$, should be denoted by q_0'; with the notation $\lambda_1=l/a_1$, the first equation has the form

(5.3) $$\sigma_{yy}(\lambda_1,0)=\sigma^0_{yy}(\lambda_1,0)+\frac{q_0}{2\lambda_1^2}+\frac{q_1}{8\lambda_1^3}=q_0'$$

while the other parameter q_1' is defined as he mean slope of stresses (5.2) along the segment $|x-l|<a_2$,

$$q_1'=\frac{1}{2}[\sigma_{yy}(l+a_2)-\sigma_{yy}(l-a_2)]$$

what yields the second equation

(5.4) $$q_1'=\frac{1}{2}[\sigma^0_{yy}(\lambda_1+\alpha_{21})-\sigma^0_{yy}(\lambda_1-\alpha_{21})]-\frac{1}{4}q_0\left[\frac{1}{(\lambda_1-\alpha_{21})^2}-\frac{1}{(\lambda_1+\alpha_{21})^2}\right]-\frac{1}{16}q_1\left[\frac{1}{(\lambda_1-\alpha_{21})^3}-\frac{1}{(\lambda_1+\alpha_{21})^3}\right].$$

Here $\alpha_{21}=a_2/a_1$.

Repeating the same procedure with respect to the other crack, another set of two equations is obtained. The complete set of four equations with four unknown parameters q_0, q_1, q_0', q_1' may be put in the form

$$
\begin{aligned}
&-q_0/2\lambda_1^2-q_1/8\lambda_1^3+q_0' = \sigma_{yy}^0(\lambda_1,0),\\
&q_0\frac{1}{4}\left[\frac{1}{(\lambda_1-\alpha_{21})^2}-\frac{1}{(\lambda_1+\alpha_{21})^2}\right]+\frac{q_1}{16}\left[\frac{1}{(\lambda_1-\alpha_{21})^3}-\frac{1}{(\lambda_1+\alpha_{21})^3}\right]+\\
&\qquad +q_1' = \frac{1}{2}\left[\sigma_{yy}^0(\lambda_1+\alpha_{21},0)-\sigma_{yy}^0(\lambda_1-\alpha_{21},0)\right],\\
&q_0-q_0'/2\lambda_2^2+q_1'/8\lambda_2^3 = \sigma_{yy}^0(0,0),\\
&q_1-\frac{q_0'}{4}\left[\frac{1}{(\lambda_2-\alpha_{12})^2}-\frac{1}{(\lambda_2+\alpha_{12})^2}\right]+\frac{q_1'}{16}\left[\frac{1}{(\lambda_2-\alpha_{12})^3}-\right.\\
&\qquad \left.-\frac{1}{(\lambda_2+\alpha_{12})^3}\right] = \frac{1}{2}\left[\sigma_{yy}^0(\alpha_{12},0)-\sigma_{yy}^0(-\alpha_{12},0)\right].
\end{aligned}
\tag{5.5}
$$

In the case of equal cracks, $a_1 = a_2$, $\alpha_{12} = 1$, $\lambda_1 = \lambda_2$ and the load symmetric with respect to the vertical line $x = l/2$, the set of Eqs. (5.5) is reduced to two equations for $q_0' = q_0$ and $q_1' = q_1$:

$$
\begin{aligned}
&\left(1-\frac{1}{2\lambda^2}\right)q_0-\frac{1}{8\lambda^3}q_1 = \sigma_{yy}^0(0,0),\\
&-\frac{1}{4}\left[\frac{1}{(\lambda-1)^2}-\frac{1}{(\lambda+1)^2}\right]q_0+\left\{1-\frac{1}{8}\left[\frac{1}{(\lambda-1)^3}-\frac{1}{(\lambda+1)^3}\right]\right\}q_1 =\\
&\qquad = \frac{1}{2}\left[\sigma_{yy}^0(1,0)-\sigma_{yy}^0(-1,0)\right]
\end{aligned}
\tag{5.6}
$$

with the solutions

$$
\begin{aligned}
q_0 &= \frac{T_0}{\Delta}\left\{1-\frac{1}{8}\left[\frac{1}{(\lambda-1)^3}-\frac{1}{(\lambda+1)^3}\right]\right\}+\frac{S_0}{\Delta}\frac{1}{8\lambda^3},\\
q_1 &= \frac{T_0}{4\Delta}\left[\frac{1}{(\lambda-1)^2}-\frac{1}{(\lambda+1)^2}\right]+\frac{S_0}{\Delta}\left(1-\frac{1}{2\lambda^2}\right).
\end{aligned}
\tag{5.7}
$$

Here

$$
T_0 = \sigma_{yy}^0(0,0), \qquad S_0 = \left[\sigma_{yy}^0(1,0)-\sigma_{yy}^0(-1,0)\right]/2,
$$

$$
\Delta = \left(1-\frac{1}{2\lambda^2}\right)\left\{1-\frac{1}{8}\left[\frac{1}{(\lambda-1)^3}-\frac{1}{(\lambda+1)^3}\right]\right\}-\frac{1}{32\lambda^3}\left[\frac{1}{(\lambda-1)^2}-\frac{1}{(\lambda+1)^2}\right].
$$

For sufficiently large values of λ (say, $\lambda > 4$), the formulae (5.7) may be simplified:

(5.8)
$$q_0 = \frac{T_0\left(1-\dfrac{3}{4\lambda^4}\right)+S_0\,\dfrac{1}{8\lambda^3}}{1-1/2\lambda^2-3/4\lambda^4},$$
$$q_1 = \frac{T_0\,\dfrac{1}{\lambda^3}+S_0\left(1-\dfrac{1}{2\lambda^2}\right)}{1-1/2\lambda^2-3/4\lambda^4}.$$

The corresponding stress intensity factors are, according to Eq. (4.4),

(5.9)
$$K_I^R = \frac{T_0\left(1+\dfrac{1}{2\lambda^3}-\dfrac{3}{4\lambda^4}\right)-S_0\left(\dfrac{1}{2}-\dfrac{1}{4\lambda^2}+\dfrac{1}{8\lambda^3}\right)}{1-1/2\lambda^2-3/4\lambda^4}\sqrt{\pi a},$$
$$K_I^L = \frac{T_0\left(1-\dfrac{1}{2\lambda^3}-\dfrac{3}{4\lambda^4}\right)-S_0\left(\dfrac{1}{2}-\dfrac{1}{4\lambda^2}+\dfrac{1}{8\lambda^3}\right)}{1-1/2\lambda^2-3/4\lambda^4}\sqrt{\pi a}.$$

Under uniform tension $T_2 = \text{const}$ applied at infinity, when $T_0 = T_2$ and $S_0 = 0$, it is seen from Eq. (5.9) that the existence of the other crack increases both stress intensity factors (2) at the first crack, the increment being greater at the "inside" crack tip, $x = a$, and smaller at the "outside" tip, $x = -a$. Table 3 presents the values of $K_I^R/T_2\sqrt{\pi a}$ and $K_I^L/T_2\sqrt{\pi a}$ calculated from Eq. (5.9), and the corresponding values as given in [9]. At larger values of λ the accuracy of approximate results is seen to be good.

Table 3. Stress intensity factors at two equal collinear cracks under uniform tension (Fig. 9).

	Formulae (5.9)		According to [9]	
$l/a = \lambda$	$K_I^L/T_0\sqrt{\pi a}$	$K_I^R/T_0\sqrt{\pi a}$	$K_I^L/T_0\sqrt{\pi a}$	$K_I^R/T_0\sqrt{\pi a}$
3.0	1.0396	1.0792	1.0517	1.1124
3.5	1.0306	1.0550	1.0373	1.0688
4.0	1.0243	1.0404	1.0280	1.0480
4.5	1.0197	1.0310	1.0220	1.0353
5.0	1.0163	1.0245	1.0179	1.0272
6.0	1.0117	1.0164	1.0125	1.0176
7.0	1.0088	1.0118	1.0091	1.0125
8.0	1.0069	1.0089	1.0071	1.0092
9.0	1.0055	1.0069	1.0057	1.0071
10.0	1.0045	1.0055	1.0046	1.0057

(2) Contrary to the remark made in [9], for small values of $a/l \ll 1$, the ratios $K_I^R/T_2\sqrt{\pi a} \approx K_I^L/T_2\sqrt{\pi a} \approx 1+a^2/2l^2$, and not $1-a/l$.

6. Other crack arrays

Let us consider infinite rows of equal cracks of lengths $2a$, either parallel like in Fig. 10a, or collinear like in Fig. 10b. Equal distances between the cracks are denoted by h and l, respectively.

Starting with the case of parallel cracks let us assume the external load to be symmetric in x, $\sigma_{yy}(x, y) = \sigma_{yy}(-x, y)$ periodic in y so that $\sigma_{ij}(x, y) = \sigma_{ij}(x, y+h)$. The latter property means that the deformation and stresses at all cracks are identical, and the same applies to the function $p_2^0(x)$, Eq. (2.4), which is the same for all cracks $|x| < a$, $y = kh$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$. With two unknown parameters q_0, q_2 the entire problem is reduced to a simple set of two equations. The stress $p_2^0(x)$ produced at the segment $|x| < a$ by all the remaining cracks must be written in the form of a series (cf. Eq. (4.11)):

$$\sigma_{yy}(x, 0) = \sigma_{yy}^0(x, 0) + 2\cdot\left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right) \sum_{k=1}^{\infty} \frac{\xi^4 + 6k^2\chi^2\xi^2 - 3k^4\chi^4}{(\xi^2 + k^2\chi^2)^3} \tag{6.1}$$

with the notation $\chi = h/a = 1/\alpha$.

Equations (4.14) and (4.15) assume the form

$$\begin{aligned} q_0 + q_2 &= \sigma_{yy}^0(0, 0) - 3\left(q_0 + \frac{3}{4} q_2\right) \frac{1}{\chi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}, \\ q_0 &= \sigma_{yy}^0(0, a) - 3\left(q_0 + \frac{3}{4} q_2\right) \sum_{k=1}^{\infty} \frac{k^4\chi^4 - 2k^2\chi^2 - 1/3}{(k^2\chi^2 + 1)^3}, \end{aligned} \tag{6.2}$$

which may easily be solved to yield the necessary parameters. The accuracy of approximation (6.2) is demonstrated on the example of uniform tension $T_2 = \text{const}$ applied at $y = \pm\infty$. Then $\sigma_{yy}^0(x, y) = T_2$, and since [10]

$$\sum_1^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \sum_1^{\infty} \frac{k^4 - 2k^2\alpha^2 - \alpha^4/3}{(k^2 + \alpha^2)^3} \approx \frac{\pi^2}{6} - \alpha^2 \frac{\pi^4}{18},$$

the solution

$$q_0 = T_2 \frac{1 + \pi^4\alpha^4/8}{1 + \pi^2\alpha^2/2 - \pi^4\alpha^4/24}, \quad q_2 = -T_2 \frac{\pi^4\alpha^4/6}{1 + \pi^2\alpha^2/2 - \pi^4\alpha^4/24}$$

yields the approximate value of the stress intensity factors

$$K_I^L = K_I^R = T_2\sqrt{\pi a}\, \frac{1 + \pi^4\alpha^4/24}{1 + \pi^2\alpha^2/2 - \pi^4\alpha^4/24}. \tag{6.3}$$

Another approach to the same problem may be based on the assumption that the stress $\sigma_{yy}(x, 0)$ produced by all the remaining cracks $|x| < a$,

$y = \pm kh$, $k = 1, 2, 3, \ldots$, is almost constant along the segment $|x| < a$, $y = 0$. This leaves us with the approximate formula (cf. Eq. (6.1))

$$\sigma_{yy}(x, 0) = T_2 + q_0 \alpha^2 \sum_{k=1}^{\infty} \frac{(x/h)^4 + 6k^2 (x/h)^2 - 3k^4}{[(x/h)^2 + k^2]^3}.$$

The mean value of $\sigma_{yy}(x, 0)$ in the interval $-a < x < a$ equals

$$\text{(6.4)} \qquad \sigma_{yy}^M = \frac{1}{2a} \int_{-a}^{a} \sigma_{yy}(x, 0)\, dx = T_2 - q_0 \alpha^2 \sum_{k=1}^{\infty} \frac{\alpha^2 + 3k^2}{(\alpha^2 + k^2)^2}.$$

Equating this expression to q_0 and taking into account that [10]

$$\text{(6.5)} \qquad \sum_{1}^{\infty} \frac{\alpha^2 + 3k^2}{(\alpha^2 + k^2)^2} = -\frac{1}{2\alpha^2} + \frac{\pi}{\alpha} \operatorname{ctgh} \pi\alpha - \frac{\pi^2}{2} \frac{1}{\sinh^2 \pi\alpha},$$

the solution is written in the form

$$q_0 = \frac{T_2}{1 + \alpha^2 S(\alpha)},$$

where $S(\alpha)$ denotes the sum (6.5). The corresponding stress intensity factor is then

$$\text{(6.6)} \qquad K_I^L = K_I^R = T_2 \sqrt{\pi a} \frac{1}{1 + \alpha^2 S(\alpha)}.$$

Table 4 presents the values of $K_I/T_2 \sqrt{\pi a}$ calculated according to the approximate formulae (6.3) and (6.6) and compared with the values taken from the literature [11].

Let us now pass to the case of collinear cracks shown in Fig. 10b. Also here, in view of the symmetry with respect to the vertical axes bisecting the cracks, all $p_2^0(x)$ — functions are the same for all cracks provided the load is periodic in x, and $\sigma_{ij}(x, y) = \sigma_{ij}(x + l, y)$. It also follows that the parameter q_1 in the expansion (4.7) must be zero, and hence the formula

$$\text{(6.7)} \qquad \sigma_{yy}(x, y) = \sigma_{yy}^0(x, y) + \left(\frac{1}{2} q_0 + \frac{3}{8} q_2\right) \frac{x^4 + 6x^2 y^2 - 3y^4}{(x^2 + y^2)^3} a^2$$

Table 4. Stress intensity factors in a plane containing an infinite number of parallel cracks (Fig. 10a) under uniform tension.

a/h	Eq. (6.3)	Eq. (6.6)	[11]
0.10	0.9537	0.9535	
0.20	0.8452	0.8409	
0.25	0.7859	0.7759	0.7896
0.30	0.7319	0.7118	0.7344
0.35	0.6873	0.6516	0.6868
0.40	0.6549	0.5969	

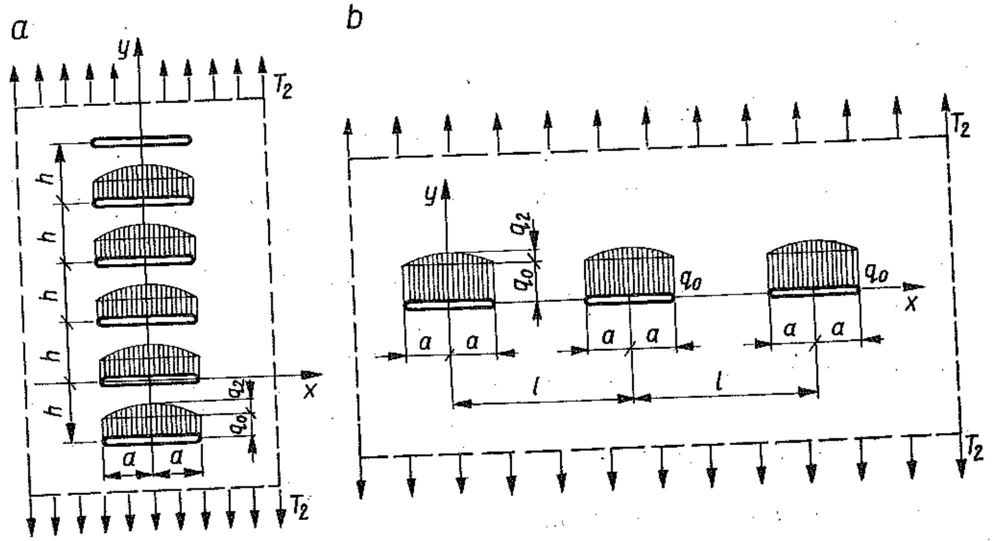

FIG. 10.

is used to calculate the stresses produced by a single crack $2a$ located at the origin of the coordinate system. Formula (6.7) is now applied, after obvious modifications, to evaluate the stresses in the interval $-a < x < a$, $y = 0$ produced by a crack $x = kl \pm a$, $y = 0$:

$$\sigma_{yy}(x,0) = \sigma_{yy}^{0}(x,0) + \left(\frac{1}{2}q_0 + \frac{3}{8}q_2\right)\frac{a^2}{(kl-x)^2}.$$

Summing up the contributions of all cracks $k = \pm 1, \pm 2, \ldots$, the stresses at $x = 0$ and $x = \pm a$ are written in the form of infinite series; it is seen from Fig. 10b that

$$\sigma_{yy}(x,0) = q_0 + q_2, \qquad \sigma_{yy}(x, \pm a) = q_0.$$

The set of equations for the two unknown parameters has the form

$$\begin{aligned} & q_0 + q_2 = \sigma_{yy}^{0}(0,0) + \left(q_0 + \frac{3}{4}q_2\right)\alpha^2 \sum_{k=1}^{\infty}\frac{1}{k^2}, \\ (6.8) \qquad & q_0 = \sigma_{yy}^{0}(\pm a, 0) + \left(\frac{1}{2}q_0 + \frac{3}{8}q_2\right)\alpha^2 \sum_{k=1}^{\infty}\left[\frac{1}{(k-\alpha)^2} + \frac{1}{(k+\alpha)^2}\right]. \end{aligned}$$

In the particular case of uniform tension T_2 applied at $y = \pm\infty$, $\sigma_{yy}^{0}(x,y) = T_2 = \text{const}$, and the solution of the system of Eqs. (6.8) is

$$q_0 = T_2 \frac{1 + \frac{3}{4}\alpha^2 \Delta}{1 - \alpha^2\left(\frac{\pi^2}{6} - \frac{\Delta}{4}\right)},$$

$$q_1 = -T_2 \frac{\alpha^2 \Delta}{1-\alpha^2\left(\frac{\pi^2}{6}-\frac{\Delta}{4}\right)}, \quad \Delta = \alpha^2 \sum_{k=1}^{\infty} \frac{3k^2-\alpha^2}{k^2(k^2-\alpha^2)^2}.$$

The corresponding stress intensity factors (4.4), the same for all crack tips, are expressed by the formula

$$K_I^L = K_I^R = T_2\sqrt{\pi a}\,\frac{1+\alpha^2 \Delta/4}{1-\alpha^2\left(\frac{\pi^2}{6}-\frac{\Delta}{4}\right)}. \tag{6.9}$$

Another approximation of the same solution may be obtained in a simpler manner by assuming the stress $p_2^0(x)$ to be constant in the interval $-a < x < a$. In the formula

$$\sigma_{yy}(x,0) = \sigma_{yy}^0(x,0) + \frac{q_0 a^2}{2}\sum_{k=1}^{\infty}\left[\frac{1}{(kl-x)^2}+\frac{1}{(kl+x)^2}\right], \tag{6.10}$$

the expression in brackets is replaced with its mean value

$$\frac{1}{2a}\int_{-a}^{a}\left[\frac{1}{(kl-x)^2}+\frac{1}{(kl+x)^2}\right]dx = \frac{2}{k^2 l^2-a^2}$$

and, since [10]

$$\sum_{k=1}^{\infty}\frac{1}{k^2-\alpha^2} = \frac{1}{2\alpha^2}-\frac{\pi}{2\alpha}\operatorname{ctg}\pi\alpha,$$

q_0 is found from the single algebraic equation

$$q_0 = \sigma_{yy}^0(x,0) + q_0\left[\frac{1}{2}+\frac{\pi\alpha}{2}\operatorname{ctg}\pi\alpha\right].$$

The stress intensity factors in the case of uniform tension $T = T_2$, $\sigma_{yy}^0(x,0) = T_2$, is

$$K_I^L = K_I^R = \frac{T_2\sqrt{\pi a}}{\frac{1}{2}(1+\pi\alpha\operatorname{ctg}\pi\alpha)}. \tag{6.11}$$

In Table 5 are presented the values of $K_I/T_2\sqrt{\pi a}$ calculated according to the formulae (6.9) and (6.11) and compared with those given in the literature [8]. Accuracy of the simplified formula (6.11) is seen to be very good.

Table 5. Stress intensity factors in a plane containing an infinite number of equal and collinear cracks, Fig. 10b, under uniform tension.

a/l	Eq. (6.9)	Eq. (6.11)	Rice [8]
1/10	1.0167	1.0168	1.0170
1/9	1.0207	1.0209	1.0211
1/8	1.0264	1.0267	1.0270
1/7	1.0347	1.0352	1.0359
1/6	1.0478	1.0488	1.0501
1/5	1.0703	1.0725	1.0753
1/4	1.1141	1.1202	1.1284

7. Mode II crack deformation. Combined modes. Interaction forces

Let us consider an infinite medium containing a single crack $|x| < a$, $y = 0$ and loaded in such a manner by external forces T that the crack deforms according to Mode II (Fig. 11). This means that in absence of the crack $\sigma^0_{yy}(x, 0) = p^0_2(x) \equiv 0$, and $\sigma^0_{xy}(x, 0) = p^0_1(x) \neq 0$, cf. formulae (2.4). Hence the action of the crack may be considered as equivalent to the action to a set of horizontal and vertical force couples distributed along the segment $|x| < a$ according to Eqs. (2.6). Since both intensities $\varrho^{12} = \varrho^{21}$, Eq. (2.8), stresses

$$\sigma_{ij}(x, y) = \sigma^0_{ij}(x, y) + \int_{-a}^{a} [\varrho^{12}(\xi)\,\bar{\sigma}^{12}_{ij}(x-\xi, y) + \varrho^{21}(\xi)\,\bar{\sigma}^{21}_{ij}(x-\xi, y)]\,d\xi$$

may be written in a compact form

$$\sigma_{ij}(x, y) = \sigma^0_{ij}(x, y) + \int_{-a}^{a} \varrho(\xi)\,\hat{\sigma}_{ij}(x-\xi, y)\,d\xi. \tag{7.1}$$

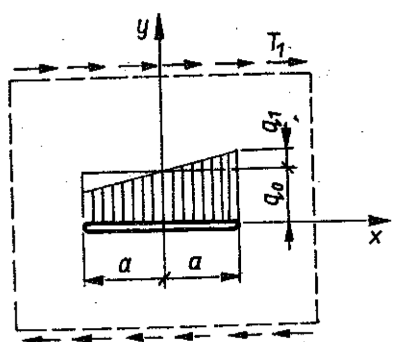

FIG. 11.

Here

$$\varrho(\xi) = \varrho^{12}(\xi) = \varrho^{21}(\xi),$$

and

$$\hat{\sigma}_{ij} = \bar{\sigma}_{ij}^{12} + \bar{\sigma}_{ij}^{21}. \tag{7.2}$$

The stresses $\bar{\sigma}_{ij}^{12}$, $\bar{\sigma}_{ij}^{21}$ are given by Eqs. (2.12) and (2.13). Expansion of $\hat{\sigma}_{ij}(x-\xi, y)$ into the power series of ξ, Eq. (3.3), and the procedure analogous to that applied in Eqs. (3.4) — (3.9) leads to the following formula for stresses in which, for take sake of simplicity, only the terms containing the parameters R_0, R_1 are preserved:

$$\sigma_{ij}(x, y) = \sigma_{ij}^{0}(x, y) + R_0\, \hat{\sigma}_{ij}(x, y) - R_1 \frac{\partial \hat{\sigma}_{ij}(x, y)}{\partial x}. \tag{7.3}$$

Here, as before, $R_i = \int\limits_{-a}^{a} \xi^i \varrho(\xi)\, d\xi$, and the suitable force couple distribution must be substituted for $\varrho(\xi)$.

Let us assume that the stress $\sigma_{xy}(x, 0)$ produced along the segment $|x| < a$ is written in the simple, linear form (cf. (4.1)):

$$\begin{aligned} p_1^0 &= p_1^{0T}(x) + q_1^0(x), \\ q_1^0 &= q_0 + q_1\, x/a. \end{aligned} \tag{7.4}$$

On substitituting Eq. $(7.4)_2$ into the formulae (2.6), the following results are obtained:

$$\begin{aligned} P_1(\xi) &= -q_0\, \xi + \frac{1}{2} q_1\, a\left(1 - \frac{2\xi^2}{a^2}\right), \\ \varrho^{12}(x) &= \varrho^{21}(x) = 2(1-\nu)\, a\left(q_0 + \frac{1}{2} q_1 \frac{x}{a}\right)\sqrt{1 - \frac{x^2}{a^2}}, \\ R_0 &= \pi a^2 q_0 (1-\nu), \qquad R_1 = \frac{1}{8} \pi a^3 q_1 (1-\nu). \end{aligned} \tag{7.5}$$

The formulae (2.14) and (2.15) lead to the stress $\hat{\sigma}_{xy}$, Eq. (7.2),

$$\hat{\sigma}_{xy} = \frac{1}{2\pi(1-\nu)} \frac{x^4 - 6x^2 y^2 + y^4}{(x^2+y^2)^3}. \tag{7.6}$$

Finally, substitution of Eqs. (7.5) and (7.6) into Eq. (7.3) yields the approximate expression for the shearing stress:

$$\sigma_{xy} = \sigma_{xy}^{0} + \frac{1}{2} q_0\, a^2 \frac{x^4 - 6x^2 y^2 + y^4}{(x^2+y^2)^3} + \frac{1}{8} q_1\, a^3 \frac{x^5 - 14x^3 y^2 + 9xy^4}{(x^2+y^2)^4}. \tag{7.7}$$

Equation (7.7) represents the Mode II counterpart of Eq. (3.13) derived in Sect. 3; it is now used to analyze the problems of several cracks subject to Mode II deformation.

Let us consider two simple examples of two equal cracks of lengths 2a in two positions: parallel (Fig. 12a) and collinear (Fig. 12b). The exter load is assumed to be represented by pure constant shear $\sigma_{12}(x, y) = T_1 = $ $= \text{const}$ applied at $y = \pm\infty$.

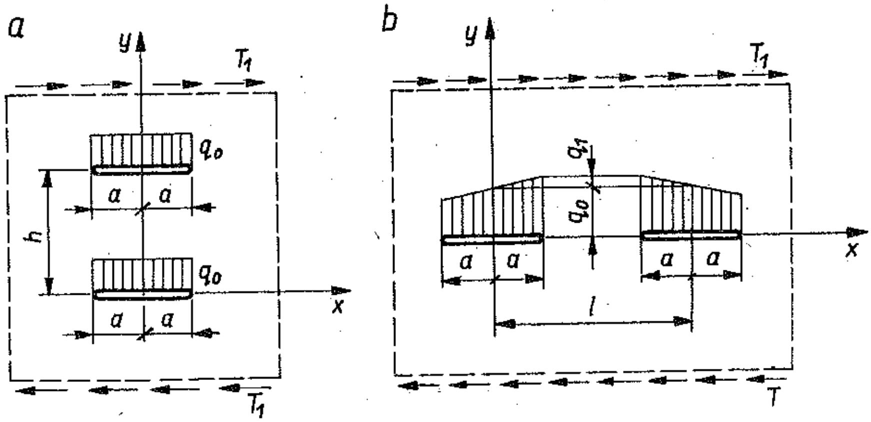

FIG. 12.

In the first case shear stresses $q_1^0(x)$ along the segments $|x| < a, y = 0$ and $y = h$ are assumed to be constant, the second order term q_2 in Eq. (7.4) being disregarded. The stress $\sigma_{xy}(x, 0)$ produced by the external load T_1 and the force couples replacing the other crack is, according to Eq. (7.1), equal to

$$\sigma_{xy}(x, 0) = T_1 + \frac{1}{2} q_0 a^2 \frac{x^4 - 6x^2 h^2 + h^4}{(x^2 + h^2)^3}. \tag{7.8}$$

Let us assume that, with $a/h \ll 1$, the stress $\sigma_{xy}(x, 0)$ in the interval $|x| < a$ is equal to its value at the center, i.e $\sigma_{xy}(x, 0) \approx \sigma_{xy}(0, 0)$; on substituting $x = 0$ into Eq. (7.8) one obtains (in the case of equal cracks) the simple equation

$$q_0 = T_1 + \frac{1}{2} q_0 \alpha^2 \tag{7.9}$$

with $\alpha = a/h$. It follows that

$$q_0 = \frac{T_1}{1 - \frac{1}{2}\alpha^2}$$

and, since (cf. Eqs. (2.5), (2.6) and (4.4))

$$K_{II}^R = \left(q_0 + \frac{1}{2} q_1\right)\sqrt{\pi a}, \quad K_{II}^L = \left(q_0 - \frac{1}{2} q_1\right)\sqrt{\pi a}, \tag{7.10}$$

the stress intensity factors are

$$K_{II}^L = K_{II}^R = T_1 \sqrt{\pi a} \frac{1}{1 - \frac{1}{2}\alpha^2}. \tag{7.11}$$

A more accurate result may again be derived if the mean value of $\sigma_{xy}(x, 0)$ over the interval $-a < x < a$ is substituted into Eq. (7.9) instead of its value at the center of the crack. Since

$$\sigma_{xy}^M x, 0 = \frac{1}{2a} \int_{-a}^{a} \sigma_{xy}(x, 0)\, dx = T_1 + \frac{q_0 a^2}{2} \frac{h^2 - a^2}{(h^2 + a^2)^2},$$

Eq. (7.9) must be replaced with

$$q_0 = T_1 + \frac{1}{2} q_0 \alpha^2 \frac{1 - \alpha^2}{(1 + \alpha^2)^2},$$

so that

$$q_0 = \frac{T_1}{1 - \frac{1}{2}\alpha^2 \frac{1 - \alpha^2}{(1 + \alpha^2)^2}}$$

and

$$K_{II}^L = K_{II}^R = T_1 \sqrt{\pi a} \frac{1}{1 - \frac{1}{2}\alpha^2 \frac{1 - \alpha^2}{(1 + \alpha^2)^2}}. \tag{7.12}$$

Several values of the ratio $K_{II}/T_1 \sqrt{\pi a}$ calculated according to the formulae (7.11) and (7.12) are given in Table 6. It is seen from the table that, in contrast to the Mode I deformation case of two parallel cracks subject to tension,

Table 6. Stress intensity factors in a plane containing two equal parallel cracks, Fig. 12a, **under constant shear.**

$\alpha = a/h$	Eq. (7.11)	Eq. (7.12)
0.05	1.0013	1.0012
0.10	1.0050	1.0049
0.15	1.0114	1.0106
0.20	1.0204	1.0181
0.25	1.0323	1.0266
0.30	1.0471	1.0357

Table 2, the stress intensity factors at two parallel cracks under shear are greater than those appearing at a single crack.

In the case of collinear cracks, Fig. 12b, the $q_1^0(x)$ — distribution along both segments may be assumed according to Eq. (7.4) with q_0, q_1 for the left-hand crack, and $q_0, -q_1$ for the right-hand crack, provided the cracks are of equal lengths and the load is symmetric: $\sigma_{xy}(x, y) = \sigma_{xy}(-x, y)$ etc. If the load T is represented by constant shear T_1, Eq. (7.7) assumes the simple form

$$\sigma_{xy}(x, 0) = T_1 + \frac{1}{2} q_0 \frac{a^2}{x^2} + \frac{1}{8} q_1 \frac{a^3}{x^3}$$

In the first approximation, the parameters q_0, q_1 are calculated from the simplified conditions

$$\sigma_{xy}(l, 0) = q_0, \qquad \frac{1}{2}[\sigma_{xy}(l+a, 0) - \sigma_{xy}(l-a, 0)] = -q_1.$$

The resulting set of equations

$$T_1 + \frac{1}{2} q_0 \alpha^2 + \frac{1}{8} q_1 \alpha^3 = q_0,$$

$$\text{(7.13)} \qquad \frac{1}{4} q_0 \alpha^2 \left[\frac{1}{(1-\alpha)^2} - \frac{1}{(1+\alpha)^2}\right] + \frac{1}{16} q_1 \alpha^3 \left[\frac{1}{(1-\alpha)^3} - \frac{1}{(1+\alpha)^3}\right] = q_1$$

yields the approximate solutions

$$\text{(7.14)} \qquad q_0 = \frac{T_1}{1 - \frac{\alpha^2}{2}}, \qquad q_1 = \frac{T_1 \alpha^3}{1 - \frac{\alpha^2}{2}}.$$

The stress intensity factors for the first crack $|x| < a$ are

$$K_{II}^L = T_1 \sqrt{\pi a} \frac{1 - \alpha^3/2}{1 - \alpha^2/2},$$

$$\text{(7.15)}$$

$$K_{II}^R = T_1 \sqrt{\pi a} \frac{1 + \alpha^3/2}{1 - \alpha^2/2}.$$

Like in the case of Mode I collinear crack deformation, the existence of the second crack increases the stress intensity factors at both crack tips $x = \pm a$, the increments at the inside crack tips being greater.

Repeating the procedure in which the value $\sigma_{yy}(l, 0)$ in Eq. (7.13) is replaced with its mean value over the interval $(l-a, l+a)$, the slightly modified results (7.14) follow,

$$q_0 = \frac{T_1}{1 - \frac{1}{2} \frac{\alpha^2}{1-\alpha^2}}, \qquad q_1 = \frac{T_1 \alpha^3}{1 - \frac{1}{2} \frac{\alpha^2}{1-\alpha^2}},$$

and the stress intensity factors at $x = -a, x = a$ are

$$K_{II}^{L} = T_1 \sqrt{\pi a} \frac{1-\alpha^3/2}{1-\alpha^2/2\,(1-\alpha^2)}, \tag{7.16}$$
$$K_{II}^{R} = T_1 \sqrt{\pi a} \frac{1+\alpha^3/2}{1-\alpha^2/2\,(1-\alpha^2)}.$$

The corresponding values of $K_{II}^L/T_1\sqrt{\pi a}$ and $K_{II}^R/T_1\sqrt{\pi a}$ following from the approximate formulae (7.15) and (7.16) are listed in Table 7.

Table 7. Stress intensity factors in a plane containing two equal collinear cracks, Fig. 11b, **under constant shear.**

$\alpha = a/l$	Eq. (7.15) $\frac{K_{II}^L}{T_1\sqrt{\pi a}}$	Eq. (7.15) $\frac{K_{II}^R}{T_1\sqrt{\pi a}}$	Eq. (7.16) $\frac{K_{II}^L}{T_1\sqrt{\pi a}}$	Eq. (7.16) $\frac{K_{II}^R}{T_1\sqrt{\pi a}}$
0.05	1.0012	1.0013	1.0012	1.0013
0.10	1.0045	1.0055	1.0046	1.0056
0.15	1.0097	1.0131	1.0099	1.0134
0.20	1.0163	1.0245	1.0172	1.0254
0.25	1.0242	1.0403	1.0264	1.0426
0.30	1.0330	1.0613	1.0378	1.0662

In the cases of regular crack arrays like those shown in Fig. 10, the problem of determining the stress intensity factors and stress distributions may be treated in the same manner as those considered in Sect. 6. The procedures outlined in the preceding sections may also be applied to the cases of arbitrary crack distributions provided the distances between individual cracks are large enough; the load may also consist of combined action of tension and shear. For instance, in the case of two inclined cracks shown in Fig. 13 in an arbitrarily loaded infinite plane, the entire problem may be reduced to the system of eight algebraic equations with eight unknown parameters: q_0, q_1, q_0', q_1' referring to the distributions of normal stresses along he corresponding cracks $2a$, $2a'$, and $\bar{q}_0, \bar{q}_1, \bar{q}_0', \bar{q}_1'$ describing the distribution of shearing stresses.

Normal stresses σ_{yy}' at points D, E, F of the second (primed) crack are now written in terms of the external loads T_1, T_2 and the unknown (unprimed) parameters $q_0, q_1, \bar{q}_0, \bar{q}_1$ of the first cracks; the necessary formulae are given in Eqs. (4.7)—(4.9). The two resulting equations have the form

$$\sigma_{yy}'(E) = q_0', \quad \frac{1}{2}[\sigma_{yy}'(F) - \sigma_{yy}'(D)] = q_1'.$$

Similar equations are written for the first crack. The remaining four equations follow from the consideration of shearing stresses. Accuracy of these results

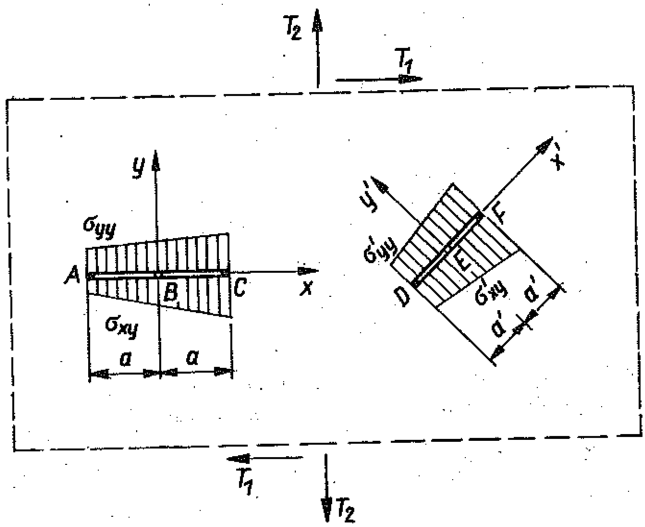

FIG. 13.

may be increased by introducing additional terms in expansions of the type of Eq. (4.5).

To conclude the considerations let us mention the problem of forces of interaction between the cracks treated more extensively in the paper [7] in connection with Mode III crack deformations. These forces are calculated formally from the energy considerations outlined in [2] and expressed in terms of the stress intensity factors. To this end let us write the formula (2.3) derived in [1], giving the horizontal component of the force

$$F^1 = \frac{1-\nu}{2\mu}\left[(K_{\rm I}^R)^2+(K_{\rm II}^R)^2-(K_{\rm I}^L)^2-(K_{\rm II}^L)^2\right]. \tag{7.17}$$

In the case of two equal and parallel cracks under constant tension T_2, the horizontal components of these forces are zero ince the cracks exhibit equal tendencies to propagate to the left and to the right. In the case of collinear cracks, however, the inside crack tip stress intensity factors are greater than the outside ones. The formula (7.17) yields the following expression for the horizontal force exerted by the right-hand crack on the left-hand one:

$$F_{12}^1 = \frac{1-\nu}{2\mu}(T_2)^2\frac{\left(1+\dfrac{1}{2\lambda^3}\right)^2-\left(1-\dfrac{1}{2\lambda^3}\right)^2}{\left(1-\dfrac{1}{2\lambda^2}\right)^2}\pi a \approx \tag{7.18}$$

$$\approx \frac{1-\nu}{\mu}(T_2)^2\frac{\pi a^4}{l^3\left(1-\dfrac{a^2}{2l^2}\right)}$$

A simple observation follows from the inspection of the approximate formula (7.18). It follows that at sufficiently large ratios l/a, that is for distant cracks, the force of interaction between them is proportional to the square of the external load T_2, to the product of squares of the crack lengths (here a^4 since the cracks are equal), and inversely proportional to the third power of the distance between them. Since the force exerted by the first crack on the other $F_{12}^1 = -F_{12}^1$ the formula (7.18) giving the force of attraction of the cracks reflects the well-known tendency of such cracks to approach each other under external tension.

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Streszczenie

O PRZYBLIŻONYM OKREŚLENIU WSPÓŁDZIAŁANIA SZCZELIN W OŚRODKACH SPRĘŻYSTYCH

Przedstawiono metodę przybliżonej analizy stanu naprężenia, a w szczególności sposób wyznaczania współczynników intensywności naprężenia w ośrodkach sprężystych poddanych płaskiemu stanowi odkształcenia i zawierających dowolny układ szczelin. W przypadku gdy szczeliny nie są rozmieszczone zbyt gęsto, proponowana metoda pozwała wyznaczyć poszukiwane parametry stanu naprężenia w sposób zbliżony do metody rozwiązania układów statycznie niewyznaczalnych w mechanice budowli.

Резюме

О ПРИБЛИЖЕННОМ ОПРЕДЕЛЕНИИ ВЗАИМОДЕЙСТВИЯ ТРЕЩИН В УПРУГИХ СРЕДАХ

Представлен метод приближенного анализа напряженного состояния, а в частности способ определения коэффициентов интенсивности напряжения в упругих средах, подвергнутых плоскому деформационному состоянию и содержавших произвольную систему трещин. В случае, когда трещины не распределены слишком густо, предлагаемый метод позволяет определить искомые параметры напряженного состояния способом сближенным к методу решения систем статически неопределенных в строительной механике.

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