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ON AN EFFECTIVE METHOD OF EVALUATION OF THE EFFECT OF CAVITIES, INCLUSIONS AND CRACKS UPON THE STRESS FIELDS IN ELASTIC MEDIA

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Cracks, holes and inclusions introduce certain disturbances into the stress fields produced in elastic media by external loads. Theoretical foundations of the analysis of such stress fields are well known and, in principle, problems of this kind may always be reduced to the solution of the corresponding sets of integral equations. However, the effective determination of stresses may prove to be not so simple, first of all in the ceses when the number of defects is high and the stress fields are singular. In such cases the finite differences or finite elements methods become impractical. This paper is aimed at presenting such an approximate method of analysis which resembles the approach known from elementary structural mechanics and which is applied to statically indeterminate structures; it reduces the problem considered to the solution of a rather simple set of algebraic equations. If the mutual distances between the elliptical inclusions and cavities are not smaller than their dimensions, the accuracy achieved will be satisfactory from the point of view of engineering applications.

1. Introduction

The problem of interaction of various defects in elastic solids subject to the action of external loads has been dealt with in numerous papers. Here let us mention several fundamental papers [1—3] published by J. D. Eshelby and an extensive study by H. Zorski [4] who considered the problem within the context of a general theory of mechanics of defects. Many other papers and books were devoted to both the theoretical and practical aspects of analysis of stresses and displacements produced in continuous media and structural elements containing defects of various kinds: cracks, cavities and inclusions in particular. For instance, let us quote here three books [5—7] published quite recently. In most cases the methods of analysis proposed there are based on various methods of analytical or numerical solution of more or less complicated systems of integral equations governing the problem under consideration.

In principle it is always possible to reduce the problem of analysis of elastic media containing holes (cavities), cracks or elastic (rigid) inclusions to the solution of the corresponding set of integral equations. An effective solution of such equations may lead, however, to serious difficulties in the cases when the number of defects is large and their distribution within the body is nonuniform. If, in addition, the defects introduce certain singularities into the stress field, the numerical methods of analysis based on finite differences or finite elements may require the introduction of a very large number of unknowns and other data and become highly time-consuming and impractical.

A slightly different approach to the problem was proposed by this author in 1975 and later [8—10]; in this approach the number of unknowns is reduced considerably in spite of a relatively high accuracy of the results obtained. A somewhat similar approach was also suggested by D. GROSS in 1982 (and by SHU-ANG ZHOU in a still unpublished paper) [11].

The present paper constitutes a continuation and generalization of the ideas discussed in papers [8—10]. We are not going to deal with the theoretical foundations of the problem which may be considered as known: our principal aim consists in presenting a simple and effective method of analysis of the state of stress produced in an elastic, infinite medium containing three types of defects: elliptical holes, elliptical inclusions (elastic or rigid) and cracks. The method presented in this paper makes it possible to apply the approach resembling the analysis of statically indeterminate systems, known from elementary structural mechanics. In the case of n defects existing in the body, the number of "redundant" elements equals n, $2n$, $4n$ or more, depending on the accuracy required. This number is considerably reduced in the case of symmetry and under the assumption that the distribution of defects is not very dense. In view of simplicity of the procedures used in the analysis, the method may prove to be useful in many engineering applications; it should also be stressed that it does not require (in most cases) any sophisticated computer equipment.

In order to make the presentation of the method as simple as possible, the considerations will be confined to the antiplane state of strain (or stationary heat flow) in unbounded, isotropic media. However, generalization of the results obtained to the plane strain and plane stress problems is not connected with any serious difficulties as it was shown in the paper [10] concerning the interaction of cracks under Mode I crack deformation conditions.

It is known that the antiplane state of strain in an isotropic elastic medium is governed by the simple Poisson equation

$$\mu\nabla^2 w(x, y) = -p(x, y), \tag{1.1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $w(x, y)$ is the displacement measured in the direction of the z-axis of a rectangular (x, y, z) or cylindrical (r, θ, z) coordinate system, μ is the elastic shear modulus, and $p(x, y)$ — intensity of the body forces parallel to the z-axis (Fig. 1). The only non-zero stress

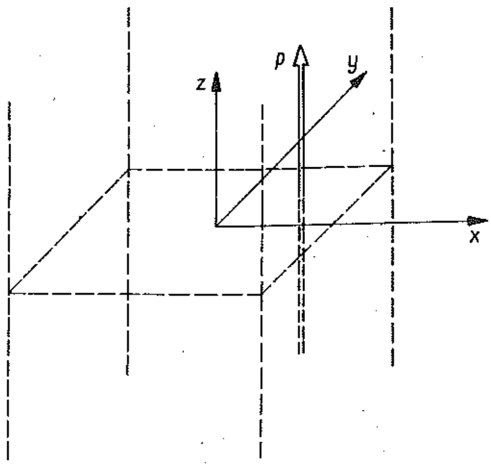

FIG. 1.

tensor components are expressed in terms of the displacement w by the formula

(1.2) $$\sigma_{xz} = \mu \frac{\partial w}{\partial x}, \qquad \sigma_{yz} = \mu \frac{\partial w}{\partial y}.$$

In cylindrical coordinates $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$, we obtain

(1.3) $$\sigma_{rz} = \mu \frac{\partial w}{\partial r}, \qquad \sigma_{\theta z} = \mu \frac{1}{r} \frac{\partial w}{\partial \theta}.$$

Equations (1.1)—(1.3) may also be used to describe (under different notation) the problem of two-dimensional, stationary heat flow in a solid characterized by the heat conduction coefficient λ and containing linear heat sources of intensity $W(x, y)$,

(1.1′) $$\lambda \nabla^2 T(x, y) = -W(x, y),$$

(1.2′) $$q_x = -\lambda \frac{\partial T}{\partial x}, \qquad q_y = -\lambda \frac{\partial T}{\partial y}.$$

Here $T(x, y)$ denotes the temperature, and q_x, q_y are the heat flux vector components. Owing to this analogy, all solutions concerning the antiplane state of stress in elastic media may also be used to describe the solution of the corresponding stationary plane heat flow problem.

Let us assume that the elastic medium is unbounded and the body forces are represented by concentrated forces $P\delta(x)\,\delta(y)$ uniformly distributed along the z-axis (Fig. 1); the solution of Eq. (1.1) satisfying the condition of vanishing of stresses at infinity is easily found to have the form

$$
\begin{aligned}
& w = \frac{P}{2\pi\mu}\log\frac{\text{const}}{r},\\
& \sigma_{xz} = -\frac{P}{2\pi}\frac{x}{r^2} = -\frac{P}{2\pi}\frac{\cos\theta}{r},\\
(1.4)\qquad & \sigma_{yz} = -\frac{P}{2\pi}\frac{y}{r^2} = -\frac{P}{2\pi}\frac{\sin\theta}{r},\\
& \sigma_{rz} = \sigma_{xz}\cos\theta+\sigma_{yz}\sin\theta = -\frac{P}{2\pi}\frac{1}{r},\\
& \sigma_{\theta z} = -\sigma_{xz}\sin\theta+\sigma_{yz}\cos\theta = 0.
\end{aligned}
$$

Let us now introduce the following notations: solutions (1.4) corresponding to unit force loading $P = 1$ are denoted by two superscripts 0, 0, so that, for instance,

$$
\begin{aligned}
& w^{0,0} = \frac{1}{2\pi\mu}\log\frac{\text{const}}{r},\\
(1.5)\qquad & \sigma_{rz}^{0,0} = -\frac{1}{2\pi}\frac{1}{r}, \qquad \sigma_{r\theta}^{0,0} = 0.
\end{aligned}
$$

Actually, P denotes the linear density of the body forces applied to the medium and has the dimension of force/length.

Applying now two concentrated forces: P at point $(0, -\delta)$ and $-P$ at point $(0, \delta)$, denoting the product $2\delta P$ by $M^{0,1}$ and passing to the limit $\delta\to 0$, we obtain the solutions which are denoted by the superscripts 0,1:

$$
w^{0,1} = \frac{\partial w^{0,0}}{\partial y}, \qquad \sigma_{xz}^{0,1} = \frac{\partial\sigma_{xz}^{0,0}}{\partial y}, \qquad \text{etc.}
$$

The magnitude $M^{0,1}$ may be interpreted as a concentrated moment (force dipole) of the first order, uniformly distributed along the z-axis. In the heat transfer theory $M^{0,1}$ should be interpreted as heat source dipole intensity (cf. [12]).

In a similar manner higher order moments (multipoles) may formally be introduced. For instance, the second order moment is obtained by applying two first order moments: $M^{0,1}$ at point $(-\delta, 0)$ and $-M^{0,1}$ at $(\delta, 0)$, denoting the product $2\delta M^{0,1}$ by $M^{1,1}$ and passing to the limit $\delta\to 0$.

Application of such moments at the respective points $(0, -\delta)$ and $(0, \delta)$ leads to another second order moment $M^{0,2}$. The corresponding solutions are obtained by simple differentiations:

$$w^{1,1} = \frac{\partial w^{0,1}}{\partial x}, \qquad w^{0,2} = \frac{\partial w^{0,1}}{\partial y}, \qquad \text{etc.}$$

The solutions corresponding to the $(m+n)$-th order moments, $M^{m,n}$, have the simple form

$$(1.6)\qquad \begin{aligned} w(x, y) &= M^{m,n} w^{m,n}(x, y), \\ \sigma_{xz}(x, y) &= M^{m,n} \sigma_{xz}^{m,n}(x, y), \qquad \text{etc.,} \end{aligned}$$

functions $w^{m,n}$ and $\sigma_{xz}^{m,n}$, $\sigma_{yz}^{m,n}$ being determined from the formulae

$$(1.7)\qquad \begin{aligned} w^{m,n} &= \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} [w^{0,0}(x, y)], \\ \sigma_{kz}^{m,n} &= \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} [\sigma_{kz}^{0,0}(x, y)], \qquad k = x, y, r, \theta. \end{aligned}$$

In Fig. 2 are shown elementary, graphical representations of several first, second and third order moments. It should be observed that the second order moments $M^{1,1}$ shown in Figs. 2c and 2d are strictly equivalent.

Substitution of the solutions (1.5) into the formula (1.7) leads to a simple form of solutions of arbitrary orders. For instance,

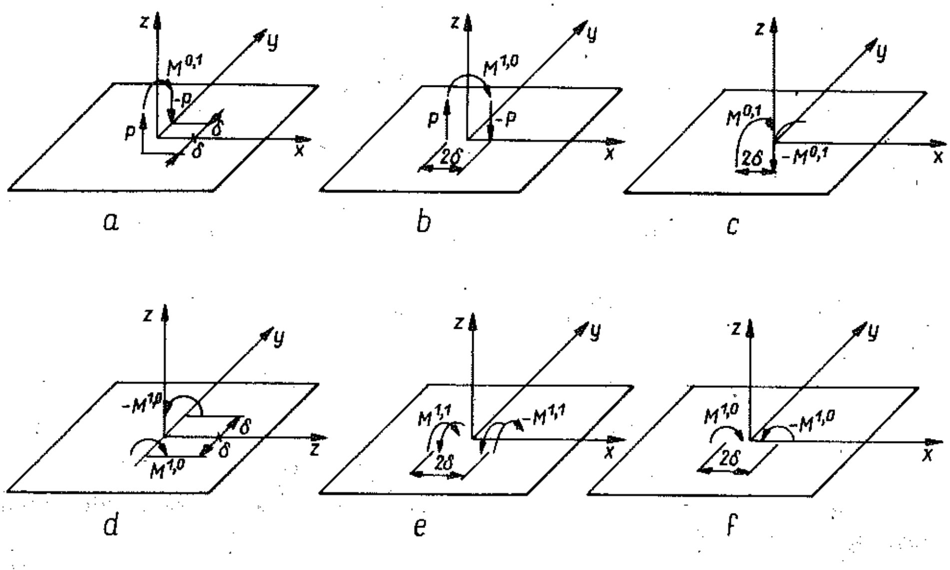

FIG. 2.

$$w^{m,1} = \frac{1}{2\pi\mu}(-1)^{m+1}\, m!\, \frac{\sin(m+1)\theta}{r^{m+1}},$$

(1.8)
$$\sigma_{xz}^{m,1} = -\frac{1}{2\pi}(-1)^{m+1}(m+1)!\, \frac{\sin(m+2)\theta}{r^{m+2}},$$

$$\sigma_{yz}^{m,1} = \frac{1}{2\pi}(-1)^{m+1}(m+1)!\, \frac{\cos(m+2)\theta}{r^{m+2}}.$$

The above solutions assume the simplest form once the complex potential $F(z)$ is introduced (cf., e.g., [13]). Equations (1.1) and (1.2) are known to be satisfied identically by the real and imaginary parts of an analytic function of complex variable $z = x+iy$ (not to be confused with the coordinate z),

(1.9)
$$\mu w(x, y) = \operatorname{Im} F(z), \qquad \Sigma(z) = \frac{dF}{dz} = \sigma_{yz} + i\sigma_{xz},$$

$$\sigma_{xz}(x, y) = \operatorname{Im}\frac{dF}{dz}, \qquad \sigma_{yz}(x, y) = \operatorname{Re}\frac{dF}{dz}.$$

The symbol $\Sigma(z)$ denotes here the complex stress tensor (vector) $\sigma_{yz}+i\sigma_{xz}$.

It is easily verified that the function

$$F^{0,0}(z) = -\frac{i}{2\pi}\log z$$

yields the results (1.4) since, according to Eqs. (1.9), we obtain

$$\operatorname{Im}\left[-\frac{i}{2\pi}\log z\right] = -\frac{1}{2\pi}\log r,$$

$$\frac{dF^{0,0}}{dz} = -\frac{i}{2\pi}\frac{1}{z} = -\frac{1}{2\pi}\frac{\sin\theta + i\cos\theta}{r}.$$

Notations

$$-\frac{i}{2\pi}\log z = F^{0,0}(z), \qquad -\frac{i}{2\pi}\frac{1}{z} = \Sigma^{0,0}(z),$$

make it possible to write down a simple general formula

(1.10)
$$\Sigma^{m,n}(z) = \frac{(-1)^{m+n+1}\, i^{n+1}}{2\pi}(m+n)!\, \frac{1}{z^{m+n+1}}.$$

Analysis of Eq. (1.10) leads to a simple conclusion that all moments of the same order $(m+n+1 = \text{const})$ lead to complex stresses differing at most by the coefficients $\pm i$ or ± 1,

$$\Sigma^{m-1,n+1} = i\Sigma^{m,n}, \qquad \Sigma^{m-2,n+2} = -\Sigma^{m,n}, \qquad \Sigma^{m-4,n+4} = \Sigma^{m,n}.$$

Let us present the explicit forms of solutions corresponding to some pairs of superscripts 0, 1, 2.

(1.11)
$$w^{0,1} = -\frac{1}{2\pi\mu}\frac{\sin\theta}{r}, \qquad w^{1,0} = -\frac{1}{2\pi\mu}\frac{\cos\theta}{r},$$
$$\sigma_{xz}^{0,1} = \frac{1}{2\pi}\frac{\sin 2\theta}{r^2}, \qquad \sigma_{xz}^{1,0} = \frac{1}{2\pi}\frac{\cos 2\theta}{r^2},$$
$$\sigma_{yz}^{0,1} = -\frac{1}{2\pi}\frac{\cos 2\theta}{r^2}, \qquad \sigma_{yz}^{1,0} = \frac{1}{2\pi}\frac{\sin 2\theta}{r^2},$$
$$\Sigma^{0,1} = -\frac{1}{2\pi}\frac{1}{z^2}, \qquad \Sigma^{1,0} = \frac{i}{2\pi}\frac{1}{z^2};$$

(1.12)
$$w^{1,1} = \frac{1}{2\pi\mu}\frac{\sin 2\theta}{r^2}, \qquad w^{2,0} = \frac{1}{2\pi\mu}\frac{\cos 2\theta}{r^2},$$
$$\sigma_{xz}^{1,1} = -\frac{1}{2\pi}\frac{2\sin 3\theta}{r^3}, \qquad \sigma_{xz}^{2,0} = -\frac{1}{2\pi}\frac{2\cos 3\theta}{r^3},$$
$$\sigma_{yz}^{1,1} = \frac{1}{2\pi}\frac{2\cos 3\theta}{r^2}, \qquad \sigma_{yz}^{2,0} = -\frac{1}{2\pi}\frac{2\sin 3\theta}{r^3},$$
$$\Sigma^{1,1} = \frac{1}{2\pi}\frac{2}{z^3}, \qquad \Sigma^{2,0} = -\frac{2i}{2\pi}\frac{1}{z^3};$$

(1.13)
$$w^{2,1} = -\frac{1}{2\pi\mu}\frac{2\sin 3\theta}{r^3}, \qquad \Sigma^{2,1} = -\frac{1}{2\pi}\frac{6}{z^4},$$
$$\sigma_{xz}^{2,1} = \frac{1}{2\pi}\frac{6\sin 4\theta}{r^4}, \qquad \sigma_{yz}^{2,1} = -\frac{1}{2\pi}\frac{6\cos 4\theta}{r^4}.$$

2. Circular hole and inclusion

Let us consider the problem of antiplane state of strain in an unbounded elastic body containing an elastic cylindrical inclusion of radius a and axis z (Fig. 3). In the x, y-plane it may be considered as a circular inclusion (or hole). Assume the medium to be loaded at infinity by the forces

$$\sigma_{xz}^{\infty} = q_0 + q_1 \frac{x}{a} + p_1 \frac{y}{a},$$

(2.1)
$$\sigma_{yz}^{\infty} = p_0 - q_1 \frac{y}{a} + p_1 \frac{x}{a},$$

$$\Sigma^{\infty} = (p_0 + iq_0) + (p_1 - iq_1) \frac{z}{a},$$

which produce at infinity the displacement w^{∞} (here $\varrho = r/a$),

$$\mu w^{\infty} = q_0 x + p_0 y + \frac{1}{2} \frac{x^2 - y^2}{a} q_1 + \frac{xy}{a} p_1,$$

$$\mu \frac{w^{\infty}}{a} = q_0 \varrho \cos\theta + p_0 \varrho \sin\theta + \frac{1}{2} q_1 \varrho^2 \cos 2\theta + \frac{1}{2} p_1 \varrho^2 \sin 2\theta.$$

Introducing the notation

(2.2)
$$\varkappa = \frac{\mu - \mu'}{\mu + \mu'},$$

where μ is the shear modulus of the medium, and μ' — the corresponding modulus of the inclusion, we obtain the elementary solutions: outside the inclusion

$$\mu \frac{w}{a} = q_0 \left(\varrho + \frac{\varkappa}{\varrho}\right) \cos\theta + p_0 \left(\varrho + \frac{\varkappa}{\varrho}\right) \sin\theta +$$
$$+ \frac{1}{2} q_1 \left(\varrho^2 + \frac{\varkappa}{\varrho^2}\right) \cos 2\theta + \frac{1}{2} p_1 \left(\varrho^2 + \frac{\varkappa}{\varrho^2}\right) \sin 2\theta,$$

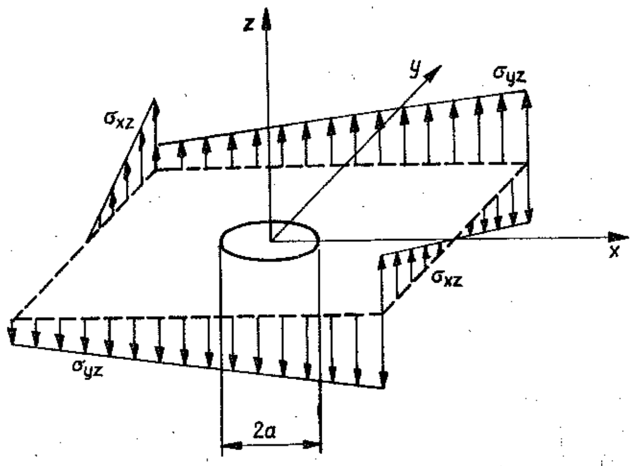

FIG. 3.

$$\sigma_{rz} = q_0\left(1-\frac{\varkappa}{\varrho^2}\right)\cos\theta + p_0\left(1-\frac{\varkappa}{\varrho^2}\right)\sin\theta + \tag{2.3}$$
$$+q_1\left(\varrho-\frac{\varkappa}{\varrho^3}\right)\cos 2\theta + p_1\left(\varrho-\frac{\varkappa}{\varrho^3}\right)\sin 2\theta,$$

$$\Sigma = \sigma_{yz}+i\sigma_{xz} = (p_0+iq_0)+(p_1+iq_1)\frac{z}{a}+$$
$$+\varkappa(p_0-iq_0)\frac{a^2}{z^2}+\varkappa(p_1-iq_1)\frac{a^3}{z^3}.$$

Inside the inclusion

$$\begin{aligned} \mu'\frac{w}{a} &= \frac{2\mu'}{\mu+\mu'}\left(q_0\frac{x}{a}+p_0\frac{y}{a}+\frac{1}{2}q_1\frac{x^2-y^2}{a^2}+p_1\frac{xy}{a^2}\right), \\ \sigma_{rz} &= \frac{2\mu'}{\mu+\mu'}(q_0\cos\theta+p_0\sin\theta+q_1\varrho\cos 2\theta+p_1\varrho\sin 2\theta). \end{aligned} \tag{2.4}$$

Comparison of the formulae $(2.1)_3$ and $(2.3)_3$ indicates that the effect of the inclusion on the state of stress for $r > a$ is reduced to the additional terms in the expression for the complex stress

$$\Delta\Sigma = \varkappa(p_0-iq_0)\frac{a^2}{z^2}+\varkappa(p_1-iq_1)\frac{a^3}{z^3}. \tag{2.5}$$

Let us now compare this result with Eqs. (1.11), (1.12); it is seen that the state of stress and displacement in the body containing an inclusion is exactly the same (for $r > a$) as the state produced in an infinite solid body (without inclusion) by loads σ_{xz}^{∞} and σ_{yz}^{∞} applied at infinity and by the first and second order moments (dipoles) applied at (0, 0),

$$\begin{aligned} M^{0,1} &= -2\pi\varkappa a^2 p_0, \quad & M^{1,1} &= \pi\varkappa a^3 p_1, \\ M^{1,0} &= -2\pi\varkappa a^2 q_0, \quad & M^{2,0} &= \pi\varkappa a^3 q_1. \end{aligned} \tag{2.6}$$

Moment $M^{2,0}$ may be replaced by the equivalent moment $-M^{0,2}$, and the entire set of moments (2.6) — by the equivalent set of two moments of complex intensities

$$\begin{aligned} M^{0,1} &= -2\pi\varkappa a^2(p_0-iq_0), \\ M^{1,1} &= \pi\varkappa a^3(p_1-iq_1). \end{aligned} \tag{2.7}$$

In the case of a circular hole, the coefficient $\varkappa = 1$ ($\mu' = 0$ in the formula (2.2)) and the equivalent moments have the intensities

$$\begin{aligned} M^{0,1} &= -2\pi a^2(p_0-iq_0), \\ M^{1,1} &= \pi a^3(p_1-iq_1). \end{aligned} \tag{2.8}$$

Equations (1.10)—(1.13), (2.6) or (2.7) allow for an exact and explicit expression of the displacements and stresses produced by the loads (2.1) at an arbitrary point of the infinite medium (for $r > a$) containing a circular inclusion.

The above solution may also be referred to the heat conduction problem in a two-dimensional region containing a circular inclusion characterized by the coefficient λ' different from that of the matrix, λ. The conditions at infinity (2.1) express then the prescribed heat fluxes

$$-q_x^\infty = q_0 + q_1\, x/a + p_1\, y/a, \qquad -q_y^\infty = p_0 - q_1\, y/a + p_1\, x/a.$$

At the interface temperature T and normal components of the heat flux vector are continuous:

$$T(a) = T'(a) \quad \text{and} \quad -\lambda \left.\frac{\partial T}{\partial r}\right|_{r=a} = -\lambda' \left.\frac{\partial T'}{\partial r}\right|_{r=a},$$

the primed quantities being referred to the interior of the inclusion. The case of a hole in the deformed elastic medium corresponds here to a perfectly insulating inclusion, $\lambda' = 0$, while a perfectly rigid inclusion — to a perfect heat conductor (constant temperature at the entire boundary of the inclusion).

3. Interaction of circular inclusions

Let us now present an approximate method of analysis of the state of displacement and stresses in an unbounded medium containing several circular inclusions. For the sake of simplicity consider the case of two identical inclusions of radii a and shear moduli μ', centered at points $(0, 0)$ and $(L, 0)$; assume that $L > a$ (with $L \gg a$ accuracy of the method will be higher). Moreover, it is assumed that the loads acting at infinity are uniform and uniaxial, $\sigma_{yz}^\infty = p_0$, $\sigma_{xz}^\infty = 0$ (Fig. 4). It should be remembered that Fig. 4 (and other figures in this paper) presents two-dimensional cross-sections of the actually three-dimensional body containing cylindrical inclusions.

The state of stress in the region outside the inclusions is represented by the sum of the states produced in the infinite region without inclusions by external loads σ_{yz}^∞ (such stresses will be denoted by $\bar{\sigma}_{kz}$, $k = x, y$ or r, θ), and the states produced in the region by the moments of unknown intensities $M_A^{i,j}$, $M_B^{i,j}$ replacing the action of inclusions A and B.

$$\sigma_{kz}(x, y) = \bar{\sigma}_{kz}(x, y) + \sum_{i,j} M_A^{i,j}\, \sigma_{kz}^{i,j}(x - x_a, y - y_A) + \sum_{i,j} M_B^{i,j}\, \sigma_{kz}^{i,j}(x - x_B, y - y_B). \tag{3.1}$$

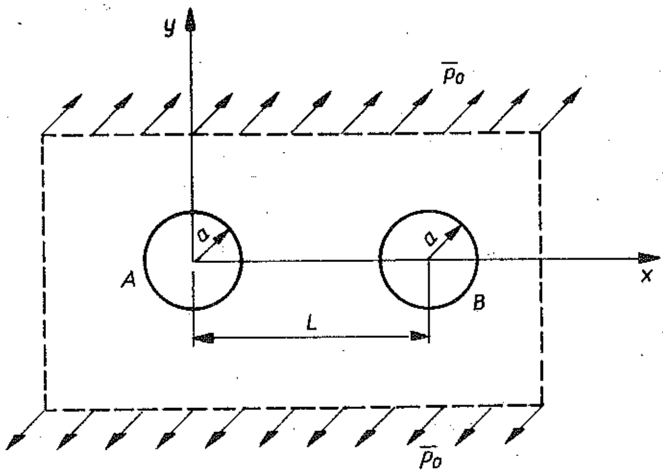

FIG. 4.

Here x_A, y_A and x_B, y_B denote the coordinates of centers of both the inclusions (hence, according to Fig. 4, $x_A = y_A = y_B = 0$, $x_B = L$), and the formula (3.1) holds true for $x^2+y^2 > a^2$ and $(x-L)^2+y^2 > a^2$.

In most cases, if L isn't too small, two or at most four terms of expansions (3.1) will be an acceptable approximation of the stresses; the terms involve the first $(M^{0,1}, M^{1,0})$ and the second order moments $(M^{1,1}, M^{2,0})$. Taking into account the relations (1.11), (1.12) and (2.6), the expansions $\Sigma M_A^{i,j} \sigma_{kz}^{i,j}$ may be expressed directly in terms of the coefficients p_0^A, q_0^A, p_1^A, q_1^A which characterize the stresses produced at point A by the external loads and by the remaining inclusions (also replaced by the corresponding moments). From Eq. (2.6) it follows that the intensity of equivalent moment $M_A^{i,j}$ depends exclusively on the values of stresses (p_0, q_0) and their derivatives (p_1, q_1) which would exist in the neighbourhood of the point x_A, y_A if the inclusion A were absent. This state of stress will be called the primary stress at A.

For $r^2 = x^2+y^2 > a^2$ and with the notation $\varrho = r/a$ the following formulae are obtained:

(3.2)

$$\sigma_{xz}(x, y) \approx -p_0^A \varkappa \frac{\sin 2\theta}{\varrho^2} - q_0^A \varkappa \frac{\cos 2\theta}{\varrho^2} - p_1^A \varkappa \frac{\sin 3\theta}{\varrho^3} - q_1^A \varkappa \frac{\cos 3\theta}{\varrho^3} + \bar{\sigma}_{xz},$$

$$\sigma_{yz}(x, y) \approx p_0^A \varkappa \frac{\cos 2\theta}{\varrho^2} - q_0^A \varkappa \frac{\sin 2\theta}{\varrho^2} + p_1^A \varkappa \frac{\cos 3\theta}{\varrho^3} - q_1^A \varkappa \frac{\sin 3\theta}{\varrho^3} + \bar{\sigma}_{yz}.$$

The derivatives of stresses (3.2) with respect to $\xi = x/a$ and $\eta = y/a$ at point (ϱ, θ) are expressed by the formulae

$$\frac{\partial \sigma_{xz}(x,y)}{\partial \xi} \approx p_0^A \frac{2\varkappa}{\varrho^3} \sin 3\theta + q_0^A \frac{2\varkappa}{\varrho^3} \cos 3\theta + \\ + p_1^A \frac{3\varkappa}{\varrho^4} \sin 4\theta + q_1^A \frac{3\varkappa}{\varrho^4} \cos 4\theta + \frac{\partial \bar{\sigma}_{xz}}{\partial \xi},$$

(3.3)

$$\frac{\partial \sigma_{xz}(x,y)}{\partial \eta} \approx -p_0^A \frac{2\varkappa}{\varrho^3} \cos 3\theta + q_0^A \frac{2\varkappa}{\varrho^3} \sin 3\theta - \\ - p_1^A \frac{3\varkappa}{\varrho^4} \cos 4\theta + q_1^A \frac{3\varkappa}{\varrho^4} \sin 4\theta + \frac{\partial \bar{\sigma}_{xz}}{\partial \eta}.$$

Obviously,

$$(3.4) \qquad -\frac{\partial \sigma_{xz}}{\partial \xi} = \frac{\partial \sigma_{yz}}{\partial \eta}, \qquad \frac{\partial \sigma_{xz}}{\partial \eta} = \frac{\partial \sigma_{yz}}{\partial \xi}.$$

Substitution of the coordinates x_B, y_B for x, y into the formulae (3.3) yields the stress parameters p_0^B, q_0^B, p_1^B, q_1^B which are necessary to determine the equivalent moments $M_B^{0,1}$, $M_B^{1,0}$, $M_B^{1,1}$, $M_B^{2,0}$:

$$(3.5) \qquad \sigma_{xz}(B) = q_0^B, \qquad \sigma_{yz}(B) = p_0^B,$$

and

$$(3.6) \qquad \left.\frac{\partial \sigma_{xz}}{\partial \xi}\right|_B = -\left.\frac{\partial \sigma_{yz}}{\partial \eta}\right|_B = q_1^B, \qquad \left.\frac{\partial \sigma_{xz}}{\partial \eta}\right|_B = p_1^B.$$

Let us now denote by $\bar{q}_0$, $\bar{q}_1$, $\bar{p}_0$, $\bar{p}_1$ the corresponding stress parameters produced by external loads:

$$(3.7) \qquad \bar{\sigma}_{xz}(B) = \bar{q}_0^B, \qquad \bar{\sigma}_{yz}(B) = \bar{p}_0^B, \qquad \frac{\partial \bar{\sigma}_{xz}}{\partial \eta} = \bar{p}_1^B, \qquad \frac{\partial \bar{\sigma}_{xz}}{\partial \xi} = \bar{q}_1^B,$$

and consider them as the values known from elementary considerations. Substitution of Eqs. (3.5)—(3.7) into Eqs. (3.2)—(3.4) leads to the following set of equations:

$$q_0^B = \bar{q}_0^B - p_0^A \varkappa \frac{\sin 2\theta}{\varrho^2} - q_0^A \varkappa \frac{\cos 2\theta}{\varrho^2} - p_1^A \varkappa \frac{\sin 3\theta}{\varrho^3} - q_1^A \varkappa \frac{\cos 3\theta}{\varrho^3},$$

$$p_0^B = \bar{p}_0^B + p_0^A \varkappa \frac{\cos 2\theta}{\varrho^2} - q_0^A \varkappa \frac{\sin 2\theta}{\varrho^2} + p_1^A \varkappa \frac{\cos 3\theta}{\varrho^3} - q_1^A \varkappa \frac{\sin 3\theta}{\varrho^3},$$

(3.8)

(3.8) [cont.]
$$q_1^B = \bar{q}_1^B + 2p_0^A \varkappa \frac{\sin 3\theta}{\varrho^3} + 2q_0^A \varkappa \frac{\cos 3\theta}{\varrho^3} + 3p_1^A \varkappa \frac{\sin 4\theta}{\varrho^4} + 3q_1^A \varkappa \frac{\cos 4\theta}{\varrho^4},$$

$$p_1^B = \bar{p}_1^B - 2p_0^A \varkappa \frac{\cos 3\theta}{\varrho^3} + 2q_0^A \varkappa \frac{\sin 3\theta}{\varrho^3} - 3p_1^A \varkappa \frac{\cos 4\theta}{\varrho^4} + 3q_1^A \varkappa \frac{\sin 4\theta}{\varrho^4}.$$

In these formulae $\varkappa$ denotes the nonhomogeneity coefficient (2.2) of inclusion A, and ϱ and θ are the coordinates of the center of B measured in the polar coordinates centered at A. Similar four equations are obtained by equating the coefficients p_0^A, q_0^A, p_1^A, q_1^A to the sum of stresses produced by external loads ($\bar{p}_0^A, \ldots, \bar{q}_1^A$) and the stresses produced by the inclusion B (or the corresponding stress gradients). Some of the terms in Eqs. (3.8) change their signs due to the change of angles $\theta_{AB} = \theta_{BA} - \pi$. As a result, we obtain a set of eight linear equations for the eight unknown moments $M_A^{i,j}$ and $M_B^{i,j}$; stresses at an arbitrary point of the medium are then calculated from Eqs. (3.1). A similar procedure is applied in the case of n inclusions; $4n$ equations with $4n$ unknowns are then obtained.

To illustrate the procedure, let us return to the case of two circular holes of equal diameters $2a$ centered at points $(0,0)$ and $(L,0)$ (Fig. 4). Due to the symmetry of the problem (external loading is uniform, $\sigma_{yz}^\infty = \bar{p}_0$), we have

$$p_0^A = p_0^B = p_0, \quad p_1^A = -p_1^B = p_1, \quad q_0^A = q_0^B = q_1^A = q_1^B = 0,$$

and the system (3.8) is reduced to a simple set of two equations with two unknowns. We obtain the solution

(3.9)
$$p_0 = \frac{\bar{p}_0}{1 - \dfrac{1}{\lambda^2} - \dfrac{2}{\lambda^6} \dfrac{1}{1-3/\lambda^4}} \approx \frac{\bar{p}_0}{1 - \dfrac{1}{\lambda^2}},$$

$$p_1 = \frac{2/\lambda^3}{1 - \dfrac{3}{\lambda^3}} \bar{p}_0 \approx \frac{2}{\lambda^3} \bar{p}_0.$$

Values of the equivalent moments are obtained from Eqs. (2.6),

$$M_A^{0,1} = M_B^{0,1} = -2\pi a^2 \frac{\bar{p}_0}{1-1/\lambda^2}, \quad M_A^{1,1} = -M_B^{1,1} = 2\pi a^3 \frac{\bar{p}_0}{\lambda^3},$$

and the approximate values of stresses (outside the holes) may be found from Eqs. (3.1) or (3.2). For instance,

$$\sigma_{yz}(\varrho, \theta) \approx \bar{p}_0 \left[1 + \frac{1}{1-1/\lambda^2} \left(\frac{\cos 2\theta_1}{\varrho_1^2} + \frac{\cos 2\theta_2}{\varrho_2^2} \right) + \frac{2}{\lambda^3} \left(\frac{\cos 3\theta_1}{\varrho_1^3} - \frac{\cos 3\theta_2}{\varrho_2^3} \right) \right]. \tag{3.10}$$

Here ϱ_1, θ_1 and ϱ_2, θ_2 are the coordinates of point (x, y) measured in the polar coordinate systems centered at $(0, 0)$ and $(L, 0)$ (Fig. 5).

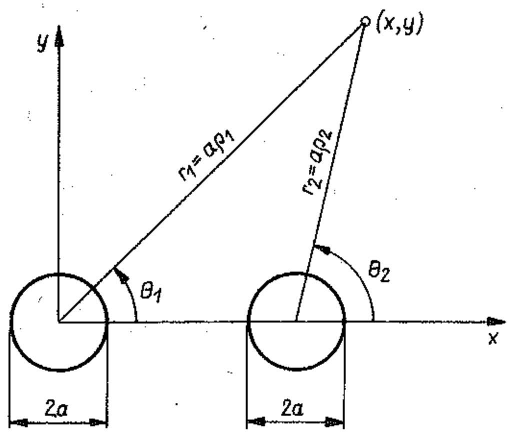

FIG. 5.

The diagrams of stresses $\sigma_{yz}(x, 0)$ calculated from the approximate formula (3.10) under the assumption that $\lambda = L/a = 5$ is shown in Fig. 6. It is interesting to note that, in spite of a rather small distance between the holes $\left(\frac{3}{2}\right.$ hole diameters$\left.\right)$, the contribution of the second order moment is almost negligible. This makes it possible to reduce the number of unknowns in the case of n different holes from $4n$ to $2n$, and under additional symmetry properties — even to a smaller number. For instance, uniaxial tension applied to a body containing an infinite row of equal and uniformly spaced holes may be reduced to two or even a single equation. Accuracy of the approximate formula (3.10) may be estimated by means of Table 1 containing the values of $\varrho = r/a$ at which the stress σ_{rz} vanishes; in the accurate solution stresses $\sigma_{rz} = 0$ at the boundary of the hole, that is at $\varrho = 1$. It should also be noted that in the case of a crack replaced by a concentrated moment [9] the approximate solution becomes fairly accurate at a larger distance from the center of the crack, $\varrho \gg 1$, while the solution (3.10) yields satisfactory results at points lying very close to the boundary of the hole.

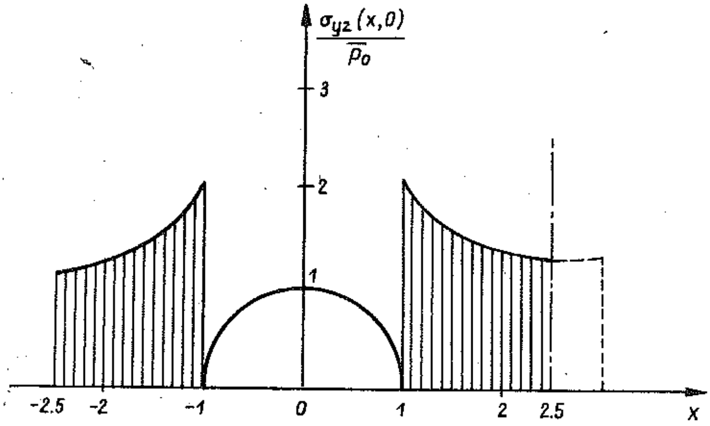

FIG. 6.

Table 1. Roots of the equation $\sigma_{rz}(r, \theta) = 0$ for several values of θ.

θ	1°	10°	30°	60°	90°	120°	150°	170°	179°
r	0.975	0.976	0.982	0.993	1.002	1.008	1.010	1.011	1.011

4. Elliptical inclusions

The method outlined in the preceding section may also be applied to the cases of elliptical inclusions and holes. In the limiting case of elliptical holes with semi-axes a, b and $\lim b/a = 0$ we may also analyze the interaction between various inclusions and cracks.

Let us consider the simple case presented in Fig. 7 and concerning an infinite elastic body subject to antiplane strain and loaded at infinity by uniaxial tension $\varrho_{yz}^{\infty} = p_0$. The body contains a cylindrical inclusion of elliptical cross-section with semi-axes a, b and $c^2 = a^2 - b^2$. The elastic moduli of the body and inclusion are μ and μ', respectively. Let us introduce the additional parameter

$$\bar{\varkappa} = \frac{\mu b - \mu' a}{\mu b + \mu' a}. \tag{4.1}$$

In the case of a circular inclusion $\bar{\varkappa} = \varkappa$ given by Eq. (2.2); for an elliptical hole $\mu' = 0$ and $\bar{\varkappa} = 1$; a rigid inclusion is characterized by $\bar{\varkappa} = -1$.

Application of the complex potential method (see, for example [13]) leads to the following solution.

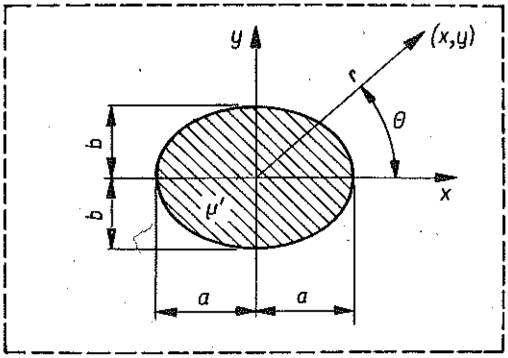

FIG. 7.

Outside the inclusion

$$F(z) = \frac{1}{2} p_0 (a+b)\left(\frac{z+\sqrt{z^2-c^2}}{a+b} - \bar{\varkappa}\frac{a+b}{z+\sqrt{z^2-c^2}}\right),$$

(4.2)
$$\mu w = \operatorname{Im} F(z),$$

$$\Sigma(z) = \sigma_{yz} + i\sigma_{xz} = \frac{p_0}{2}\left[1+\bar{\varkappa}\frac{(a+b)^2}{(z+\sqrt{z^2-c^2})^2}\left(1+\frac{z}{\sqrt{z^2-c^2}}\right)\right].$$

Separation of the real and imaginary parts in Eq. $(4.2)_3$ yields the stresses σ_{yz} and σ_{xz}. At the boundary of the inclusion variable z assumes the values

(4.3) $$\hat{z} = a\cos\vartheta + ib\sin\vartheta,$$

ϑ denoting a parameter. With the additional notation

$$h^2 = a^2\sin^2\vartheta + b^2\cos^2\vartheta,$$

the formulae for stresses σ_{xz} and σ_{yz} may be written, in view of the identity

$$\hat{z} + \sqrt{z^2-a^2+b^2} = (a+b)(\cos\vartheta + i\sin\vartheta),$$

in the form

$$\sigma_{xz}(\hat{z}) = -\frac{p_0(a+b)}{2h^2}[a-b+\bar{\varkappa}(a+b)]\sin\vartheta\cos\vartheta,$$

(4.4)

$$\sigma_{yz}(\hat{z}) = \frac{p_0(a+b)}{2h^2}[a\sin^2\vartheta + b\cos^2\vartheta + \bar{\varkappa}(-a\sin^2\vartheta + b\cos^2\vartheta)].$$

Stresses σ_{nz} and σ_{tz} (in the directions normal and tangent to the elliptic boundary) are calculated from the formulae

$$\tag{4.5} \begin{aligned} \sigma_{nz} &= \sigma_{xz}\frac{b}{h}\cos\vartheta+\sigma_{yz}\frac{a}{h}\sin\vartheta, \\ \sigma_{tz} &= -\sigma_{xz}\frac{a}{h}\sin\vartheta+\sigma_{yz}\frac{b}{h}\cos\vartheta. \end{aligned}$$

Substituting the stresses (4.4) into Eq. (4.5), we obtain the particularly simple formulae

$$\tag{4.6} \begin{aligned} \sigma_{nz} &= p_0\frac{a+b}{2h}(1-\bar{\varkappa})\sin\vartheta, \\ \sigma_{tz} &= p_0\frac{a+b}{2h}(1+\bar{\varkappa})\cos\vartheta. \end{aligned}$$

It is seen that, in the case of a hole ($\bar{\varkappa}=1$), stress $\sigma_{nz}(\hat{z})=0$.

Displacement w at the boundary of the inclusion is expressed by the formula

$$\tag{4.7} \mu w(\hat{z})=\frac{p_0}{2}(a+b)(1+\bar{\varkappa})\sin\vartheta.$$

In the case of a perfectly rigid inclusion, $\bar{\varkappa}=-1$, the displacement vanishes, $w(\hat{z})=0$, in agreement with the physical sense of the problem.

Inside the inclusion the displacement w is a linear function of y, and the stresses are constant,

$$\tag{4.8} w'=p_0\frac{(a+b)\,y}{\mu' a+\mu b}, \qquad \sigma'_{yz}=p_0\frac{\mu'(a+b)}{\mu' a+\mu b}, \qquad \sigma'_{xz}=0.$$

It is easily verified that at $y=b\sin\vartheta$ Eqs. (4.8) lead to identical results as Eqs. (4.6) and (4.7), so that the normal stresses and displacements are continuous at the boundary of the inclusion. Since, in addition, Eq. $(4.2)_3$ at $|z|\to\infty$ yields the constant value p_0,

$$\lim_{|z|\to\infty}\Sigma(z)=p_0,$$

the formulae (4.2) and (4.8) are seen to represent the accurate solution to the problem shown in Fig. 7.

In order to determine the equivalent concentrated moments corresponding to the elliptical inclusion in the stress field considered, the function $\Sigma(z)$ given by Eq. (4.2) must be expanded into a power series of $1/z$; disregarding the terms of orders greater than four we obtain

$$\tag{4.9} \Sigma(z)=\frac{p_0}{4}[\bar{\varkappa}(a+b)^2+c^2]\frac{1}{z^2}+\frac{3}{16}p_0[\bar{\varkappa}(a+b)^2+c^2]\frac{c^2}{z^4}.$$

Let us now compare the terms of the expansion (4.9) with the corresponding terms of the formula for $\Sigma(z)$ resulting from the application of concentrated moments $M^{0,1}$, $M^{1,1}$, $M^{2,1}$, Eq. (1.10):

$$\Sigma(z)=\frac{1}{2\pi}\left[-1!\,\frac{M^{0,1}}{z^2}+2!\,\frac{M^{1,1}}{z^3}-3!\,\frac{M^{2,1}}{z^4}+\ldots\right.$$
$$\left.\ldots+(-1)^{m+1}(m+1)!\,\frac{M^{m,1}}{z^{m+2}}\right]. \tag{4.10}$$

From the comparison it follows that the stresses produced in the body (outside the inclusion) by uniformly distributed loads $\sigma_{yz}^{\infty}=p_0$ at infinity, may be expressed by the sum of p_0 and the stresses produced by two concentrated moments of the first and third orders:

$$M^{0,1}=\frac{\pi}{2}K_a p_0,\qquad M^{2,1}=-\frac{\pi}{16}K_a c^2 p_0,$$
$$M^{1,1}=M^{3,1}=0. \tag{4.11}$$

with the notation

$$K_a=\varkappa(a+b)^2+c^2=\frac{2ab(a+b)(\mu-\mu')}{\mu b+\mu' a}. \tag{4.12}$$

Confining our approximation to a single first order moment $M^{0,1}$ (Fig. 2a), the stresses in a body containing an elliptical inclusion are calculated from the formula

$$\Sigma(z)=p_0\left(1+\frac{1}{4}K_a\frac{1}{z^2}\right),$$

$$\sigma_{yz}=p_0\left(1+\frac{1}{4}K_a\frac{\cos 2\theta}{r^2}\right),\qquad \sigma_{xz}=-\frac{p_0}{4}K_a\frac{\sin 2\theta}{r^2}.$$

Accuracy of the formulae derived may be illustrated by the following comparison. With $x=0$, the accurate formula (4.2) in the case of an elliptical hole leads to the stresses

$$\sigma_{yz}^{1}(0,y)=\frac{p_0}{2}(a+b)\left[f(y)-\frac{1}{f(y)}\right]\frac{1}{\sqrt{y^2+c^2}},$$

where

$$f(y)=\frac{y+\sqrt{y^2+c^2}}{a+b},\qquad y>b,\qquad a\geqslant b,$$

Table 2. Single elliptical hole; comparison of accurate and approximate results.

a/b	y/b	$\sigma^{\mathrm{I}}_{yz}(0, y)$	$\sigma^{\mathrm{II}}_{yz}(0, y)$	Error
1.2	1.2	0.251	0.293	17%
	1.5	0.487	0.499	2.4%
	2.0	0.695	0.697	0.3%
	3.0	0.8585	0.8587	0.02%
2	2.0	0.513	0.672	31%
	2.5	0.644	0.693	7%
	3.0	0.732	0.750	2.5%
	5.0	0.8898	0.8908	0.01%
10	10.0	0.677	0.858	26%
	20.0	0.884	0.888	0.5%
	30.0	0.9435	0.9439	0.04%
100	100.0	0.704	0.873	24%
	200.0	0.893	0.897	0.5%
	300.0	0.9481	0.9485	0.04%

while the approximate formula involving two moments $M^{0,1}$, $M^{2,1}$ yields

$$\sigma^{\mathrm{II}}_{yz}(0, y) = p_0\left[1-\frac{1}{2}\,\frac{a(a+b)}{y^2}+\frac{3}{8}\,\frac{a(a+b)(a^2-b^2)}{y^4}\right].$$

Both results are compared in Table 2 for various a/b ratios and several values of y. It is seen that the approximate formula yields fairly accurate results outside the circle of radius a circumscribing the elliptical hole; for $y = 1.75\,a$ the relative error is about 1%, for $y = 2a$ — about 0.5%, and for $y = 3a$ the error drops below 0.05%. Similar results are obtained from the analysis of stress variation along the x-axis: $\sigma_{yz}(\xi, 0)$, $\xi = x/a > 1$. With the notation $\beta = b/a$ the approximate formula reads

$$\sigma_{yz}^{\mathrm{II}}(\xi, 0) = p_0\left[1+\frac{1}{2}\,\frac{1+\beta}{\xi^2}+\frac{3}{8}\,\frac{(1+\beta)(1-\beta^2)}{\xi^4}\right].$$

The values of $\sigma_{yz}^{\mathrm{II}}$ are compared with the accurate values

$$\sigma^{\mathrm{I}}_{yz}(\xi, 0) = \frac{p_0}{2}\left[1+\varkappa\left(\frac{1+\beta}{\xi+\sqrt{\xi^2+\beta^2-1}}\right)^2\right]\left(1+\frac{\xi}{\sqrt{\xi^2+\beta^2-1}}\right),$$

in Table 3.

In the neighbourhood of point $x = a$, $y = 0$ the approximate formula yields inaccurate results so that at points lying near the boundary of the inclusion or hole other approximate formulae must be used. In the case

Table 3. Single elliptical hole; comparison of accurate $\sigma^{I}_{yz}(\xi, 0)$, (I), and approximate $\sigma^{II}_{yz}(\xi, 0)$, (II). (At the boundary an approximate formula is not applicable).

$\xi = \frac{x}{a}$		1.0	1.75	2.0	3.0
$\beta = 1/2$	I II error	3 2.1118 29%	1.8016 1.2900 0.9%	1.2188 1.2139 0.4%	1.0889 1.0885 0.04%
$\beta = 1/10$	I II error	11 1.9584 82%	1.2395 1.2231 1.3%	1.1696 1.1630 0.5%	1.0667 1.0661 0.05%
$\beta = 1/100$	I II error	101 1.8837 98%	1.2208 1.2053 1.3%	1.1560 1.1499 0.5%	1.0613 1.0608 0.05%

of a crack, the stresses must be calculated from the known near-tip expansions (cf. [10], p. 125). In the case of an elliptical hole or inclusion, the known values of normal and circumferential stresses, Eq. (4.6), may be used to determine the behaviour of stresses near the boundary. For instance,

$$\sigma_{yz}\,[(1+\varepsilon)\,a,\,0] = \sigma_{tz}\,(1-\varepsilon a^2/b^2) = \sigma_{tz}\,(1-\varepsilon a/R),$$

where $R = b^2/a$ is the radius of curvature of the ellipse at $x = a$, $y = 0$. In Fig. 8 the solid line represents the variation of stress $\sigma_{yz}(x, 0)$ according to the accurate formula, and the dashed lines a and b are the "near field" and "far field" approximations of the same stress.

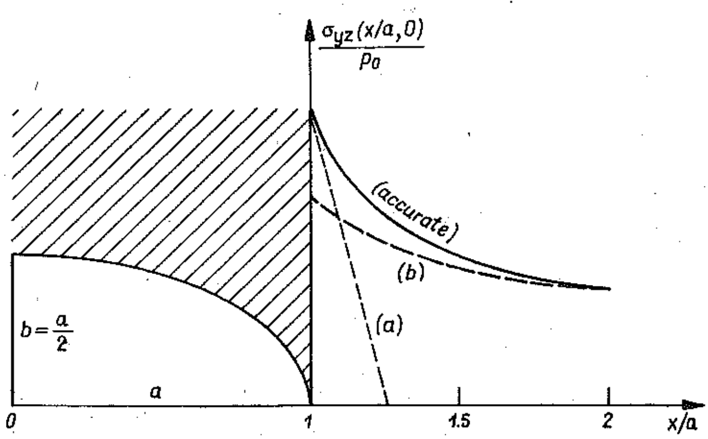

FIG. 8.

Let us now return to the general solution of the problem of stress distribution in an infinite body loaded at infinity and assume that the load is represented by horizontal tension $\sigma_{xz}^{\infty} = q_0$. The complex potential is then assumed in the form

$$F(z) = \frac{q_0}{2} i(a+b)\left(\frac{z+\sqrt{z^2-a^2+b^2}}{a+b} - \hat{\varkappa}\frac{a+b}{z+\sqrt{z^2-a^2+b^2}}\right),$$

what leads to the complex stress tensor

$$\Sigma(z) = \frac{q_0}{2} i\left[1+\hat{\varkappa}\frac{(a+b)^2}{(z+\sqrt{z^2-c^2})^2}\right]\left(1+\frac{z}{\sqrt{z^2-c^2}}\right),$$

with the notation

$$\hat{\varkappa} = \frac{\mu' b - \mu a}{\mu' b + \mu a}, \qquad c^2 = a^2 - b^2.$$

At the boundary of the inclusion, $z = \hat{z}$, the stresses are

$$\sigma_{nz} = \frac{q_0}{2}\frac{a+b}{h}(1+\hat{\varkappa})\cos\vartheta, \qquad \sigma_{tz} = \frac{q_0}{2}\frac{a+b}{h}(-1+\hat{\varkappa})\sin\vartheta,$$

and the displacements

$$w(\hat{z}) = \frac{1}{\mu}\operatorname{Im} F(z) = \frac{q_0}{2\mu}(a+b)(1-\hat{\varkappa})\cos\vartheta.$$

Inside the inclusion

$$w' = \frac{q_0}{2\mu}\frac{a+b}{a}(1-\hat{\varkappa})x, \qquad \sigma'_{xz} = \mu'\frac{\partial w'}{\partial x}, \qquad \sigma'_{yz} = 0.$$

The conditions of continuity of normal stresses and displacements are fulfilled at the boundary of the inclusion, and the complex stress tensor reduces at infinity to the required value,

$$\lim_{|z|\to\infty} \Sigma(z) = q_0 i.$$

Expansion of $\Sigma(z)$ into a series and comparison of the first three terms of expansion with the corresponding terms of the series

$$\Sigma(z) = \frac{1}{2\pi}\left[1!\, i\frac{M^{1,0}}{z^2} + 2!\frac{M^{1,1}}{z^3} - 3!\, i\frac{M^{1,2}}{z^4} + \ldots\right],$$

yields the values of the equivalent moments

$$M^{1,0} = -\frac{\pi}{2}K_b q_0, \qquad M^{1,1} = 0, \qquad M^{1,2} = -\frac{\pi}{16}K_b c^2 q_0,$$

with the notation

$$K_b = \frac{2ab\,(a+b)\,(\mu-\mu')}{\mu a+\mu' b}.$$

Summing up, we may conclude that in considering an unbounded elastic medium loaded at infinity by uniformly distributed forces $\sigma_{xz}^{\infty} = q_0$, $\sigma_{yz}^{\infty} = p_0$, the action of an elliptic inclusion $(a; b)$ may be replaced by the action of a system of concentrated moments:

of the first order

(4.13) $$M^{0,1} = -\frac{\pi}{2}\,p_0\,K_a, \qquad M^{1,0} = -\frac{\pi}{2}\,q_0\,K_b,$$

and of the third order

(4.13.1) $$M^{2,1} = -\frac{\pi}{16}\,K_a\,(a^2-b^2)\,p_0, \qquad M^{1,2} = -\frac{\pi}{16}\,K_b\,(a^2-b^2)\,q_0.$$

The coefficients $K_a(a, b) = K_b(b, a)$, and

$$K_a = \frac{2ab\,(a+b)\,(\mu-\mu')}{\mu b+\mu' a}, \qquad K_b = \frac{2ab\,(a+b)\,(\mu-\mu')}{\mu a+\mu' b}.$$

Let us consider several special cases.

(a) If the body is homogeneous and $\mu' = \mu$ (no inclusion), then $K_a = K_b = 0$, $M^{0,1} = M^{1,0} = 0$ and the equivalent moments vanish.

(b) In the case of an elliptical hole, $\mu' = 0$, $\mu \neq 0$, $K_a = 2a\,(a+b)$, $K_b = 2b\,(a+b)$, and

(4.13.2) $$M^{0,1} = -\pi a\,(a+b)\,p_0, \qquad M^{1,0} = -\pi b\,(a+b)\,q_0.$$

(c) If, in addition, the hole is circular, $b = a$, then

(4.13.3) $$M^{0,1} = -2\pi a^2\,p_0, \qquad M^{1,0} = -2\pi a^2\,q_0;$$

this solution is accurate. An accurate solution is also obtained

(d) in the case of a circular inclusion, $K_a = K_b = 4\varkappa a^2$, $\varkappa = (\mu-\mu')/(\mu+\mu')$, and

(4.13.4) $$M^{0,1} = -2\pi a^2\,\varkappa p_0, \qquad M^{1,0} = -2\pi a^2\,\varkappa q_0.$$

(e) In the case of a horizontal crack, $b/a \to 0$,

(4.13.5) $$M^{0,1} = -\pi a^2\,p_0, \qquad M^{1,0} = 0.$$

This result is identical with that given in [9] (p. 100), and with the known phenomenon that a crack parallel to the direction of tension applied to the body does not disturb the uniform state of stresses.

(f) In the case of a perfectly rigid inclusion, $\mu/\mu' \to 0$, we obtain $K_a = -2b(a+b)$, $K_b = -2a(a+b)$, and

$$M^{0,1} = \pi b(a+b)\, p_0, \quad M^{1,0} = \pi a(a+b)\, q_0. \tag{4.13.6}$$

Once the values of the equivalent moments are determined, the stresses (in the medium outside the inclusions) are found from the formulae

$$\sigma_{xz} = q_0 + \frac{M^{0,1}}{2\pi}\frac{\sin 2\theta}{r^2} + \frac{M^{1,0}}{2\pi}\frac{\cos 2\theta}{r^2} + \frac{M^{2,1}}{2\pi}\frac{6\sin 4\theta}{r^4} - \frac{M^{1,2}}{2\pi}\frac{6\cos 4\theta}{r^4}, \tag{4.14}$$

$$\sigma_{yz} = p_0 - \frac{M^{0,1}}{2\pi}\frac{\cos 2\theta}{r^2} + \frac{M^{1,0}}{2\pi}\frac{\sin 2\theta}{r^2} - \frac{M^{2,1}}{2\pi}\frac{6\cos 4\theta}{r^4} - \frac{M^{1,2}}{2\pi}\frac{6\sin 4\theta}{r^4}. \tag{4.15}$$

The formulae (4.14) and (4.15) containing four equivalent moments constitute a good approximation for stresses produced by uniform loads in an infinite elastic body containing a single elliptical inclusion. The accurate solution of such a problem is known, and thus the approximate formulae will be used in the analysis of more complicated cases when an infinite (or large enough) body contains a larger number of inclusions. In order to simplify the considerations, let us discuss the problem of two elliptical inclusions (a_1, b_1) and (a_2, b_2) centered at the respective points $A(0, 0)$ and $B(L\cos\varphi, L\sin\varphi)$, and take into account the first order moments only: $M^{0,1}$ and $M^{1,0}$, Fig. 9.

The values of equivalent moments $M_B^{0,1}$, $M_B^{1,0}$ replacing the inclusion at B may be expressed, as it follows from Eqs. (4.13), in terms of the "primary" stresses which would act in the region of B if the inclusion did not exist and the body were homogeneous. The "primary" stresses are produced by the prescribed external loads applied to the body (at infinity or far enough from the region), and by the concentrated moments $M_A^{0,1}$, $M_A^{1,0}$ replacing the other inclusion (their values are not known yet). The primary stresses are not constant in the region occupied by the inclusion B, but they may be assumed to be approximately equal to their values at the center of B (the error will be small provided the inclusions are not too close to each other). Denoting the first component of the primary stresses (due to external loads) by $\bar{q}_0 = \bar{\sigma}_{xz}(B)$ and $\bar{p}_0 = \bar{\sigma}_{yz}(B)$, the complete formulae for the primary stresses at B are written in the form

$$\sigma_{xz}(B) = \bar{q}_0 + \frac{M_A^{0,1}}{2\pi}\frac{\sin 2\varphi}{L^2} + \frac{M_A^{1,0}}{2\pi}\frac{\cos 2\varphi}{L^2}, \tag{4.16}$$

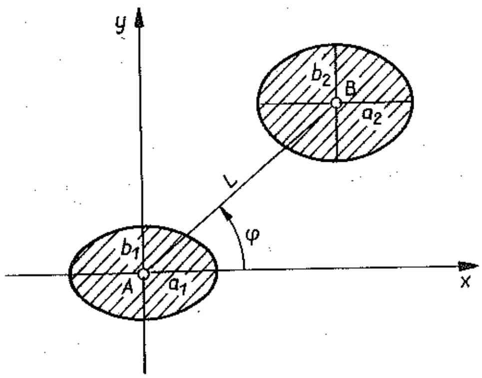

FIG. 9.

$$\sigma_{yz}(B) = \bar{p}_0 - \frac{M_A^{0,1}}{2\pi}\frac{\cos 2\varphi}{L^2} + \frac{M_A^{1,0}}{2\pi}\frac{\sin 2\varphi}{L^2}. \tag{4.16}$$

The values of equivalent moments replacing the inclusion at B are calculated from Eqs. (4.13):

$$M_B^{0,1} \approx -\frac{\pi}{2} K_a^B \sigma_{yz}(B), \quad M_B^{1,0} \approx -\frac{\pi}{2} K_b^B \sigma_{xz}(B). \tag{4.17}$$

Substitution of Eqs. (4.17) into Eq. (4.16) leads to a set of two equations with four unknowns:

$$\begin{aligned} &\frac{M_B^{1,0}}{2\pi}\frac{4}{K_b^B} + \frac{M_A^{0,1}}{2\pi}\frac{\sin 2\varphi}{L^2} + \frac{M_A^{1,0}}{2\pi}\frac{\cos 2\varphi}{L^2} = -\bar{q}_0, \\ &\frac{M_B^{0,1}}{2\pi}\frac{4}{K_a^B} - \frac{M_A^{0,1}}{2\pi}\frac{\cos 2\varphi}{L^2} + \frac{M_A^{1,0}}{2\pi}\frac{\sin 2\varphi}{L^2} = -\bar{p}_0. \end{aligned} \tag{4.18}$$

The two other equations are obtained by expressing the moments $M_A^{0,1}$, $M_A^{1,0}$ in terms of the stresses produced at A by the external loads and by the moments $M_B^{0,1}$ and $M_B^{1,0}$; the four unknown moments are then determined from the complete set of four equations.

If the characteristics of both the inclusions A and B are identical, that is $K_a^A = K_a^B = K_a$ and $K_b^A = K_b^B = K_b$, the corresponding moments will also be equal: $M_A^{0,1} = M_B^{0,1} = M^{0,1}$ and $M_A^{1,0} = M_B^{1,0} = M^{1,0}$; their values will be determined from the set of two algebraic equations, and

$$\begin{aligned} &\frac{M^{0,1}}{2\pi} = \frac{1}{D}\left[-\bar{p}_0\left(\frac{4}{K_b} + \frac{\cos 2\varphi}{L^2}\right) + \bar{q}_0\frac{\sin 2\varphi}{L^2}\right], \\ &\frac{M^{1,0}}{2\pi} = \frac{1}{D}\left[\bar{p}_0\frac{\sin 2\varphi}{L^2} - \bar{q}_0\left(\frac{4}{K_a} - \frac{\cos 2\varphi}{L^2}\right)\right], \end{aligned} \tag{4.19}$$

with the additional notation

$$D = \frac{16}{K_a K_b} + \frac{4\cos 2\varphi}{L^2}\left(\frac{1}{K_a} - \frac{1}{K_b}\right) - \frac{1}{L^4}.$$

Let us now consider several particular cases; assume the infinite body to be subject to uniaxial vertical shear, so that $\bar{q}_0 = 0$, $\bar{p}_0 \neq 0$. If the body contains two identical elliptical inclusions (a, b) located at the x-axis at a distance of L from each other, (Fig. 10a), we obtain

$$M^{1,0} = 0, \qquad M^{0,1} = -\frac{\pi}{2}\,\frac{\bar{p}_0 K_a}{1 - K_a/4L^2}.$$

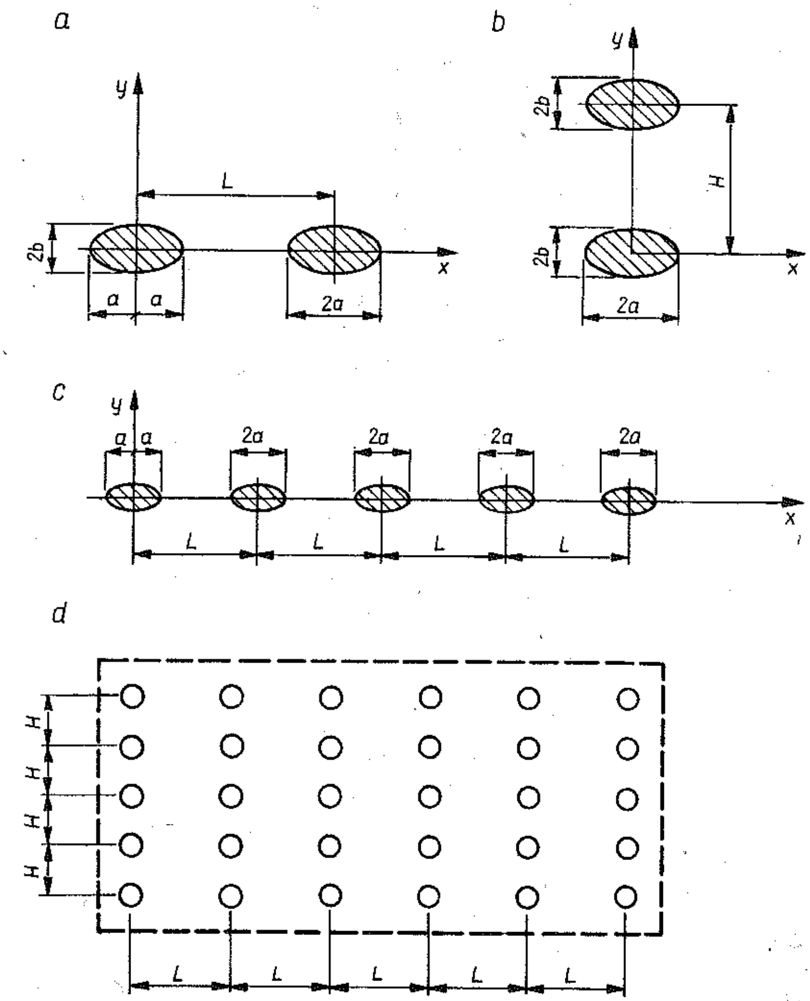

FIG. 10.

Two inclusions located at the vertical axis (Fig. 10b) lead to the equivalent moments

$$M^{1,0} = 0, \qquad M^{0,1} = -\frac{\pi}{2}\,\frac{\bar{p}_0\,K_a}{1+K_a/4L^2}.$$

An infinite row of uniformly spaced elliptical inclusions (Fig. 10c) may be replaced by an infinite row of equal moments

$$M^{1,0} = 0, \qquad M^{0,1} = -\frac{\pi}{2}\,\frac{\bar{p}_0\,K_a}{1-\dfrac{\pi^2}{12}\,\dfrac{K_a}{L^2}}.$$

In the case of an infinite, regular array of elliptical inclusions (Fig. 10d), each inclusion may be replaced by the moment

$$M^{0,1} = -\frac{\pi}{2}\,\frac{\bar{p}_0\,K_a}{1-S\,\dfrac{K_a}{4L^2}}, \qquad M^{1,0} = 0.$$

where $S = \frac{\pi^2}{3}\left(1-\frac{1}{\alpha^2}\right)+4\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{n^2-k^2\alpha^2}{(n^2+k^2\alpha^2)^2}$, and $\alpha = H/L$. If $L=H$, $S=0$ and $M^{0,1} = -\pi\bar{p}_0\,K_a/2$. Finally, in the case of two horizontal, collinear cracks of lengths $2a$ and distance L between them, we have $K_a = 2a^2$, and

$$M^{0,1} = -\frac{\bar{p}_0\,a^2\,\pi}{1-1/2\lambda^2}, \qquad M^{1,0} = 0, \qquad \lambda = L/a.$$

5. Elliptical holes, linear stress variation

Let us consider the problem of an infinite body containing a cylindrical, elliptical cavity (hole) and assume a more general case of external loading at infinity, expressed by the linear functions

$$\sigma_{xz}^{\infty} = p_1\,\frac{2y}{a+b}, \qquad \sigma_{yz}^{\infty} = p_1\,\frac{2x}{a+b}. \tag{5.1}$$

The solution of such a problem (combined with the relations (5.7)) will enable us to achieve a better accuracy in replacing the holes by second order moments and taking account of variable primary stresses in the region contained within the hole boundary.

The corresponding complex potential must be assumed in the form

$$\Sigma(z) = \sigma_{yz}(z)+i\sigma_{xz}(z) = \tag{5.2}$$
$$= \frac{p_1}{2}\left[\frac{z+\sqrt{z^2-c^2}}{a+b}+\frac{(a+b)^3}{(z+\sqrt{z^2-c^2})^3}\right]\left(1+\frac{z}{\sqrt{z^2-c^2}}\right).$$

Displacements w are determined from the formula

$$\mu w(z)=\frac{p_1(a+b)}{4}\operatorname{Im}\left[\left(\frac{z+\sqrt{z^2-c^2}}{a+b}\right)^2-\left(\frac{a+b}{z+\sqrt{z^2-c^2}}\right)^2\right]. \tag{5.3}$$

From Eq. (5.2) it follows that at infinity the stresses tend to

$$\Sigma^{\infty}(z)=p_1\frac{2z}{a+b},$$

in agreement with the assumption (5.1). At the boundary of the hole $\hat{z}=a\cos\vartheta+ib\sin\vartheta$ the displacement (5.3) becomes

$$\mu w(\hat{z})=p_1(a+b)\sin\vartheta\cos\vartheta, \tag{5.4}$$

while in the case of a circular hole we had (Eq. $(2.3)_1$)

$$\mu w(\hat{z})=2ap_1\sin\vartheta\cos\vartheta.$$

The normal component of the stress at the boundary of the hole vanishes since from Eq. (5.2) we obtain for $z=\hat{z}$

$$\sigma_{xz}(\hat{z})=\frac{-p_1(a+b)}{2h^2}2a\sin\vartheta\cos2\vartheta,$$

$$\sigma_{yz}(\hat{z})=\frac{p_1(a+b)}{2h^2}2b\cos\vartheta\cos2\vartheta,$$

and substitution of these stresses into Eq. (4.5) yields

$$\sigma_{nz}(\hat{z})=0,\qquad \sigma_{tz}(\hat{z})=\frac{p_1(a+b)\cos2\vartheta}{h}. \tag{5.5}$$

In order to determine the equivalent moments, the function $\Sigma(z)$ from Eq. (5.2) is expanded into a series. The first two terms of this expansion have the form

$$\Sigma(z)=\frac{2p_1}{a+b}\left[z+\frac{(a+b)^4+c^4}{16}\frac{1}{z^3}+\ldots\right].$$

The term with z^{-2} is missing from this expansion, like in Eq. (2.3). With $a=b$ this formula yields the expression

$$\Sigma(z)=p_1\frac{z}{a}+p_1\frac{a^3}{z^3}+\ldots$$

identical with Eq. $(2.3)_3$ for $q_0=p_0=q_1=0$ and $\varkappa=1$. The equivalent moment is obtained by equating $\Sigma^{1,1}$ from the formula (1.12) to the corresponding term of the above expansion,

$$M^{1,1} = \frac{1}{8}\pi p_1 \frac{(a+b)^4+c^4}{a+b}. \tag{5.6}$$

Reducing the elliptical hole to a crack ($b = 0$, $c = a$) we obtain the result complying with that given in [9], p. 100.

To make the analysis complete let us finally consider the case of loading the body at infinity by the following forces:

$$\sigma_{xz}^{\infty} = q_1 \frac{2x}{a+b}, \quad \sigma_{yz}^{\infty} = -q_1 \frac{2y}{a+b}. \tag{5.7}$$

The displacement and complex stress must be assumed in the form

$$\begin{aligned} \mu w &= \frac{1}{4} q_1 (a+b) \,\mathrm{Im}\, i\left[\left(\frac{z+\sqrt{z^2-c^2}}{a+b}\right)^2 + \left(\frac{a+b}{z+\sqrt{z^2-c^2}}\right)^2\right], \\ \Sigma(z) &= \frac{1}{2} q_1 i\left[\frac{z+\sqrt{z^2-c^2}}{a+b} - \left(\frac{a+b}{z+\sqrt{z^2-c^2}}\right)^3\left(1+\frac{z}{\sqrt{z^2-c^2}}\right)\right]. \end{aligned} \tag{5.8}$$

At the boundary of the hole, $z = \hat{z}$, we obtain

$$\mu w(\hat{z}) = \frac{1}{2} q_1 (a+b) \cos 2\vartheta,$$

$$\sigma_{nz}(\hat{z}) = 0, \quad \sigma_{tz}(\hat{z}) = -q_1 \frac{a+b}{h} \sin 2\vartheta.$$

Expansion of the function (5.8) into a power series

$$\Sigma(z) = \frac{2q_i i}{a+b}\left[z - \frac{(a+b)^4-c^4}{16}\frac{1}{z^3} + \ldots\right],$$

and comparison with Eq. (1.13) yields the equivalent moment

$$M^{2,0} = \frac{1}{8}\pi q_1 \frac{(a+b)^4-c^4}{a+b}. \tag{5.9}$$

Substitution of the values of $M^{1,1}$ and $M^{2,0}$, Eqs. (5.6) and (5.9), into the formulae for stresses leads to the expressions

$$\begin{aligned} \sigma_{xz} &= \frac{2p_1}{a+b}\left[r \sin\theta - \frac{(a+b)^4+c^4}{16}\frac{\sin 3\theta}{r^3}\right] + \\ &\quad + \frac{2q_1}{a+b}\left[r\cos\theta - \frac{(a+b)^4-c^4}{16}\frac{\cos 3\theta}{r^3}\right], \\ \sigma_{yz} &= \frac{2p_1}{a+b}\left[r\cos\theta + \frac{(a+b)^4+c^4}{16}\frac{\cos 3\theta}{r^3}\right] + \\ &\quad + \frac{2q_1}{a+b}\left[r\sin\theta - \frac{(a+b)^4-c^4}{16}\frac{\sin 3\theta}{r^3}\right]. \end{aligned} \tag{5.10}$$

which describe the stress field in an infinite body containing an elliptical hole and loaded at infinity by the forces (5.1) and (5.7). However, in the case of a single hole and linear loading at infinity, accurate solutions are known and there is no need to apply the approximate formulae (5.10); they will be used mainly in analyzing the more complicated cases of several holes and arbitrarily variable stress fields, the coefficients p_1, q_1 (and p_0, q_0 discussed in the preceding section) playing the role of characteristics of the "primary" stress field necessary to determine the values of equivalent moments.

6. Interaction of elliptical holes

Summing up the results derived above, let us outline the procedure of evaluation of the equivalent moments replacing the elliptical holes in a body subject to arbitrary loads. Each hole is replaced by a set of four moments $M_A^{0,1}$, $M_A^{1,0}$, $M_A^{1,1}$, $M_A^{2,0}$ (here subscript A corresponds to hole A). In order to determine their values, the primary stress field in the region occupied by hole A must be determined. It consists of the component due to the external load applied to the body (either at infinity or, at least, far away from the region), and of the components due to the action of other moments $M_B^{i,j}$, $M_C^{i,j}$, ... replacing the holes B, C, ... The primary stress must be expanded into a power series of x and y in the neighbourhood of point A (center of hole A) with the coordinates x_A, y_A:

$$\begin{aligned} \sigma_{xz}^A(x, y) &\approx q_0^A + \frac{2q_1^A}{a+b}(x-x_A) + \frac{2p_1^A}{a+b}(y-y_A), \\ \sigma_{yz}^A(x, y) &\approx p_0^A - \frac{2q_1^A}{a+b}(y-y_A) + \frac{2p_1^A}{a+b}(x-x_A). \end{aligned} \tag{6.1}$$

The formulae (6.1) may also be useful in the case of a single elliptical hole in an infinite region, provided the stress field produced by the external loads is not linear. Instead of using the complicated potentials $F(z)$ corresponding to such stress states, the average values of p_0, q_0, p_1, q_1 in the region of A may be determined and then used to evaluate the set of four equivalent moments. The approximate stress distribution is found from the formulae

$$\begin{aligned} \sigma_{xz}(r, \theta) = \bar{\sigma}_{xz}(r, \theta) &- \frac{1}{4} K_a p_0 \frac{\sin 2\theta}{r^2} - \frac{1}{4} K_b q_0 \frac{\cos 2\theta}{r^2} - \\ &- \frac{1}{8} K_1 p_1 \frac{\sin 3\theta}{r^3} - \frac{1}{8} K_2 q_1 \frac{\cos 3\theta}{r^3}, \end{aligned} \tag{6.2}$$

$$\sigma_{yz}(r,\theta)=\bar{\sigma}_{yz}(r,\theta)+\frac{1}{4}K_a p_0\frac{\cos 2\theta}{r^2}-\frac{1}{4}K_b q_0\frac{\sin 2\theta}{r^2}+$$
$$+\frac{1}{8}K_1 p_1\frac{\cos 3\theta}{r^3}-\frac{1}{8}K_2 q_1\frac{\sin 3\theta}{r^3}.$$

The following notations are introduced here:

$$K_a=2a(a+b),\qquad K_b=2b(a+b),$$
$$K_1=\frac{(a+b)^4+c^4}{a+b},\qquad K_2=\frac{(a+b)^4-c^4}{a+b},$$

and $\bar{\sigma}_{xz}(r,\theta)$, $\bar{\sigma}_{yz}(r,\theta)$ are the stresses produced by external loads. In the case of a circular hole $a=b$ and linear stress distribution at infinity, the formulae (6.2) become accurate and reduce to Eqs. (2.3); in such a case $K_a=K_b=4a^2$ and $K_1=K_2=8a^3$.

In the case of several holes, Eqs. (6.1) are used to construct the necessary set of equations for the unknown values of $M^{i,j}$. The primary stresses and stress gradients at A expressed in terms of the external loads and the remaining moments $M_B^{i,j}$, $M_C^{i,j}$, ..., enable us to write down the four equations required (cf. Eqs. (5.1), (5.7)),

$$\sigma_{xz|A}=q_0^A,\qquad \sigma_{yz|A}=p_0^A,$$
$$\left.\frac{\partial\sigma_{xz}}{\partial x}\right|_A=q_1^A\frac{2}{a+b},\qquad \frac{\partial\sigma_{xz}}{\partial y}=p_1^A\frac{2}{a+b};$$

note that, for obvious reasons,

$$\frac{\partial\sigma_{xz}}{\partial y}=\frac{\partial\sigma_{yz}}{\partial x}\quad\text{and}\quad\frac{\partial\sigma_{xz}}{\partial x}=-\frac{\partial\sigma_{yz}}{\partial y}.$$

Consequently, the case of n holes will be reduced to a system of $4n$ equations with the same number of unknowns. In the case of additional symmetry properties, this number may be reduced, thus making the solution fairly simple.

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Streszczenie

O EFEKTYWNEJ METODZIE OKREŚLANIA WPŁYWU OTWORÓW, SZCZELIN I INKLUZJI NA STAN NAPRĘŻENIA W OŚRODKACH SPRĘŻYSTYCH

Defekty różnego rodzaju wprowadzają zaburzenia do pola naprężeń wywołanego w ośrodku sprężystym przez obciążenia zewnętrzne. Teoretyczne podstawy analizy takiego pola naprężeń są dobrze znane i w zasadzie, zagadnienia takie sprowadzić można do rozwiązywania odpowiednich układów równań całkowych. Jednak problem efektywnego wyznaczenia wpływu defektów na pola naprężeń napotyka znaczne trudności, zwłaszcza gdy mamy do czynienia z dużą liczbą defektów wprowadzających nieskończone koncentracje naprężeń i wymagających np. zastosowania wielkiej liczby elementów skończonych. Celem tej pracy jest właśnie przedstawienie takiej metody przybliżonej, która przypomina znaną z elementarnej mechaniki budowli metodę rozwiązywania układów statycznie niewyznaczalnych i pozwala sprowadzić omawiane zagadnienie do rozwiązania prostego układu równań algebraicznych. Jeśli wzajemne odległości otworów i inkluzji eliptycznych nie są mniejsze od ich średnic, dokładność uzyskanych wyników okazuje się zadowalająca z punktu widzenia zastosowań inżynierskich.

Резюме

ОБ ЭФФЕКТИВНОМ МЕТОДЕ ОПРЕДЕЛЕНИЯ ВЛИЯНИЯ ОТВЕРСТИЙ, ТРЕЩИН И ВКЛЮЧЕНИЙ НА НАПРЯЖЕННОЕ СОСТОЯНИЕ В УПРУГИХ СРЕДАХ

Дефекты разного рода вводят возмущения в поле напряжений, вызванное в упругой среде внешним нагружением. Теоретические основы анализа такого поля напряжений хорошо известны и в принципе такие задачи можно свести к решению соответствующих систем интегральных уравнений. Однако проблема эффективного определения влияния дефектов на поля напряжений встречает значительные трудности, особенно, когда имеем

дело с большим количеством дефектов, вводящих бесконечные концентрации напряжений и требующих, например, применения большого количества конечных элементов. Целью настоящей работы является именно представление такого приближенного метода, который напоминает известный из элементарной строительной механики метод решения статически неопределимых систем и позволяет свести обсуждаемую задачу к решению простой системы алгебраических уравнений. Если взаимные расстояния отверстий и эллиптических включений не меньше чем их диаметры, точность полученных результатов оказывается достаточной с точки зрения инженерских применений.

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