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THERMOELASTIC CYLINDER SUBJECTED TO SUDDENLY APPLIED RADIAL SYMMETRIC PRESSURE

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Stresses, temperature and oscillations of a thermoelastic cylinder suddenly exposed to the radial symmetric pressure are analyzed in the paper. Solution of the corresponding eigenvalue problem is obtained by means of the perturbation technique, so that all the results are expressed as function of eigenvalues and eigenfunctions of the corresponding uncoupled (elastic) eigenvalue problem.

1. Introduction

At the instant $t = 0$, the perfectly insulated thermoelastic cylinder, having been previously in a homogeneous temperature field $T(r, \varphi, z, t) = T_0 = \text{const}$ $(t < 0)$, is exposed to the radial symmetric pressure Fig. 1. This pressure initiates a stress wave which, propagating through the body, disturbs its mechanical and thermal equilibrium. The object of this work is to find stresses, temperature and oscillations of the cylinder as functions of time and space variables.

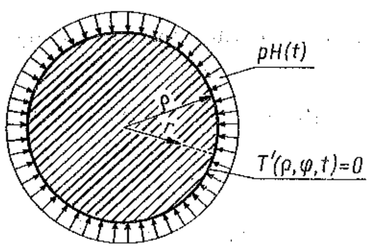

Fig. 1.

We assume that temperature changes and deformations are small, so that the problem may be analyzed within the linearized coupled thermoelasticity theory.

2. Partial differential equations of the problem

If the scales for length, time, temperature and stress are taken as $\frac{a}{c_1}$, $\frac{a}{c_1^2}$, $\frac{1-\nu}{\alpha_T(1+\nu)}$ and $\frac{(1-\nu)^E}{(1+\nu)(1-2\nu)}$, where a represents the thermal conductivity, c_1 — the isothermal velocity of the longitudinal wave, ν — the Poisson's ration, α_T — the thermal dilatation coefficient and E — the modulus of elasticity, we get the following set of partial differential equations, initial and boundary conditions of this coupled thermoelasticity problem in the dimensionless form:

$$u''+\frac{u'}{r}-\frac{u}{r^2}-\theta'-\ddot{u}=0, \tag{2.1}$$

$$\theta''+\frac{\theta'}{r}-\mathring{\theta}-\varepsilon\left(\dot{u}'+\frac{\dot{u}}{r}\right)=0, \tag{2.2}$$

$$u(r,0)=\dot{u}(r,0)=\theta(r,0)=\theta'(r,0)=0, \tag{2.3}$$

$$\text{for}\quad r=0:\quad |u|,\ |\theta|,\ |u'|,\ |\theta'|,\ |\dot{u}|,\ \ldots<\infty,$$

$$\text{for}\quad r=\varrho:\quad u'(\varrho,t)+\nu_1\cdot\frac{u(\varrho,t)}{\varrho}-\theta(\varrho,t)=-pH(t), \tag{2.4}$$

$$\theta'(\varrho,t)=0.$$

In the above equations u represents the radial displacement, $\theta = T-T_0$ — the increase of the temperature, r — the space coordinate, t — the time, p — the radial pressure, $H(t)$ — the Heaviside unit function, $\varepsilon=\frac{(1+\nu)\alpha_r^2 ET_0}{(1-\nu)(1-2\nu)c_\varepsilon}$ — the coupling coefficient, c_ε — the specific heat of undeformed material and $\nu_1=\frac{\nu}{1-\nu}$. A comma and a dot at the symbol denote partial differentiation of the corresponding variable in space and time, respectively.

Differentiation of the equation (2.2) with respect to r and introduction of the operator

$$\nabla=\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\,\partial r}-\frac{1}{r^2},$$

and the set of new variables u, $\dot{u}$ and θ', leads to the following system of partial differential equations in the matrix form:

$$\frac{\partial}{\partial t}\begin{matrix} u\\ \dot{u}\\ \theta'\end{matrix}=\begin{bmatrix}0 & 1 & 0\\ \nabla & 0 & -1\\ 0 & -\varepsilon\nabla & \nabla\end{bmatrix}\begin{bmatrix}u\\ \dot{u}\\ \theta'\end{bmatrix}, \tag{2.5}$$

which can be written simply

$$\dot{\mathbf{U}}(r, t) = L\mathbf{U}(r, t),$$

where the meanings of the vector function $\mathbf{U}(r, t)$ and the matrix differential operator L are evident. Here, we emphasize the fact that assumption of small strain and temperature amplitudes makes the operator L linear, bounded and continuous [3]. The procedure of solving the problem posed above can be simplified if by using the extended definition of the operator [3], this homogeneous system of p.d.e. with nonhomogeneous boundary conditions is replaced by an equivalent system of nonhomogeneous p.d.e. with homogeneous boundary conditions:

$$\dot{\mathbf{U}}(r, t) = L\mathbf{U}(r, t) + [\theta(\varrho, t) - pH(t)]\,\delta(r-\varrho)\begin{bmatrix}0\\1\\0\end{bmatrix}, \tag{2.6}$$

$$\mathbf{U}(r, 0) = \dot{0}, \tag{2.7}$$

$$|\mathbf{U}(0, t)| < \infty,$$

$$\begin{aligned} u_1'(\varrho, t) + \nu_1 \frac{u_1(\varrho, t)}{\varrho} &= 0,\\ u_2'(\varrho, t) + \nu_1 \frac{u_2(\varrho, t)}{\varrho} &= 0, \end{aligned} \tag{2.8}$$

$$u_3(\varrho, t) = 0,$$

where $\delta(r-\varrho)$ is Dirac's symbolic function.

3. Eigenvalues, eingenvectors and adjoint eingenvectors

Separation of time and space differential operators in the equation (2.6) allows for the reduction of the posed problem to the eigenvalue one

$$L\phi(r) = \lambda\phi(r), \tag{3.1}$$

with boundary conditions of the type (2.8).

In the equation (3.1) ϕ is the eigenvector and λ — the eigenvalue of the problem.

As known, the exact solution of this equation imposes considerable mathematical difficulties. In the simpler problem of a thermoeleastic layer exposed to the symmetric pressure [7], the exact solution of its eigenvalue problem leads to the practically unsolvable transcendental characteristic equation.

The perturbation method proposed in this paper is based on the fact that any spatial differential operator L of the linearized coupled thermoelasticity problem can be represented as

$$L = L_0 + \varepsilon \Delta L, \tag{3.2}$$

where L_0 is the differential operator of the corresponding uncoupled problem, ΔL — the coupling operator and ε — the coupling coefficient.

Generally, the coupling coefficient of a thermoelastic material is a small number. For example, if the temperature of the natural state is $T_0 = 293°K$, we obtain $\varepsilon = 0.028$ for aluminium, 0.011 for steel and even 0.001 for concrete. This is the main reason why the solutions of the classical elasticity theory (with $\varepsilon = 0$), although qualitatively unsatisfactory and disregarding the influence of displacement field on the temperature (and, consequently, thermoelastic damping of any mechanical process), are quantitatively so "close" to the corresponding coupled thermoelasticity solutions.

Suppose now that the solution of the eigenvalue equation

$$L_0 \psi(r) = \omega \psi(r), \tag{3.3}$$

of the uncoupled problem with boundary conditions of the type (2.8) is known, i.e., the eigenvalue spectrum ω_m $(m = 1, 2, \ldots)$ together with the corresponding eigenvector functions $\psi_m(r)$ are at our disposal. Suppose also that the set of eigenvectors $\psi_m^*(r)$ of the adjoint operator L_0^* is known. Let the sets ψ_m^* and ψ_n be complete and orthonormal, so that

$$\langle \psi_m^*, \psi_n \rangle = \delta_{mn},$$

where the sign $\langle , \rangle$ indicates the complex type scalar product $\int_S z (\bar{\psi}_m^*)' \psi_n dr$, r is the weight function, s denotes the region of integration and δ_{mn} is the Kronecker symbol.

Using Eq. (3.2), we can write for each eigenvector ϕ_i of the operator L

$$(\lambda_i - L_0) \phi_i = \varepsilon \Delta L \phi_i, \quad i = 1, 2, \ldots. \tag{3.4}$$

If ψ_m and ψ_m^* are assumed as the basis and the reciprocal basis of the vector space under consideration ϕ_i and $\Delta L \phi_i$ can be represented in the spectral form as

$$\phi_i(r) = \sum_m \alpha_{im} \psi_m(r), \tag{3.5}$$

and

$$\Delta L \phi_i(r) = \sum_m \beta_{im} \psi_m(r), \tag{3.6}$$

where α_{im} and β_{im} are the complex constants.

Making the left-hand scalar product of the equation (3.6) with adjoint vectors ψ_m^*, we get

$$\beta_{im} = \langle \psi_m^*, \Delta L \phi_i \rangle. \tag{3.7}$$

Using the same procedure with the equation

$$\sum_m (\lambda_i - \omega_m)\, \alpha_{im}\, \psi_m = \varepsilon \sum_m \langle \psi_m^*, \Delta L \phi_i \rangle\, \psi_m, \tag{3.8}$$

we find

$$\alpha_{im} = \varepsilon \frac{\langle \psi_m^*, \Delta L \phi_i \rangle}{\lambda_i - \omega_m}, \tag{3.9}$$

and hence

$$\phi_i(r) = \varepsilon \sum_m \frac{\langle \psi_m^*, \Delta L \phi_i \rangle}{\lambda_i - \omega_m}\, \psi_m(r). \tag{3.10}$$

We make an assumption here that λ_i and ϕ_i are close to ω_i and ψ_i, respectively. Then we can look for eigenvalues and eigenvectors of the posed problem in the following form of power series in the coupling coefficient ε:

$$\lambda_i = \omega_i + \nu_{i1}\, \varepsilon + \nu_{i2}\, \varepsilon^2 + \ldots, \tag{3.11}$$

$$\phi_i = \psi_i + \mathbf{N}_{i1}\, \varepsilon + \mathbf{N}_{i2}\, \varepsilon^2 + \ldots. \tag{3.12}$$

Clearly, the leadings terms obtainable upon setting $\varepsilon = 0$, are solutions of the corresponding uncoupled eigenvalue problem. The complex constants ν_{ij} and the complex vector functions $\mathbf{N}_{ij}(r)$ are to be determined.

We substitute λ_i and ϕ as given by Eq. (3.12) in Eq. (3.10) and obtain

$$\psi_i + \mathbf{N}_{i1}\, \varepsilon + \mathbf{N}_{i2}\, \varepsilon^2 + \ldots = \varepsilon \sum_m \frac{\langle \psi_m^*, \Delta L (\psi_i + \mathbf{N}_{i1}\, \varepsilon + \mathbf{N}_{i2}\, \varepsilon^2 + \ldots)\rangle}{\omega_i + \nu_{i1}\, \varepsilon + \nu_{i2}\, \varepsilon^2 + \ldots - \omega_m}\, \psi_m. \tag{3.13}$$

Comparing the coefficient of ψ_i on the left side with the one on the right-hand side of this equation, we find

$$\nu_{i1}\, \varepsilon + \nu_{i2}\, \varepsilon^2 + \ldots = \varepsilon \langle \psi_i^*, \Delta L (\psi_i + \mathbf{N}_{i1}\, \varepsilon + \mathbf{N}_{i2}\, \varepsilon^2 + \ldots)\rangle,$$

and therefore

$$\nu_{i1} = \langle \psi_i^*, \Delta L \psi_i \rangle, \tag{3.14}$$

$$\nu_{i2} = \langle \psi_i^*, \Delta L \mathbf{N}_{i1} \rangle, \ldots. \tag{3.15}$$

On the other hand, if all the other terms of the perturbation series on the left and right-hand sides of the Eq. (3.13) are equated, the following result is obtained

$$\mathbf{N}_{i1}\, \varepsilon + \mathbf{N}_{i2}\, \varepsilon^2 + \ldots = \varepsilon \Sigma' \frac{\langle \psi_m^*, \Delta L (\psi_i + \mathbf{N}_{i1}\, \varepsilon + \mathbf{N}_{i2}\, \varepsilon^2 + \ldots)\rangle}{(\omega_i - \omega_m)\left(1 + \dfrac{\nu_{i1}}{\omega_i - \omega_m}\, \varepsilon + \dfrac{\nu_{i2}}{\omega_i - \omega_m}\, \varepsilon^2 + \ldots\right)}\, \psi_m,$$

where symbol Σ' indicates that the sum is to be taken over all values of m except $m = i$. This equation can be written in the following way as well

$$(3.16)\quad \mathbf{N}_{i1}\,\varepsilon+\mathbf{N}_{i2}\,\varepsilon^2+\ldots=\varepsilon\sum_m{}'\left[\frac{1}{\omega_i-\omega_m}-\frac{\nu_{i1}}{(\omega_i-\omega_m)^2}\,\varepsilon+\ldots\right]\times$$
$$\times\langle\psi_m^*,\,\Delta L(\psi_i+\mathbf{N}_{i1}\,\varepsilon+\ldots\rangle\,\psi_m.$$

Equating the coefficients of ε, $\varepsilon^2, \ldots$, we find

$$(3.17)\qquad \mathbf{N}_{i1}=\sum_m{}'\,\frac{\langle\psi_m^*,\,\Delta L\psi_i\rangle}{\omega_i-\omega_m}\,\psi_m,$$

$$(3.18)\qquad \mathbf{N}_{i2}=\sum_m{}'\left[\frac{\langle\psi_m^*,\,\Delta L\mathbf{N}_{i1}\rangle}{\omega_i-\omega_m}-\frac{\nu_{i1}\,\langle\psi_m^*,\,\Delta L\psi_i\rangle}{(\omega_i-\omega_m)^2}\right]\psi_m,\ \ldots.$$

Solution of the problem (2.6)–(2.8) requires also the knowledge of adjoint eigenvectors $\phi_i^*(r)$.

It is easy to conclude that the adjoint operator L^* may be written in the same way as L in Eq. (3.2),

$$(3.19)\qquad L^* = L_0^*+\varepsilon\,(\Delta L)^*.$$

In this relation, L_0^* is the adjoint operator of the corresponding uncoupled problem, and $(\Delta L)^*$ is the adjoint of the coupling operator.

Since λ_i represent the eigenvalue spectrum of the operator L, the complex conjugate

$$(3.20)\qquad \bar{\lambda}_i=\bar{\omega}_1+\bar{\nu}_{i1}\,\varepsilon+\bar{\nu}_{i2}\,\varepsilon^2+\ldots,$$

is the eigenvalue spectrum of the adjoint operator L^*.

Repeating the already used procedure we can find the adjoint vector functions $\phi_i^*(r)$. For that purpose, expansions of ϕ_i^* and $(\Delta L)^*\phi_i^*$ in terms of ψ_m^* are required, together with expansion of ϕ_i^* into the perturbation series in terms of the coupling coefficient

$$(3.21)\qquad \phi_i^*=\psi_i^*+\mathbf{M}_{i1}\,\varepsilon+\mathbf{M}_{i2}\,\varepsilon^2+\ldots.$$

Consequently, the identities are obtained

$$\bar{\nu}_{i1}=\langle\psi_i,(\Delta L)^*\psi_i^*\rangle\equiv\overline{\langle\psi_i^*,\,\Delta L\psi_i\rangle},$$

$$\bar{\nu}_{i2}=\langle\psi_i,(\Delta L)^*\mathbf{M}_{i1}\rangle\equiv\overline{\langle\psi_i^*,\,\Delta L\mathbf{N}_{i1}\rangle},\ldots,$$

followed by the functional coefficients of Eq. (3.21)

$$(3.22)\quad \mathbf{M}_{i1}=\sum_m{}'\,\frac{\langle\psi_m,(\Delta L)^*\,\psi_i^*\rangle}{\bar{\omega}_i-\bar{\omega}_m}\,\psi_m^*,$$

$$(3.23)\quad \mathbf{M}_{i2}=\sum_m{}'\left[\frac{\langle\psi_m,(\Delta L)^*\,\mathbf{M}_{i1}\rangle}{\bar{\omega}_i-\bar{\omega}_m}-\frac{\nu_{i1}\,\langle\psi_m,(\Delta L)^*\,\psi_i^*\rangle}{(\bar{\omega}_i-\bar{\omega}_m)^2}\right]\psi_m^*,\ \ldots.$$

Therefore, we write the space operator of our problem in the form

$$L = L_0 + \varepsilon \Delta L = \begin{bmatrix} 0 & 1 & 0 \\ \Delta & 0 & -1 \\ 0 & 0 & \Delta \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\Delta & 0 \end{bmatrix}. \tag{3.24}$$

Eigenvalues ω_k of the operator L_0 are grouped in sets of three ($k = 1, 2, 3; 4, 5, 6; 7, 8, 9; \ldots$). Two imaginary, mutually conjugate eigenvalues $\omega_1 = \bar{\omega}_2$, $\omega_4 = \bar{\omega}_5, \ldots$ are successive solutions of the equation

$$i\omega\varrho J_0(i\omega\varrho) = (1 - \nu_1) J_1(i\omega\varrho), \tag{3.25}$$

and the third $\omega_3, \omega_6, \ldots$ (index divisible by 3) are successive solutions of

$$J_1(i\sqrt{\omega}\,\varrho) = 0. \tag{3.26}$$

In the above equations i is the imaginary unit and J_0, J_1 are Bessel's functions of the first kind.

Eigenvectors ψ_k are also grouped in sets of three. The first and the second (1, 2; 4, 5; ...) ones are the complex conjugate. For example,

$$\psi_1(z) = \bar{\psi}_2(r) = a_1 \begin{bmatrix} 1 \\ \omega_1 \\ 0 \end{bmatrix} J_1(i\omega_1 z). \tag{3.27}$$

The third eigenvector in every set (3, 6, 9, ...) is real. For example,

$$\psi_3(r) = a_3 \left[\begin{bmatrix} 1 \\ \omega_3 \\ 0 \end{bmatrix} J_1(i\omega_3 r) + \frac{(1-\nu_1)\dfrac{J_1(i\omega_3\varrho)}{\varrho} - i\omega_3 J_0(i\omega_3\varrho)}{i\sqrt{\omega_3}\, J_0(i\sqrt{\omega_3}\,\varrho)} \times \begin{bmatrix} 1 \\ \omega_3 \\ \omega_3 - \omega_3^2 \end{bmatrix} J_1(i\sqrt{\omega_3}\, r) \right]. \tag{3.28}$$

Relation

$$\langle \mathbf{G}, L\mathbf{F} \rangle = \langle L^* \mathbf{G}, \mathbf{F} \rangle, \tag{3.29}$$

leads to the adjoint operator

$$L^* = \begin{bmatrix} 0 & \nabla & 0 \\ 1 & 0 & -\varepsilon\nabla \\ 0 & -1 & \nabla \end{bmatrix}, \tag{3.30}$$

with the boundary conditions

$$\begin{gathered} |\bar{\mathbf{V}}(0)| < \infty, \\ \bar{v}_2'(\varrho) + \nu_1 \frac{\bar{v}_2(\varrho)}{\varrho} = 0, \\ \bar{v}_3(\varrho) = 0, \quad \varepsilon \bar{v}_3'(\varrho) = 0, \end{gathered} \tag{3.31}$$

obtained from the requirement that the transformation of the left-hand side of the Eq. (3.29) into the right-hand side, should be homogeneous.

So we can see that, according to (3.19),

$$(3.32)\qquad L_0^* = \begin{bmatrix} 0 & \nabla & 0 \\ 1 & 0 & 0 \\ 0 & -1 & \nabla \end{bmatrix} \quad \text{and} \quad (\Delta L)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\nabla \\ 0 & 0 & 0 \end{bmatrix}.$$

Knowing that the eigenvalue spectrum of L_0^* is ω_k ($k = 1, 2, 3; 4, 5, 6; \ldots$), we can easily find the adjoint eigenvectors ψ_k^*. Again, in each of the three vectors, the first and the second ones are the complex conjugate and the third one is real. For example,

$$(3.33)\qquad \bar{\psi}_1^*(r) = \bar{\psi}_2^*(r) = b_1 \begin{bmatrix} \omega_1^2(\omega_1-1) \\ \omega_1(\omega_1-1) \\ 1 \end{bmatrix} J_1(i\omega_1 r) - \frac{J_1(i\omega_1 \varrho)}{J_1(i\sqrt{\omega_1}\,\varrho)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} J_1(i\sqrt{\omega_1}\,r),$$

$$(3.34)\qquad \bar{\psi}_3^*(r) = b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} J_1(i\sqrt{\omega_3}\,r).$$

The bar over ψ_3^* is purely formal, due to its subsequent use in the complex scalar products.

Coefficients a_k and b_k of the eigenvectors and the adjoint eigenvectors, obtained from the condition

$$\langle \psi_i^*, \psi_j \rangle = \delta_{ij},$$

are

$$a_1 = b_1 = \bar{a}_2 = \bar{b}_2 = [(\omega_1 - 1)(\omega_1^2 \varrho^2 - 1 - \nu_1^2)\, J_1^2(\omega_1 \varrho)]^{-\frac{1}{2}},$$

and

$$a_3 = b_3 = \sqrt{2}\,\{i\sqrt{\omega_3}\,\varrho\,(1-\omega_3)\,[i\omega_3\,\varrho\, J_0(i\omega_3\,\varrho) - (1-\nu_1)\, J_1(i\omega_3\,\varrho)]\, J_o(i\sqrt{\omega_3}\,\varrho)\}^{-\frac{1}{2}},$$

and so on.

Keeping in Eq. (3.11) the linear terms only, we find now the eigenvalues of the problem (3.1) in the following form

$$(3.35)\qquad \lambda_1 = \bar{\lambda}_2 \approx \omega_1 + \varepsilon \left\{ \frac{\omega_1}{2(1-\omega_1)} + \frac{\omega_1^2 [(1-\nu_1)\, J_1(i\sqrt{\omega_1}\,\varrho) - i\sqrt{\omega_1}\,\varrho\, J_0(i\sqrt{\omega_1}\,\varrho)]}{(1-\omega_1)^2 (\omega_1^2 \varrho^2 + 1 - \nu_1^2)\, J_1(i\sqrt{\omega_1}\,\varrho)} \right\},$$

$$(3.36)\qquad \lambda_3 \approx \omega_3 - \varepsilon \left\{ \frac{\omega_3}{1-\omega_3} + \right.$$

$$\left. + \frac{2\omega_3^2 J_1(i\omega_3 \varrho)}{(1-\omega_3)^2 [i\omega_3 \varrho J_0(i\omega_3 \varrho) - (1-\nu_1) J_1(i\omega_3 \varrho)]} \right\},$$

and so on.

Taking into account the properties of some real thermoelastic material, one can verify these results: two complex conjugate eigenvalues λ_1 and λ_2 have negative real parts, what is essential for the manifestation of thermoelastic damping. The third eigenvalue remains real and negative.

Eigenvectors and the adjoint eigenvectors of our problem are, according to Eqs. (3.12) and (3.21),

$$(3.37)\qquad \phi_k(r) \approx c_k \left[\psi_k(r) + \varepsilon \Sigma' \frac{\langle \psi_i^*, \Delta L \psi_k \rangle}{\omega_k - \omega_i} \psi_i(r) \right],$$

$$(3.38)\qquad \phi_k^*(r) \approx d_k \left[\bar{\psi}_k^*(r) + \varepsilon \Sigma' \frac{\langle \bar{\psi}_i, (\Delta L)^* \bar{\psi}_k^* \rangle}{\omega_k - \omega_i} \bar{\psi}_i^*(r) \right].$$

As stated above, the prime at the summation symbol means that the term with the index $i = k$ is to be omitted.

4. Solution

Now we seek a solution of Eqs. (2.6)–(2.8) in the form

$$(4.1)\qquad \mathbf{U}(r, t) = \sum_{k=1}^{\infty} \alpha_k(t) \phi_k(r),$$

where α_k are, for the time being, certain unknown complex functions of time. Using Eq. (3.1), we also have

$$(4.2)\qquad L\mathbf{U}(r, t) = \sum_{k=1}^{\infty} \lambda_k \alpha_k(t) \phi_k(r).$$

Free term of the equation (2.6) $\mathbf{F}(r) = [\theta(\varrho, t) - pH(t)]\, \delta(r-\varrho) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

expanded in the eigenvectors has the form

$$(4.3)\qquad F(r, t) = \sum_{k=1}^{\infty} \beta_k(t) \phi_k(r),$$

with the coefficients

$$(4.4)\quad \beta_k(t) = \langle \phi_k^*, \mathbf{F}\rangle = \varrho c_k [\theta(\varrho, t) - pH(t)] \times \times \left[\bar{\psi}_{k2}^*(\varrho) + \varepsilon \sum_{k=1}^{\infty}{}' \frac{\langle \bar{\Psi}_i, (\Delta L)^* \bar{\Psi}_k^* \rangle}{\omega_k - \omega_i} \bar{\psi}_{i2}^*(\varrho)\right].$$

Symbol ψ_{k2}^* denotes the second coordinate of the respective vector.

Introducing those expansions into Eq. (2.6) and taking into account that the eigenvectors are linearly independent, we get the infinite set of the first order differential equations with respect to time:

$$(4.5)\quad \dot{\alpha}_k(t) - \lambda_k \alpha_k(t) - \beta_k(t) = 0, \quad k = 1, 2, 3, \ldots,$$

with initial conditions resulting from (2.7) $\alpha_k(0) = 0$.

Applying the Laplace transforms to these equations, we get

$$(4.6)\quad \begin{gathered} s\hat{\alpha}_k(s) - \alpha_k(0) - \lambda_k \hat{\alpha}_k(s) - \beta_k(s) = 0, \\ \hat{\alpha}_k = \frac{\hat{\beta}_k}{s - \lambda_k} = \frac{\hat{\theta}(\varrho, s) - \dfrac{p}{s}}{s - \lambda_k} e_k, \quad k = 1, 2, 3, \ldots, \end{gathered}$$

where the symbol $(\hat{\cdot})$ means $\mathscr{L}(\cdot)$ and

$$e_k = \varrho c_k \left[\bar{\psi}_{k2}^*(\varrho) + \varepsilon \sum_{k=1}^{\infty}{}' \frac{\langle \bar{\Psi}_i, (\Delta L)^* \bar{\Psi}_k^* \rangle}{\omega_k - \omega_i} \bar{\psi}_{i2}^*(\varrho)\right].$$

Taking the Laplace transform of the expansion (4.1) we arrive at

$$(4.7)\quad \hat{\mathbf{U}}(r, s) = \sum_{k=1}^{\infty} \hat{\alpha}_k(s) \phi_k(r) = \left[\hat{\theta}(\varrho, s) - \frac{p}{s}\right] \sum_{k=1}^{\infty} \frac{e_k}{s - \lambda_k} \phi_k(r).$$

Integrating the third coordinate of this vector with respect to r, we obtain the Laplace transform of the temperature increase in the form

$$(4.8)\quad \hat{\theta}(r, s) = \left[\hat{\theta}(\varrho, s) - \frac{p}{s}\right] \sum_{k=1}^{\infty} \frac{f_k(r)}{s - \lambda_k} + \hat{c}(s),$$

with the notation

$$f_k(r) = e_k \int \varphi_{k3}(r)\, dz.$$

$\hat{c}(s)$ being the integration function. This function must be equal to zero because, for $p = 0$, $\theta(r, t) = \theta(\varrho, t) = \hat{\theta}(r, s) = \hat{\theta}(\varrho, s) \equiv 0$.

For $r = \varrho$, we obtain

$$(4.9)\quad \hat{\theta}(\varrho, s) = \frac{p}{s} \cdot \frac{\displaystyle\sum_{k=1}^{\infty} \frac{f_k(\varrho)}{s - \lambda_k}}{\displaystyle\sum_{k=1}^{\infty} \frac{f_k(\varrho)}{s - \lambda_k} - 1}.$$

Using this result, together with the final-value theorem it is easy to find the limit temperature increase in the cylinder

$$\lim_{t\to\infty}\theta(\varrho,t)=\lim_{t\to\infty}\theta(r,t)=\lim_{s\to 0}s\hat{\theta}(\varrho,s)=p\,\frac{\sum_{k=1}^{\infty}\frac{f_k(\varrho)}{\lambda_k}}{\sum_{k=1}^{\infty}\frac{f_k(\varrho)}{\lambda_k}+1}. \tag{4.10}$$

The expression (4.9) may also be written in the following form,

$$\hat{\theta}(\varrho,s)=\frac{p}{s}\,\frac{P(s)}{Q(s)},$$

$$P(s)=\prod_{j=1}^{\infty}{}'(s-\lambda_j)\sum_{k=1}^{\infty}f_k(\varrho),$$

$$Q(s)=\prod_{j=1}^{\infty}{}'(s-\lambda_j)\sum_{k=1}^{\infty}[f_k(\varrho)+\lambda_k-s].$$

The symbol $\prod_{j=1}^{\infty}{}'$ represents the product of all factors $s-\lambda_j$, except the one with index $j=k$.

The degree of the polynomial $P(s)$ is lower by one than the degree of the polynomial $Q(s)$. Suppose that $Q(s)$ has infinitely many distinct zeros α_i, $i=1,2,3,\ldots$. Then, making use of the Heaviside expansion formula, together with the convolution theorem, we can find the boundary temperature increase in the form

$$\theta(\varrho,t)=p\sum_{i=1}^{\infty}\frac{P(\alpha_i)}{Q'(\alpha_i)}(e^{\alpha_i t}-1), \tag{4.11}$$

with the abbreviation $Q'=\dfrac{dQ(s)}{ds}$.

Using the convolution theorem again, it is easy to obtain the inverse Laplace transform of the expression (4.8), that is, the temperature increase at any point of the cylinder

$$\theta(r,t)=p\sum_{k=1}^{\infty}\left\{\sum_{i=1}^{\infty}\frac{e^{\alpha_i t}-e^{\lambda_k t}}{\alpha_i-\lambda_k}\cdot\frac{P(\alpha_i)}{Q'(\alpha_i)}+\frac{1-e^{\lambda_k t}}{\lambda_k}\left[1+\sum_{i=1}^{\infty}\frac{P(\alpha_i)}{Q'(\alpha_i)}\right]\right\}f_k(r). \tag{4.12}$$

Applying the same procedure to find the inverse Laplace transform of the first coordinate of the vector (4.7), we get the radial displacement

$$u(r,t)=p\sum_{k=1}^{\infty}e_k\left\{\sum_{i=1}^{\infty}\frac{e^{\alpha_i t}-e^{\lambda_k t}}{\alpha_i-\lambda_k}\cdot\frac{P(\alpha_i)}{Q'(\alpha_i)}+\right.$$
$$\left.+\frac{1-e^{\lambda_k t}}{\lambda_k}\left[1+\sum_{i=1}^{\infty}\frac{P(\alpha_i)}{Q'(\alpha_i)}\right]\right\}\varphi_{k1}(r). \tag{4.13}$$

Finally, the radial stress may be represented in the form

$$(4.14) \quad \sigma(r,t) = u'(r,t) - \nu_1 \frac{u(r,t)}{r} - \theta(r,t) =$$

$$= p \sum_{k=1}^{\infty} e_k \left\{ \sum_{i=1}^{\infty} \frac{e^{\alpha_i t} - e^{\lambda_k t}}{\alpha_i - \lambda_k} \cdot \frac{P(\alpha_i)}{Q'(\alpha_i)} + \frac{1 - e^{\lambda_k t}}{\lambda_k} \times \right.$$

$$\left. \times \left[1 + \sum_{i=1}^{\infty} \frac{P(\alpha_i)}{Q'(\alpha_i)} \right] \right\} \left[\varphi'_{k1}(r) - \nu_1 \frac{\varphi_{k1}(r)}{r} - \int \varphi_{k3}(r)\, dr \right].$$

It is clear (see Sect. 2) that all these quantities are dimensionless.

If we introduce $\varepsilon = 0$ in those solutions, we obtain undamped, "elastic" solutions, with $\theta(r,t)$ (homogeneous function of the coupling coefficient), equal to zero.

5. Conclusions

In this paper, the thermoelastic cylinder exposed to a suddenly applied radial symmetric pressure is examined.

The assumption is made that the temperature increase and the deformations are small so that the problem belongs to the linearized coupled thermoelasticity theory.

The corresponding eigenvalue problem is solved by means of the perturbation method and all the results, representing stresses, temperature and oscillations as functions of time and space variables, are expressed in terms of the eigenvalues and eigenfunctions of the corresponding uncoupled eigenvalue problem.

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Streszczenie

WALEC TERMOSPRĘŻYSTY POD DZIAŁANIEM NAGLE PRZYŁOŻONEGO SYMETRYCZNEGO OBCIĄŻENIA PROMIENIOWEGO

W pracy omówiono problem wyznaczania naprężeń, temperatury i drgań walca termosprężystego poddanego nagle przyłożonemu, symetrycznemu obciążeniu radialnemu. Rozwiązanie odpowiedniego problemu wartości własnych otrzymano za pomocą metody perturbacji; w ten sposób rozwiązanie wyrazić można w postaci funkcji wartości i funkcji własnych odpowiedniego niesprężonego problemu dla ciała sprężystego.

Резюме

ТЕРМОУПРУГИЙ ЦИЛИНДР ПОД ДЕЙСТВИЕМ ВНЕЗАПНО ПРИЛОЖЕННОЙ СИММЕТРИЧНОЙ РАДИАЛЬНОЙ НАГРУЗКИ

В работе рассматривается задача по определению напряжений, температуры и колебаний термоупругого цилиндра, подверженного действию внезано приложенной, симметричной радиальной нагрузки. Решение соответствующей задачи собственных значений было получено с помощью метода петрубации; таким образом решение можно выразить в виде функции собственных значений соответствующей несопряженной задачи для упругого тела.

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