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ON DONATI'S THEOREM IN SHELL THEORY

Z. RYCHTER (BIAŁYSTOK)

The Donati theorem has been formulated in connection with three-dimensional problems of the linear theory of elasticity. In this paper a simular theorem is considered for two-dimensional problems encountered in the linear theory of shells. As closely related topics the stress functions and compatibility equations in the shell theory are also studied. A shell is assumed to be deformed in accordance with the hypothesis of linear distribution of the displacement vector across the shell thickness. Thus, six local degrees of freedom of the shell are taken into account. The results obtained in the paper include, as a special case, the well-known stress functions and compatibility equations of the Kirchhoff-Love and Reissner theories.

1. Introduction

Donati's theorem (see [1] where a concise formulation of the theorem and related references can be found) provides a simple method of deriving the three-dimensional compatibility equations in the linear theory of elasticity. The method only requires a representation of the stress tensor in terms of stress functions. As regards the linear theory of shells, its equations being essentially two-dimensional, an analogue of the Donati theorem has so far not been formulated. Nevertheless, a similar approach based on a virtual work principle with appropriate stress functions was used in [2] to derive the compatibility equations in the framework of the Reissner-type shell theory.

The present paper concerns a more general linear theory of shells referred to as a six-parametric theory (SP). Such a theory results from the assumption of linear distribution of the displacement vector across the shell thickness [3–7]. Alternatively, SP is obtained when the shell is thought to be a Cosserat surface with six local kinematical degrees of freedom [8].

The aim of this paper is to provide a formulation and proof of the Donati theorem for SP. Also, a general solution of the equations of equilibrium of SP in terms of stress functions is given. From the Donati theorem presented, the compatibility equations for SP are deduced in a form equivalent to those found in [8] for a Cosserat surface. Under appropriate restricions, the results mentioned reduce to the stress functions and compatibility equations known in the Kirchhoff-Love and the Reissner shell theories, see e.g. [2, 9–11].

2. Notation. Geometry of a shell

The paper utilises a standard in shell theory tensor and vector-tensor notation (see e.g. [6, 8, 10]). For future use some definitions and relations are recorded without proof below.

Vectors (possibly vectors with vector components) and their components are denoted by the same kernel symbol, the former in semi-bold type, the latter and scalars is normal type. Whatever scalar or vector, all the fields are defined on the middle surface τ of an underformed shell, their sole arguments being curvilinear coordinates $\{x^\alpha\}=\{x^1, x^2\}$ on τ. The Greek letters serve as sub- or superscripts ranging over the integers $\{1, 2\}$. Any index appearing twice, i.e. as a sub- and superscript, means a summation. The symbols $(\ )_{,\alpha}$ and $(\ )_{|\alpha}$ denote partial and surface covariant derivatives. A dot and a cross between two symbols indicate a scalar and a vector product.

The geometry of a shell can be completely characterized from a radius vector $\mathbf{r}(x^\alpha)$ of the midsurface τ. Then the following relations hold:

$$\mathbf{g}_\alpha=\mathbf{r}_{,\alpha}=d\mathbf{r}/dx^\alpha, \quad \mathbf{g}^\beta\cdot\mathbf{g}_\alpha=\delta^\beta_\alpha, \quad \mathbf{g}_3=\frac{1}{2}\varepsilon_{\alpha\beta}\,\mathbf{g}^\alpha\times\mathbf{g}^\beta, \tag{2.1}$$

$$g_{\alpha\beta}=\mathbf{g}_\alpha\cdot\mathbf{g}_\beta, \quad b_{\alpha\beta}=\mathbf{g}_{\alpha,\beta}\cdot\mathbf{g}_3, \quad \varepsilon_{\alpha\beta}=\sqrt{g}\,e_{\alpha\beta}, \tag{2.2}$$

$$g=\det(g_{\alpha\beta}), \quad e_{11}=e_{22}=0, \quad e_{12}=-e_{21}=1, \tag{2.3}$$

defining the natural base vectors $\mathbf{g}_\alpha$ tangent to τ, their duals $\mathbf{g}^\beta$ (δ^β_α being the Kronecker delta), a unit vector $\mathbf{g}_3$ normal to τ, the components $g_{\alpha\beta}$, $b_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ of the first and second metric and the alternating (Ricci) tensor on τ. Some properties of these vectors and tensors read

$$\mathbf{g}_\alpha\times\mathbf{g}_\beta=\varepsilon_{\alpha\beta}\,\mathbf{g}_3, \quad \mathbf{g}_3\times\mathbf{g}_\alpha=\varepsilon_{\alpha\beta}\,\mathbf{g}^\beta, \tag{2.4}$$

$$\begin{gathered}\mathbf{g}_{\alpha|\beta}=b_{\alpha\beta}\,\mathbf{g}_3, \quad \mathbf{g}_{3|\beta}=-b^\alpha_\beta\,\mathbf{g}_\alpha, \quad \varepsilon^{\alpha\beta}\,\mathbf{g}_{\alpha|\beta}=0, \quad \varepsilon^{\alpha\beta}{}_{|\lambda}=0,\\ \varepsilon^{\alpha\beta}\,\mathbf{B}_{|\alpha\beta}=0, \quad \varepsilon^{\gamma\eta}\,b_{\beta\gamma}\,b_{\eta\delta}\,\theta^\delta=\varepsilon^{\gamma\eta}\,\theta_{\beta|\eta\gamma}\end{gathered} \tag{2.5}$$

the last two in the relations (2.5) being true for arbitrary $\mathbf{B}$ and θ^δ. Of special use will be the following integration by parts formula:

$$\int_\tau \mathbf{B}\cdot\mathbf{A}^\alpha|_\alpha\,da=-\int_\tau \mathbf{B}|_\alpha\cdot\mathbf{A}^\alpha\,da+\int_{\partial\tau}\mathbf{B}\cdot\mathbf{A}^\alpha\,n_\alpha\,ds, \tag{2.6}$$

valid for any vectors $\mathbf{B}$ and $\mathbf{A}^\alpha$ with n_α denoting the components of a unit vector outward normal to the edge $\partial\tau$ of τ.

If τ is supposed to be an open set with a closure $\bar{\tau}$, then a suitable form of the Dubois–Raymond lemma may be stated as.

Lemma. If $\int_\tau \mathbf{B}\cdot\mathbf{A}^\alpha\,da=0$ for any $\mathbf{B}$ of class C^∞ on $\bar{\tau}$ that vanishes near $\partial\tau$, then $\mathbf{A}^\alpha=0$; (the lemma and Eq. (2.6) are proved in a general form in [1]).

3. Basic equations of SP

All the variants of equations of the six-parametric shell theory (SP) have in common the assumption of linear distribution of the displacement vector across the shell thickness. Differences only appear when strain measures are concerned. They are generally defined as symmetric or non-symmetric. In the present paper the latter variant will be dealt with. It has been discussed at length in [8], from where we cite the equations of SP needed for our purposes, to wit an energy functional

$$I=\int_{\tau}(N^{\beta\alpha}\gamma_{\alpha\beta}+N^{3\alpha}\gamma_{3\alpha}+M^{\beta\alpha}\varkappa_{\alpha\beta}+M^{3\alpha}\varkappa_{3\alpha}+N^{33}\gamma_{33})\,da \tag{3.1}$$

and the homogeneous equations of equilibrum

$$\begin{gathered} N^{\beta\alpha}|_{\beta}-b^{\alpha}_{\nu}N^{3\nu}=0, \quad N^{3\alpha}|_{\alpha}+b_{\alpha\beta}N^{\beta\alpha}-N^{33}=0, \\ M^{\beta\alpha}|_{\alpha}-N^{3\alpha}=0, \quad M^{3\alpha}|_{\alpha}+b_{\alpha\beta}N^{\beta\alpha}-N^{33}=0. \end{gathered} \tag{3.2}$$

Here γ and $\varkappa$ represent the strain measures in SP whereas N and M are the stress resultants and couples energetically compatible with γ and $\varkappa$, respectively. Roughly speaking, $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ are changes in $g_{\alpha\beta}$ and $b_{\alpha\beta}$, $\gamma_{3\alpha}$ and $\varkappa_{3\alpha}$ characterize the transverse shear deformation, γ_{33} describes a uniform stretch in the $\mathbf{g}_3$ direction. Under $\varkappa_{3\alpha}=N^{33}=M^{3\alpha}=0$ reduction of the above to the Reissner theory along with the additional requirement $\gamma_{3\alpha}=0$ to the Kirchhoff–Love theory is obtained.

It will prove convenient to rewrite Eqs. (3.1) and (3.2) in the following vector-tensor form:

$$I=\int_{\tau}(\mathbf{N}^{\alpha}\cdot\boldsymbol{\gamma}_{\alpha}+\mathbf{M}^{\alpha}\cdot\boldsymbol{\varkappa}_{\alpha}+M^{3\alpha}\varkappa_{3\alpha}+N^{33}\gamma_{33})\,da \tag{3.3}$$

and

$$\mathbf{N}^{\alpha}|_{\alpha}=0, \quad \mathbf{g}_{\alpha}\times\mathbf{N}^{\alpha}+\mathbf{M}^{\alpha}|_{\alpha}=0, \quad M^{3\alpha}|_{\alpha}+\mathbf{b}_{\alpha}\cdot\mathbf{M}^{\alpha}-N^{33}=0, \tag{3.4}$$

where

$$\begin{gathered} \mathbf{N}^{\alpha}=N^{\alpha\beta}\mathbf{g}_{\beta}+N^{3\alpha}\mathbf{g}_3, \quad \mathbf{M}^{\alpha}=\varepsilon_{\lambda\eta}M^{\alpha\lambda}\mathbf{g}^{\eta}, \\ \boldsymbol{\gamma}_{\alpha}=\gamma_{\beta\alpha}\mathbf{g}^{\beta}+\gamma_{3\alpha}\mathbf{g}_3, \quad \boldsymbol{\varkappa}_{\alpha}=\varepsilon^{\nu\lambda}\varkappa_{\nu\alpha}\mathbf{g}_{\lambda}, \quad \mathbf{b}_{\alpha}=\varepsilon^{\nu\lambda}b_{\nu\alpha}\mathbf{g}_{\lambda}. \end{gathered} \tag{3.5}$$

On substituting Eq. (3.5) into Eq. (3.4) one may check with the aid of Eq. (2.4) and (2.5) that Eqs. (3.2) and (3.4) are equivalent. As for Eqs. (3.1) and (3.3) their equivalence is obvious.

4. Stress functions

Owing to the simplicity of the equations of equilibrum (3.4), their general solution in terms of two vector and two scalar stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\theta}$, ψ and μ can easily be found in the form

$$\begin{gathered} \mathbf{N}^{\alpha}=\varepsilon^{\alpha\beta}\boldsymbol{\varphi}|_{\beta}, \quad \mathbf{M}^{\alpha}=\varepsilon^{\alpha\beta}(\boldsymbol{\theta}|_{\beta}+\mathbf{g}_{\beta}\times\boldsymbol{\varphi}), \\ M^{3\alpha}=\varepsilon^{\alpha\beta}\psi|_{\beta}+\mu|^{\alpha}, \quad N^{33}-\mathbf{b}_{\alpha}\cdot\mathbf{M}^{\alpha}=\mu|^{\alpha}_{\alpha}, \end{gathered} \tag{4.1}$$

where

$$\boldsymbol{\varphi}=\varphi^{\alpha}\,\mathbf{g}_{\alpha}+\varphi^{3}\,\mathbf{g}_{3}, \qquad \boldsymbol{\theta}=\theta^{\alpha}\,\mathbf{g}_{\alpha}+\theta^{3}\,\mathbf{g}_{3}. \tag{4.2}$$

On substituing the relations (4.1) with due allowance for Eqs. $(2.5)_{3-5}$, Eqs. (3.4) become identities. Reduction of the relations (4.1) to component form yields

$$\begin{aligned} &N^{\beta\alpha}=\varepsilon^{\beta\gamma}\,(\varphi^{\alpha}|_{\gamma}-b^{\alpha}_{\gamma}\,\varphi^{3}), \qquad N^{3\alpha}=\varepsilon^{\alpha\beta}\,(\varphi^{3}|_{\beta}+b_{\gamma\beta}\,\varphi^{\gamma}),\\ &M^{\beta\alpha}=\varepsilon^{\alpha\gamma}\,\varepsilon^{\beta\eta}\,(\theta_{\gamma}|_{\eta}-b_{\gamma\eta}\,\theta^{3}+\varepsilon_{\gamma\eta}\,\varphi^{3}), \qquad M^{3\alpha}=\varepsilon^{\alpha\beta}\,\psi|_{\beta}+\mu|^{\alpha}, \qquad N^{33}-b_{\alpha\beta}\,M^{\beta\alpha}=\mu|^{\alpha}_{\alpha}, \end{aligned} \tag{4.3}$$

where eight scalar stress functions appear, i.e. φ^{α}, φ^{3}, θ^{α}, θ^{3}, ψ, μ. However, only six functions are independent for the orthogonality condition $\mathbf{M}^{\alpha}\cdot\mathbf{g}_{3}=0$ (resulting from Eq. $(3.5)_2$) with $\mathbf{M}^{\alpha}$ replaced by Eq. $(4.1)_2$ implies

$$\varphi^{\alpha}=\varepsilon^{\alpha\beta}\,(\theta^{3}|_{\beta}+b_{\gamma\beta}\,\theta^{\gamma}). \tag{4.4}$$

Thus φ^{α} may be eliminated from Eqs. $(4.3)_{1,2}$. When, in addition to that, Eq. $(2.5)_6$ is taken into account, Eqs. $(4.3)_{1,2}$ assume their final form

$$\begin{aligned} &N^{\beta\alpha}=\varepsilon^{\beta\gamma}\,[\varepsilon^{\alpha\eta}\,(\theta^{3}|_{\eta}+b_{\eta\delta}\,\theta^{\delta})|_{\gamma}\times b^{\alpha}_{\gamma}\,\varphi^{3}],\\ &N^{3\alpha}=\varepsilon^{\alpha\beta}\,[\varphi^{3}|_{\beta}+\varepsilon^{\gamma\eta}\,(b_{\gamma\beta}\,\theta^{3}|_{\eta}+\theta_{\beta}|_{\eta\gamma})]. \end{aligned} \tag{4.5}$$

The above results are novel for SP. With $\psi=\mu=0$ they simplify to the well-known stress functions in the Kirchhoff–Love and the Reissner theories [10, 2].

5. Donati's theorem. Compatibility equations

The Donati theorem was formulated within the framework of the linear theory of elasticity as regards three-dimensional problems [1]. In this section a similar theorem related to the two-dimensional equations of the linear six-parametric shell theory (SP) is given. To this end consider the functional (3.3) together with the equations of equilibrium (3.4).

THEOREM. *Let $\boldsymbol{\gamma}_{\alpha}$, $\boldsymbol{\varkappa}_{\alpha}$, $\varkappa_{3\alpha}$ and γ_{33} be of class C^1 on τ. Further, suppose that*

$$I=\int_{\tau}(\mathbf{N}^{\alpha}\cdot\boldsymbol{\gamma}_{\alpha}+\mathbf{M}^{\alpha}\cdot\boldsymbol{\varkappa}_{\alpha}+M^{3\alpha}\,\varkappa_{3\alpha}+N^{33}\,\gamma_{33})\,da=0 \tag{5.1}$$

for every class C^{∞} tensor fields $\mathbf{N}^{\alpha}$, $\mathbf{M}^{\alpha}$, $M^{3\alpha}$, N^{33} on $\bar{\tau}$ that vanish near $\partial\tau$ and satisfy the relations (3.4). *Then $\boldsymbol{\gamma}_{\alpha}$, $\boldsymbol{\varkappa}_{\alpha}$, $\varkappa_{3\alpha}$ and γ_{33} satisfy the equations of compatibility of* SP.

Proof. Let the stress functions $\boldsymbol{\varphi}$, $\boldsymbol{\theta}$, ψ and μ be of class C^{∞} on $\bar{\tau}$ and vanish near $\partial\tau$. Then, by the relations (4.1) $\mathbf{N}^{\alpha}$, $\mathbf{M}^{\alpha}$, $M^{3\alpha}$ and N^{33} are of class C^{∞} on $\bar{\tau}$ and vanish near $\partial\tau$. On substituting the relations (4.1) into Eq. (5.1), integrating by parts according to Eq. (2.6) with the line integrals on $\partial\tau$ disappearing as the stress functions vanish near $\partial\tau$, making use of Eq. $(2.5)_4$ and the identity $(\mathbf{g}_{\beta}\times\boldsymbol{\varphi})\cdot\boldsymbol{\varkappa}_{\alpha}=(\boldsymbol{\varkappa}_{\alpha}\times\mathbf{g}_{\beta})\cdot\boldsymbol{\varphi}$ (the latter holds for any mixed product of vectors), we get an equation of the form

$$\int_{\tau}[\boldsymbol{\varphi}\cdot(\ldots)+\boldsymbol{\theta}\cdot(\ldots)+\psi\,(\ldots)+\mu\,(\ldots)]\,da=0. \tag{5.2}$$

The above meets the assumptions of the Dubois–Rymond lemma, so the multipliers by the stress functions in Eq. (5.2) must be zero. Thus we arrive at

$$(5.3)\qquad \begin{gathered} \varepsilon^{\alpha\beta}\left[-\boldsymbol{\gamma}_{\alpha}|_{\beta}+(\boldsymbol{\varkappa}_{\alpha}+\mathbf{b}_{\alpha}\,\gamma_{33})\times\mathbf{g}_{\beta}\right]=0, \qquad \varepsilon^{\alpha\beta}(\boldsymbol{\varkappa}_{\alpha}+\mathbf{b}_{\alpha}\,\gamma_{33})|_{\beta}=0, \\ \varepsilon^{\alpha\beta}\,\varkappa_{3\alpha}|_{\beta}=0, \qquad (\varkappa_{3\alpha}-\gamma_{33}|_{\alpha})\,|^{\alpha}=0. \end{gathered}$$

These equations are identical with those (see [8]) ensuring the existence of a unique displacement field in shell deforming in conformity with the kinematical assumptions of SP. Consequently, Eqs. (5.3) are the needed compatibility equations of SP in vector form.

According to the Donati theorem, the compatibility equations (5.3) are related to the stress functions in the form (4.1). In the same way, the component form of Eqs. (5.3) can be ascribed to component representation (4.3) of the stress functions. Similarly, another component version of the compatibility equations will be obtained using the Donati theorem with the stress functions (4,5), to wit

$$(5.4)\qquad \begin{gathered} \varepsilon^{\beta\eta}\left[\varepsilon^{\alpha\gamma}(\varkappa_{\alpha\beta}+b_{\alpha\beta}\,\gamma_{33}-\gamma_{3\alpha}|_{\beta})|_{\eta}+\varepsilon^{\alpha\delta}\,b^{\gamma}_{\delta}\,\gamma_{\alpha\beta}|_{\eta}\right]=0, \\ \varepsilon^{\alpha\gamma}\,\varepsilon^{\beta\eta}\left[\gamma_{\alpha\beta}|_{\eta\gamma}-b_{\gamma\eta}\,(\varkappa_{\alpha\beta}+b_{\alpha\beta}\,\gamma_{33}-\gamma_{3\alpha}|_{\beta})\right]=0, \\ \varepsilon^{\alpha\beta}(\varkappa_{\alpha\beta}+b_{\alpha\beta}\,\gamma_{33}-\gamma_{3\alpha}|_{\beta}-b^{\beta}_{\lambda}\,\gamma_{\lambda\alpha})=0, \qquad \varepsilon^{\alpha\beta}\,\varkappa_{3\alpha}|_{\beta}=0, \qquad (\varkappa_{3\alpha}-\gamma_{33}|_{\alpha})^{\alpha}=0. \end{gathered}$$

These equations of compatibility are clearly equivalent to those given in [8] for a Cosserat surface. Assuming $\varkappa_{3\alpha}=\gamma_{33}=0$, Eqs. (5.4) become identical with the Reissner-type compatibility equations [2]; also when, $\gamma_{3\alpha}=0$, then Eqs. (5.4) are the same as the Kirchhoff–Love-type compatibility equations derived in [9].

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STRESZCZENIE

O TWIERDZENIU DONATIEGO W TEORII POWŁOK

Twierdzenie Donatiego zostało sformułowane w związku z trójwymiarowymi zagadnieniami liniowej teorii sprężystości. W niniejszej pracy rozważono podobne twierdzenie w zastosowaniu do dwuwymiarowych zagadnień liniowej teorii powłok. Przedyskutowano również ściśle z nim związane funkcje naprężeń i warunki zgodności. Założono, że powłoka deformuje się zgodnie z hipotezą liniowego rozkładu wektora przemieszczenia względem grubości powłoki. Zgodnie z powyższym wzięto pod uwagę sześć lokalnych stopni swobody powłoki. Otrzymane wyniki zawierają jako przypadek szczególny dobrze znane funkcje naprężeń i warunki zgodności w teorii Kirchhoffa–Love'a i Reissnera.

Резюме

О ТЕОРЕМЕ ДОНАТИ В ТЕОРИИ ОБОЛОЧЕК

Теорема Донати сформулирована в связи с трехмерными задачами линейной теории упругости. В настоящей работе рассмотрена аналогичная теорема в применении к двумерным задачам линейной теории оболочек. Обсуждены тоже, тесно с ними связанные, функции напряжений и условия совместности. Предполагается, что оболочка дефонмируетея согласно с гипотезой линейного наспределения перемешений на толщине оболочки. Согласно с вышеупомянутым взяты во внимание шесть локалвных степеней свободы оболочки. Полученные результаты содержат, как частный случай, хорошо известные функции напряжений и условия совместности в теории Кирхгоффа–Лява и Рейсснера.

TECHNICAL UNIVERSITY OF BIAŁYSTOK

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