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THE POST-BUCKLING BEHAVIOUR OF THIN-WALLED COLUMNS IN THE ELASTO-PLASTIC RANGE

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The post-buckling behaviour in the elasto-plastic range of thin-walled, steel columns subject to uniform compression is analysed. The problem is solved using the Rayleigh–Ritz variational method involving plasticity. The plastic stress-strain relations are described by Prandtl–Reuss equations. Numerical calculations have been carried out for different geometrical parameters and material properties. The load-shortening curves and the spreads of plasticity regions in component plates of a column are presented in figures.

Notation

$S^i_j, S_{\alpha\beta}, S_{33}$	deviatoric stress tensor,
$\tau^i_j, \tau_{\alpha\beta}$	Cauchy's stress tensor,
δ^i_j	Kronecker's delta,
$\varepsilon_{\alpha\beta}, \varepsilon_{33}$	in-plane direct and shear strain,
μ, λ	Lamé constants,
$d\zeta$	plastic strain increment factor,
U	total potential energy,
T_i	vector components of the edge forces per unit area,
u_i	displacement component,
v_α	middle surface in-plane displacement,
$w(\theta_\alpha)$	out-plane displacement,
$n_{\alpha\beta}, n_{xx}, n_{yy}, n_{xy}$	membrane sectional forces per unit length,
$m_{\alpha\beta}, m_{xx}, m_{yy}, m_{xy}$	bending sectional moments per unit length,
ϕ	Airy's stress function,
x, y, z	coordinate system for a plate,
a	length, width of a plate,
h	thickness of a plate,
u, v, w	in-plane and out-plane displacements,
f_1, f_2	independent parameters of the deflection function,
δ	amplitude of local buckle deflection in a center of a plate,
S	applied compressive strain,
E	Young's modulus,
ν	Poisson's ratio,
σ_Y	yield stress in simple tension,
σ_{av}	average stress corresponding to S,

σ_{cr} elastic critical buckling stress $\sigma_{cr} = \dfrac{\pi^2 E}{3(1-\nu^2)}\left(\dfrac{h}{\alpha}\right)^2$.

The superscripts e, p denote the elastic and plastic parts of strain increments.

1. Introduction

It is well known that the collapse analysis of a plate loaded in-plane is relevant to all forms of steel-plated constructions, e.g. box-girder bridges, ships and cranes. In spite of the fact that a great deal of attention has been given to the elastic analysis of a plate buckling locally under in-plane compression, relatively fewer attempts have been made to account for plasticity in the post-buckling range.

The rigorous numerical solutions to the plate collapse analysis problem have been done by Graves-Smith [1], Moxham [2], Crisfield [3], Frieze and others [4], Little [5], using energy methods, finite element formulation or the finite difference dynamic approach. Recently, Bradfield and Stonor [6] presented a simplified elastic-plastic analysis for plates uniaxially loaded, in which the full section yield criterion (Ilyushin criterion) was used.

In this work a more precise approach to the problem solved by T. R. Graves–Smith in his pioneering studies is presented.

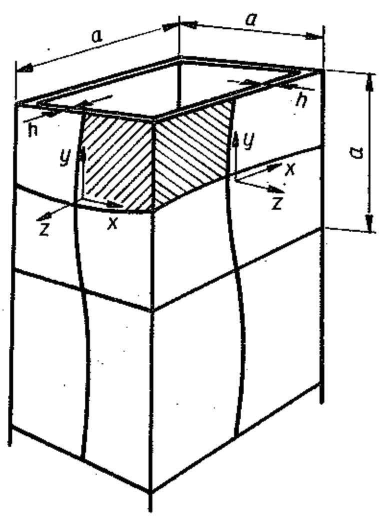

Fig. 1. Square section with local buckles.

The thin-walled box column (Fig. 1) of a square cross-section, subject to uniform compression, is considered. The column sections are of such proportions that local buckling occurs entirely elastically and then the interaction between large deflections and the spread of yielding through the volume of the plate has to be taken into account.

2. Yield criterion

The plastic yield of the isotropic material is governed by the Huber–Mises yield criterion, which may be written in the form

$$S^i_j\, S^i_j = \frac{2}{3}\,\sigma^2_y, \tag{2.1}$$

where the deviatoric stress tensor component is defined by the equation

$$S^i_j = \tau^i_j - \frac{1}{3}\,\delta^i_j\, \tau^k_k, \tag{2.2}$$

and σ_y is the experimental material yield stress. For a plate we have the system of Cartesian coordinates θ_α $(\alpha = 1, 2)$ and θ_3, where θ_3 is orthogonal to θ_α so that the yield criterion can be expressed in terms of in-plane components of the deviatoric stress tensor as follows:

$$\begin{aligned} S_{\alpha\beta}\, S_{\alpha\beta} + S^2_{\gamma\gamma} &= \frac{2}{3}\,\sigma^2_y, \qquad (\alpha, \beta = 1, 2), \\ S_{\gamma\gamma} &= -S_{33}. \end{aligned} \tag{2.3}$$

3. Plastic stress-strain relations

It is assumed that after yielding differential increments of strain, $d\varepsilon$ are a total of elastic and plastic components $d\varepsilon^e$ and $d\varepsilon^p$. Applying the usual elastic stress-strain relations to the problem of a plate (Cartesian coordinates), the equations for in-plane increments become

$$d\tau_{\alpha\beta} = 2\mu\, d\varepsilon_{\alpha\beta} + \lambda\delta_{\alpha\beta}\,(d\varepsilon_{\gamma\gamma} + d\varepsilon_{33}) - 2\mu\, d\varepsilon^p_{\alpha\beta}, \quad \alpha, \beta = 1, 2. \tag{3.1}$$

In these equations the fact that there is no change in volume as a result of plastic strain increments has been taken into account.

Further it is assumed that strain hardening effects are neglected and the material being elastic-perfectly plastic is governed by the Prandtl-Reuss equations

$$d\varepsilon^p_{\alpha\beta} = d\zeta\, S_{\alpha\beta}. \tag{3.2}$$

The scalar $d\zeta$ is determinate $(d\zeta > 0)$ when a known strain increment is applied to the plastic material in such a way that unloading from the yield surface does not occur, thus the corresponding stress changes must satisfy the differential form of the yield criterion (2.3).

According to (3.2) the equations for in-plane stress increments become

$$d\tau_{\alpha\beta} = 2\mu\, d\varepsilon_{\alpha\beta} + \frac{2\mu\lambda}{2\mu+\lambda}\,\delta_{\alpha\beta}\, d\varepsilon_{33} - d\zeta\left[2\mu\, S_{\alpha\beta} + \frac{2\mu}{2\mu+\lambda}\,\delta_{\alpha\beta}\, S_{33}\right]. \tag{3.3}$$

In the elastic region the stress-strain relations take the form

$$\tau_{\alpha\beta} = 2\mu\, \varepsilon_{\alpha\beta} + \frac{2\mu\lambda}{2\mu+\lambda}\, \delta_{\alpha\beta}\, \varepsilon_{33}. \tag{3.4}$$

The scalar $d\zeta$ is determined by

$$d\zeta = \frac{S_{\alpha\beta}\, d\varepsilon_{\alpha\beta} + S_{33}\, d\varepsilon_{33}\, [\lambda/(2\mu+\lambda)]}{S_{\alpha\beta}\, S_{\alpha\beta} + S_{33}^2\, [\lambda/(2\mu+\lambda)]}. \tag{3.5}$$

In a practical calculation, finite increments of strain are applied to the elasto-plastic body and the stress increments $\Delta\tau$ can be found approximately by substituting Δ for d in the equations. The deviatoric stress tensor is determined after each strain application by making a running total of the stress increments as they are calculated, noting that

$$\Delta S_{\alpha\beta} = \Delta\tau_{\alpha\beta} - \frac{1}{3}\, \delta_{\alpha\beta}\, \Delta\tau_{33}. \tag{3.6}$$

4. The variational principle

Let the potential energy in the elasto-plastic range have the form

$$U = \int\limits_E \frac{\tau_{\alpha\beta}\, \varepsilon_{\alpha\beta}}{2}\, dE + \int\limits_P \left[\left(\tau_0^{\alpha\beta} + \frac{\Delta\tau^{\alpha\beta}}{2}\right) \Delta\varepsilon_{\alpha\beta} + K\right] dP - \int\limits_c \int\limits_{-h/2}^{h/2} T_i\, u_i\, d\theta_3\, dC, \tag{4.1}$$

$\int\limits_E$, $\int\limits_P$ represent volume integrals taken over the elastic and plastic parts of the column plates.

K is some scalar constant representing the integral $\int\limits_0^{\varepsilon_0} \tau_{\alpha\beta}\, d\varepsilon_{\alpha\beta}$ taken in the plastic zones up to the stress and strain levels τ_0, ε_0, existing prior to the current strain increment (for the purposes of minimization this strain energy may be arbitrarily put equal to zero), T_i are the vector components of the edge forces per unit area.

The functional (4.1) can be expressed in terms of displacements using the second order in-plane strain displacement relations:

$$2\varepsilon_{\alpha\beta} = v_\alpha|_\beta + v_\beta|_\alpha - 2\theta_3\, w|_{\alpha\beta} + w|_\alpha\, w|_\beta, \tag{4.2}$$

where $v_\alpha(\theta_1, \theta_2)$ are the middle surface in-plane displacements and $w(\theta_1, \theta_2)$ is the out-of-plane displacement.

It was proved by Graves-Smith that by equating to zero the variation of a potential energy functional, with respect to virtual displacements satisfying the geometric boundary conditions, we can obtain Euler equations representing the equations of equilibrium and the static boundary conditions of an elasto-plastic plate undergoing finite deflections.

The equilibrium equations for the plate have the form

$$n^{\alpha\beta}|_{\beta} = 0, \tag{4.3}$$

$$m^{\alpha\beta}|_{\alpha\beta} + (n^{\alpha\beta} w|_{\alpha})|_{\beta} = 0, \tag{4.4}$$

where

$$n^{\alpha\beta} = \int_{-h/2}^{h/2} \tau^{\alpha\beta} d\theta_3, \quad m^{\alpha\beta} = \int_{-h/2}^{h/2} \theta_3 \tau_{\alpha\beta} d\theta_3.$$

A stress function $\phi(\theta_1, \theta_2)$ can be introduced from which $n^{\alpha\beta}$ are derived that satisfy Eq. (4.3) identically.

For Cartesian coordinates x, y we have

$$n_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad n_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad n_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y}. \tag{4.5}$$

The compatibility condition may be derived from Eqs. (4.2) (3.4) by eliminating v_{α}.

The compatibility condition and the second equation of equilibrium are the well-known von Kármán equations.

5. Extent of plastic zones

The extent of plastic zones in the cross-section can be determined by establishing the values z/h for which the elastic stresses satisfy the Huber–Mises Criterion (Fig. 2). Substituting into Eq. (2.3) the stresses expressed by the strains (3.4), we obtain the values e_1 and e_2 which are therefore the roots of the equation of the second degree.

If $e_2 < z/h < e_1$ plate is elastic,
$z/h > e_1$, $z/h < e_2$ plate is plastic,

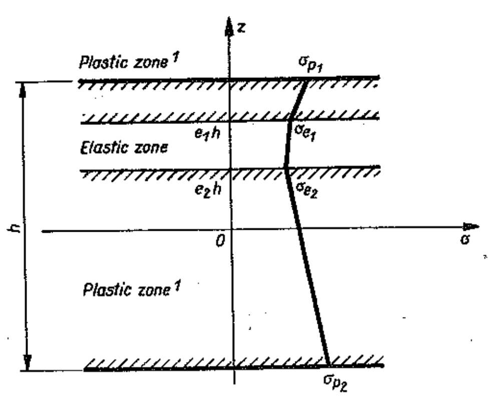

Fig. 2. Extent of plastic zones in the cross-section and assumed distribution of stress.

$|e_i| > 0{,}5$ $(i = 1, 2)$ plate is entirely elastic and
e_i — complex plate is entirely plastic.

In order to integrate the energy functional through the thickness, an approximate method is used in which the stresses are found accurately on the surface and are then assumed to vary linearly with depth until reaching their values at the elasto-plastic interfaces.

6. Boundary conditions

In the problem considered the square built-up column consists of four identical plates. The boundary conditions correspond to those occurring in a square plate simply supported along all edges. The unloaded edges are assumed to be stress-free and the loaded edges to remain straight, so the change in distance between them is proportional to the applied compressive strain S. It should also be noted that a plate is initially flat and stress-free.

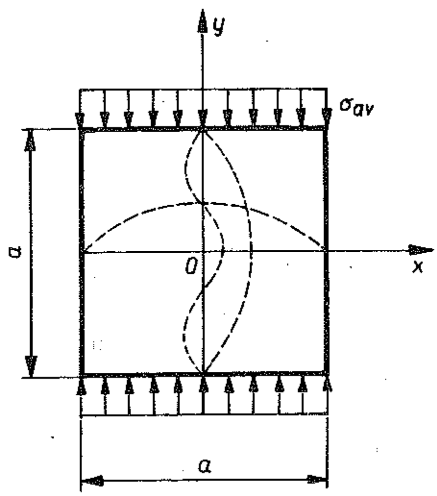

Fig. 3. Plate geometry with applied loading and boundary conditions.

The general arrangement of a single plate, loading, deflection functions, coordinate systems and boundary conditions are given in Fig. 3.

7. Airy stress function and displacement functions

In Graves-Smith's work [1] and many others, the deflection function of a single plate was assumed as follows:

$$w = \delta \cos \frac{\pi y}{a} \cos \frac{\pi x}{a}. \tag{7.1}$$

It is well known [7] that the deflection function of a form

$$(7.2)\qquad w = f_1 \cos\frac{\pi y}{a}\cos\frac{\pi x}{a} + f_2 \cos\frac{3\pi y}{a}\cos\frac{\pi x}{a},$$

gives more "exact" values for the so-called equivalent width of a post buckled plate in the elastic range. Little [5] has suggested that for plates with a small ratio h/a the representation (7.2) seems to be more appropriate to predict the ultimate load in the elasto-plastic range.

In this paper the function of out-plane deflection is chosen in terms of two independent parameters f_1 and f_2.

The stress function ϕ, satisfying the static boundary conditions (Fig. 3), is found using von Kármán's equation.

$$(7.3)\qquad \phi = (H_1 \cosh 2\gamma x + H_2\, x \sinh 2\gamma x)\cos 2\gamma y + (H_3 \cosh 4\gamma x + H_4\, x \sinh 4\gamma x)\cos 4\gamma y + (H_5 \cosh 6\gamma x + H_6\, x \sinh 6\gamma x)\cos 6\gamma y + E\left\{-\frac{1}{32} f_1^2 (\cos 2\gamma x + \cos 2\gamma y) - \frac{1}{32} f_2^2\left(9\cos 2\gamma x + \frac{1}{9}\cos 6\gamma y\right) - \frac{1}{16} f_1 f_2\left(\cos 2\gamma y + \frac{1}{4}\cos 4\gamma y + \cos 2\gamma x \cos 2\gamma y + \frac{1}{25}\cos 2\gamma x \cos 4\gamma y\right) + C\frac{x^2}{2}\right\},$$

where $\gamma = \dfrac{\pi}{a}$, C, from H_1 to H_6 — constants known from the boundary conditions.

Next, for the elastic case, the in-plane displacement functions u, v, fulfilling kinematic boundary conditions are determined by applying Eqs. (3.4) and (4.2). The forms of these functions in the elasto-plastic state are assumed to be the same, but their amplitudes may take any values without resulting in kinematically inadmissible displacements.

The deflection parameters f_1 and f_2 can be found, according to the Rayleigh–Ritz variational method, by minimizing the functional of a potential energy U (4.1).

Knowing the extent of plastic zones throughout the cross-section (values e^1 and e^2) and applying analytical integration over the thickness, the volume integrals in Eq. (4.1) may be changed into surface integrals.

The method requires now the repeated evaluation of integrals over the surface of a plate. In order to accomplish this only a quadrant of the plate surface is considered, enclosed by a mesh and all integrations are performed numerically.

To provide the convergence of the procedure, the parameters e_1 and e_2 are assumed to be invariant during each minimization. In order to satisfy the requirements of the plastic theory, the analysed system is subjected to a load which increases in small increments, thus the response of the column to the increment of the longitudinal strain ΔS is examined.

The average stress is found numerically, using the principle of virtual work

$$\sigma_{av} = \frac{1}{4a^2 h} \frac{\partial U}{\partial S}. \tag{7.4}$$

It should be underlined that the average stress σ_{av} corresponds directly to the carrying capacity load of a column.

8. Results of numerical calculations

Numerical calculations have been carried out for the columns of the square sections of the geometrical parameters h/a (thickness to width) varying from 0.005 to 0.017. The material properties have been used as follows: $E = 2.06 \cdot 10^5$ MPa, $\nu = 0.3$, $\sigma_y = 256,372,496$ MPa.

Theoretical curves showing the relation average stress σ_{av} versus the compressive strain S are given in Fig. 4 and the corresponding amplitudes

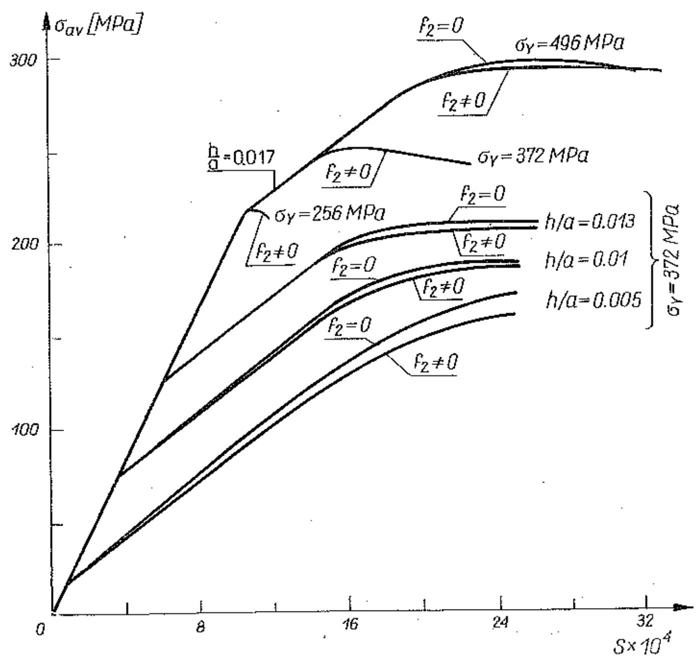

FIG. 4. Stress-strain curves.

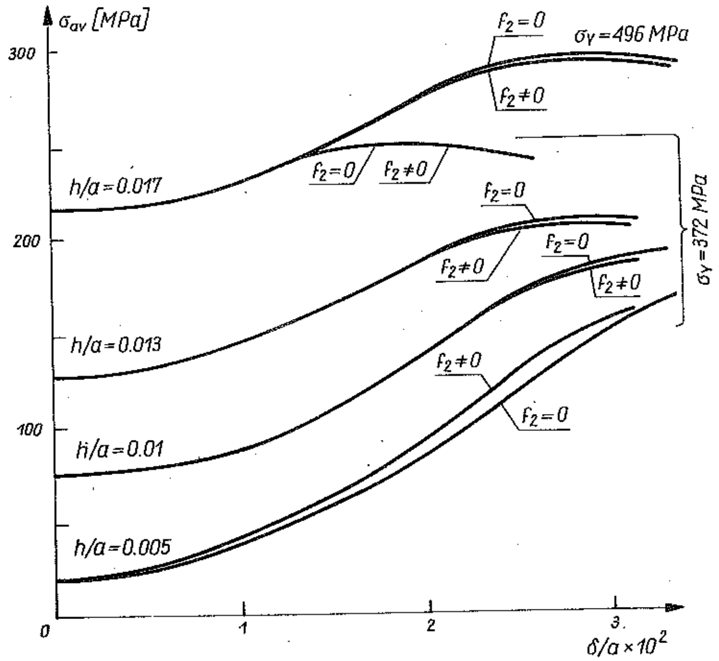

FIG. 5. Amplitude of local buckles.

of local buckles are plotted in Fig. 5. The spreads of plasticity regions are presented in Figs. 6, 7 and 8. In all cases calculations have been carried out using deflection functions with one independent parameter ($f_2 = 0$) and two parameters ($f_2 \neq 0$).

Discussing the results of numerical calculations we should notice that the character of "load-shortening" curves depends on the yield stress and on the ratio thickness to the width of a plate. When σ_y is constant (in our case equal 372 MPa), it has been found that the remarkable influence of the second parameter of the deflection function on the average stress occurs only for thin plates. For plates of $h/a = 0.13, 0.01$ the decreasing of the average stress is in the range from 2 to 3 per cent and for plates of $h/a = 0.005$ is about 5 per cent. For thick plates ($h/a = 0.017$), such as were examined by Graves-Smith, the discrepancy in average stresses (applying the deflection function with one or two independent parameters) appears for larger values of σ_y.

The influence of the yield limit on a character of the curves σ_{av} versus S, for plates of $h/a = 0.017$, has been analysed. When the yield stress is near the critical stress (e.g. $\sigma_{cr}/\sigma_y = 0.85$) of a plate, the collapse load is reached very soon and after the rapid decrease of loading capacity is observed. For the ratio σ_{cr} to σ_y equal 0.39 the load-shortening curve has a long "plateau".

The effect of different ratios thickness to width of a plate on the character of the load-shortening relation has been investigated for plates of $h/a = 0.017, 0.013, 0.01, 0.005$ with $\sigma_y = 372$ MPa. It is easy to notice that only the curve for a thick plate reaches maximum, for plates of $h/a = 0.013$ and 0.01 the curves have the long plateau and for a thin plate ($h/a = 0.005$) the relation σ_{av} versus S is still increasing. It should be underlined that numerical calculations have been stopped when the plate edges of a width equal to 1/8 "a" became completely plastic.

If the amplitudes of local buckles are concerned (Fig. 5), it may be found that for the fixed load ($\sigma_{av} = \text{const}$) the introduction of the second parameter causes the increase of the amplitude for plates of $h/a = 0.013, 0.01$. For a plate of $h/a = 0.005$ the situation is quite opposite, it means that the shape of the deflected plate becomes more flat.

The spreads of plasticity regions have been determined for each examined plate. Because of symmetry, only a quadrant of the plate is considered. In thick plates the plastic zones appear in the middle of the plate (Fig. 6) and when the longitudinal strain increases, almost the whole plate is in the elasto-plastic state. Thin plates ($h/a = 0.005$) behave in a different way: plasticity is encountered only in the neighbourhood of the unloaded edges, which become completely plastic, and the middle of a plate is entirely elastic (Fig. 7).

It has been found that for thick plates there is no influence of a second parameter of the deflection function on the spreads of plasticity regions,

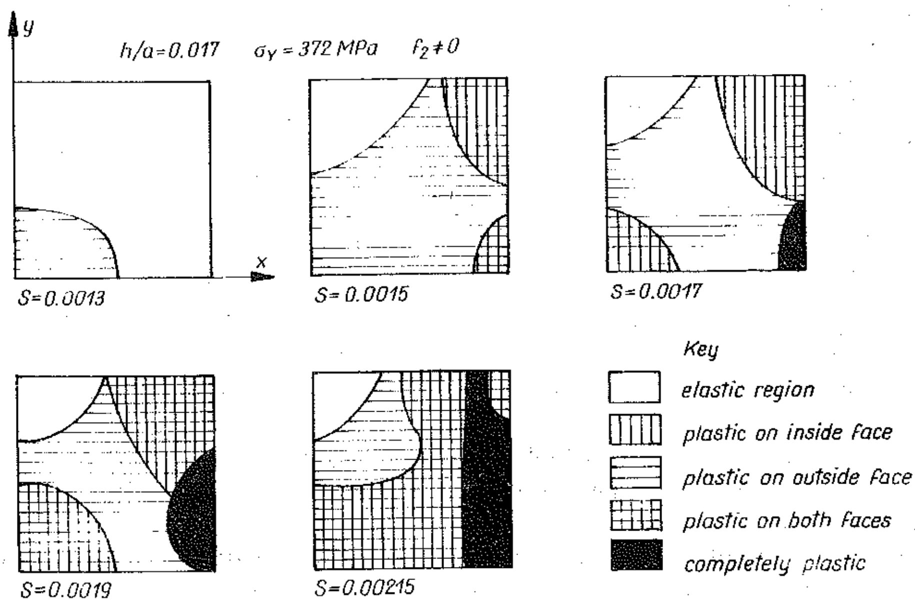

FIG. 6. Spread of plastic zones in thick plate.

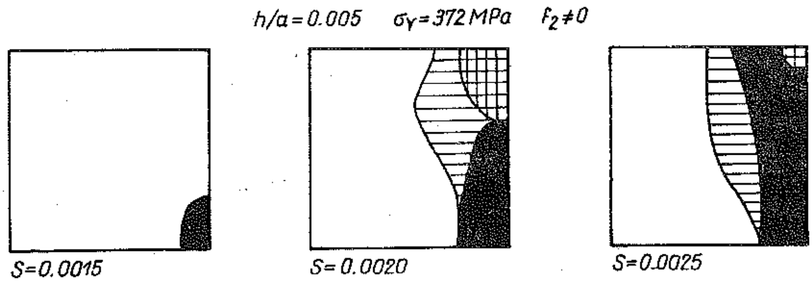

FIG. 7. Spread of plastic zones in thin plate.

small differences appear for plates of $h/a = 0.013$ and the distinct ones may be observed for plates of $h/a = 0.01$ (see Fig. 8).

The results of theoretical calculations have been compared with experimental tests done by MOXHAM [8]. These experiments seem to have been carried out very carefully and the results in the elastic range have the great accuracy. Moxham tested long, rather thick plates ($h/a > 0.0125$), initially flat, with the yield limit equal to 240 MPa.

Comparison can be done only for plates for which the yield stress is larger than the critical stress and their values differ slightly (e.g. $h/a = 0.017$, $\sigma_{cr} = 215$ MPa, $\sigma_y = 256$ MPa). In such cases the experimental and theoretical results are in good agreement, regarding the character of load-shortening curves and the values of the average stress and the longitudinal strain in the maximum point.

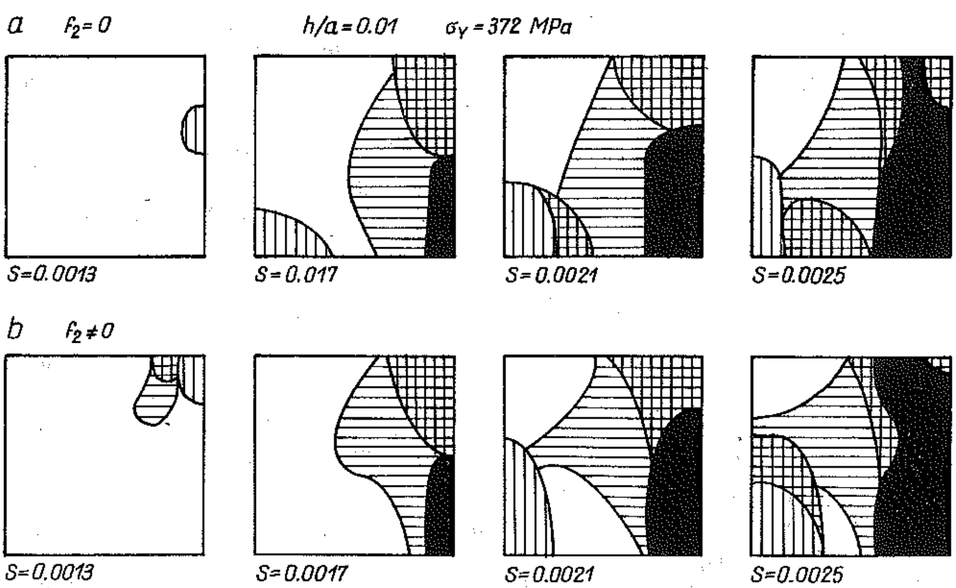

FIG. 8. Influence of a deflection function on the spreads of plasticity regions.

During the calculations the effect of the increment size ΔS on the accuracy has been analysed. Because of the assumptions of the plastic theory, a small shortening increment would appear imperative. This has been tested by investigating the same two plates using successively larger increment. It has been shown that although it is important to use relatively small increments in order to pin-point the maximum load, the increment size may then be increased significantly to calculate post-collapse behaviour. In this paper the fairly small increments have been used, sometimes unnecessarily small.

The problem is solved upon the assumption that the application of ΔS does not cause a strain reversal at local plastic zones. For a chosen few plates, after each minimization, the sign of $d\zeta$ (3.5) has been checked at the points in question. In the cases examined it has found that no unloading from the yield surface occurs.

9. Final remarks

Comparing the results given in this paper with previous theoretical works, we must admit that the direct comparison can be made only for precisely the same pair of E and σ_y values. This has been done only with Graves–Smith's work for plates of $h/a = 0.017$ and the results are in a good agreement. Also the character of load-shortening curves for different ratios h/a agress well with Little's results [5].

In most papers the ultimate load of plate corresponding to the stress at which the purely compressive stiffness is zero, has been found. From the results considered here it follows that the ultimate load can be determined only in the case of plates for which the load-shortening curves get the maximum point or the long plateau.

It sems that the deflection function with one parameter is sufficiently accurate for most engineering purposes provided h/a is not small; for small h/a the representation with two independent parameters is more appropriate.

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Streszczenie

STAN ZAKRYTYCZNY CIENKOŚCIENNYCH SŁUPÓW W OBSZARZE SPRĘŻYSTO-PLASTYCZNYM

W pracy zbadano stan zakrytyczny cienkościennych stalowych słupów poddanych równomiernemu ściskaniu, pracujących w obszarze sprężysto-plastycznym. Problem został rozwiązany metodą wariacyjną Rayleigha–Ritza z uwzględnieniem plastyczności. Związki naprężenie-odkształcenie w obszarze plastycznym opisane są równaniami Prandtla–Reussa. Obliczenia numeryczne przeprowadzono dla różnych parametrów geometrycznych i własności materiałowych. Wyniki w postaci zależności obciążenie jako funkcja skrócenia słupa przedstawiono na wykresach. Na wybranych przykładach pokazano również rozwój stref plastycznych w płytach składowych słupa.

Резюме

ЗАКРИТИЧЕСКОЕ СОСТОЯНИЕ ТОНКОСТЕННЫХ ОПОР В УПРУГО-ПЛАСТИЧЕСКОЙ ОБЛАСТИ

В работе исследовано закритическое состояние тонкостенных стальных опор подвергнутых равномерному сжатию, работающих в упруго-пластической области. Задача решена вариационным методом Релея-Ритца с учетом пластичности. Соотношения напряжение — деформация в пластической области описаны уравнениями Прандтля–Рейса. Численные расчеты проведены для разных геометрических параметров и матерйальных свойств. Результаты в виде нагружения как функции сокращения опоры представлены на графиках. На избранных примерах показано тоже развитие пластических зон в плитах составляющих опоры.

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