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FIRST AND SECOND ORDER SENSITIVITY DERIVATIVES OF MECHANICAL SYSTEMS TRANSFER MATRIX

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Sensitivity analysis in the frequency domain is considered in the paper. The first and second order sensitivity matrices of the mechanical lumped system are determined utilizing the transfer matrix partial derivatives. Examples presented in the paper show the effects of the parameter variation on amplitude-frequency characteristics.

1. INTRODUCTION

When the influence of parameter variation on system dynamic characteristic is considered, parameter sensitivity methods are very useful.

Sensitivity analysis was first applied in optimal control and automated structural optimization where gradient methods were used to find search directions toward optimum solutions. Less well known is the use of sensitivity methods to mechanical systems; however, in the last years papers have appeared in this area. HAUG and ROUSSELET [1] and HAUG, KOMKOV and CHOI [2] use the sensitivity analysis for the static structural response of mechanical systems. The adjoint variable method to dynamic proceses in planar mechanism was used by HAUG, WEHAGE and BARMAN [3]. Also RAY, PISTER and POLAK [4] use this method for the analysis of the hysteretic damping system. The sensitivity method for cam mechanism has been applied by YOUNG and SHOUP [5]. Sensitivity analysis for vehicle system dynamics has been performed by WATARI and IWAMOTO [6]. Also MIKULCIK [7] presents the application of the sensitivity method to car-trailer stability. The parametric sensitivity analysis of the eigenvalues of a linear model of a merchandise wagon has been carried out by KISILOWSKI [8].

Although the first order sensitivity functions give valuable information on the influence of parameter variation on system dynamic characteristics, the second and higher order sensitivity derivatives are required if large parameter variations are involved. On the other hand, the second and higher order sensitivity derivatives are unfrequently, as yet, applicable, and methods are much less developed, particularly for mechanical systems. The flutter eigenvalue problem using second order sensitivity derivatives was studied by

RUDISILL and BHATIA [9]. Second order sensitivity analysis was applied by HAUG [10] and HAUG and EHLE [11] where the adjoint variable technique was proposed. VAN BELLE [12] derived also second derivatives of flexibility matrices. The adjoint variable method has been used by DEMS and MRÓZ [13] who investigated higher order sensitivity analysis of elastic structures. The second order sensitivity of eigenvalue has also been considered by BRANDON [14]. A valuable survey paper in the area of sensitivity analysis was published by ADELMAN and HAFTKA [15].

Sensitivity analysis of a dynamic system can be considered in the time domain or in the frequency domain. One of the shortcomings of the sensitivity functions defined in the time domain is that they depend on the actual form of the input signal. The sensitivity functions defined as partial derivatives of the transfer matrix of the system with respect to one of the parameters are independent of the form of the input signal. This property is very useful in practice.

A procedure of sensitivity analysis of the mechanical lumped system utilizing the first and second order transfer matrix derivatives is presented in the paper. The two examples, one for the two-degree-of-freedom model of front vehicle suspension, and the second for the five-degree-of-freedom car model, illustrate the practical application of parametrical sensitivity analysis. The examples shown indicate that the proposed second order logarithmic sensitivity functions preserve the qualitative character of first order sensitivity functions and simultaneously are more "sensitive" to system parameter changes.

2. SENSITIVITY MATRICES

A multi-degree-of-freedom kinematically excited mechanical system can be expressed in the well-known form of the differential matrix equation:

$$M\ddot{z}+C\dot{z}+Kz = M'\ddot{u}+C'\dot{u}+K'u, \tag{2.1}$$

where M, C, K are square, n-dimensional, positive definite and symmetric matrices of inertia, damping and stiffness, respectively; z is the n-dimensional displacement vector; M', C', K' are $(n \times m)$-dimensional matrices connected with the m-dimensional vector of kinematic excitation u.

Assuming that all components of the vectors z and u are Fourier transformable, Eq. (2.1) can be written in the form

$$z^* = Hu^*, \tag{2.2}$$

where

$$H = A^{-1}B = [K-\omega^2 M+j\omega C]^{-1}[K'-\omega^2 M'+j\omega C'], \tag{2.3}$$

where H is the $(n\times m)$-dimensional transfer matrix; z^* and u^* are Fourier transforms of z and u, respectively; ω is angular frequency; $j=\sqrt{-1}$. Equation (2.1) occurs frequently in modelling the behaviour of vehicles and in vibration isolation problems.

It can be noticed that the n-degree-of-freedom lumped mechanical system is very frequently described by the equation

$$M\ddot{z}+C\dot{z}+Kz=F, \tag{2.4}$$

where F is the n-dimensional force vector. The transfer matrix of Eq. (2.4) can be obtained from Eq. (2.3) under the assumption that B is the unit matrix. Thus the model (2.1) and transfer matrix (2.3) have a more general form. For this reason Eqs. (2.1) and (2.3) will be taken into consideration in further analysis.

Let us assume that the each component of the matrices M, C, K, M', C', K' depend on the parameter vector $p=(p_1, p_2, ..., p_r)$ which appears in a lumped system (2.1). As regards Eq. (2.3) the transfer matrix depends on the parameters p, i.e. $H=H(p)$.

The first and second order sensitivity matrices W_i and $W_{i,j}$ can be written in the form (using the rule of differentiating the inverse matrix)

$$W_i=\left.\frac{\partial H}{\partial p_i}\right|_{p_0}, \qquad W_{i,j}=\left.\frac{\partial^2 H}{\partial p_i\,\partial p_j}\right|_{p_0}, \tag{2.5}$$

where p_0 means the nominal parameter vector. The parameter vector after variation, called actual parameter vector, is $p=p_0+\Delta p$, where Δp is a small parameter variation. Equations (2.5) show that W_i and $W_{i,j}$ are calculated at the nominal parameter values. For simplicity this subscript will be neglected in further considerations.

In accordance with Eq. (2.3) the sensitivity matrices given by Eqs. (2.5) can be written in the form (using the rule of differentiating the inverse matrix)

$$W_i=\frac{\partial}{\partial p_i}(A^{-1}B)=-A^{-1}\frac{\partial A}{\partial p_i}A^{-1}B+A^{-1}\frac{\partial B}{\partial p_i}=A^{-1}(B_i-A_i H), \tag{2.6}$$

where

$$A_i=\frac{\partial A}{\partial p_i}, \qquad B_i=\frac{\partial B}{\partial p_i}, \tag{2.7}$$

$$\begin{aligned} W_{i,j}=\frac{\partial W_i}{\partial p_j}=\frac{\partial}{\partial p_j}(B_i-A_i H)= \\ =A^{-1}A_j A^{-1}A_i H-A^{-1}A_{i,j}H+A^{-1}A_i A^{-1}A_j A^{-1}B- \\ -A^{-1}A_i A^{-1}B_j-A^{-1}A_j A^{-1}B_i+A^{-1}B_{i,j}= \\ =-2A^{-1}A_i W_i-A^{-1}A_{i,j}H+A^{-1}B_{i,j}, \end{aligned} \tag{2.8}$$

where

$$A_{i,j} = \frac{\partial^2 A}{\partial p_i \partial p_j}, \quad B_{i,j} = \frac{\partial^2 B}{\partial p_i \partial p_j}. \tag{2.9}$$

If the parameters of the system depend linearly on dynamic characteristics, the matrices A_i and B_i have constant components and, in consequence, the second derivatives $A_{i,j}$ and $B_{i,j}$ have zero components. In this case Eq. (2.8) can be simplified:

$$W_{i,j} = -2A^{-1} A_i W_i. \tag{2.10}$$

The nondimensional logarithmic sensitivity functions which possess normalizing coefficients are very useful when sensitivity analysis is carried out in the frequency domain. If $h_{k,l}$ represents k, the l-th element of the transfer matrix H, then the logarithmic sensitivity function is defined as [16]

$$s_{i(k,l)} = \frac{\partial \ln h_{k,l}}{\partial \ln p_i} = \frac{\partial h_{k,l}}{\partial p_i} \frac{p_i}{h_{k,l}} = w_{i(k,l)} \frac{p_i}{h_{k,l}}, \tag{2.11}$$

where $w_{i(k,l)}$ in the k, l-th element of the first order sensitivity matrix W_i. In a similar way the second order logarithmic sensitivity function may be defined as

$$s_{i,j(k,l)} = \frac{\partial \ln w_{i(k,l)}}{\partial \ln p_j} = \frac{\partial w_{i(k,l)}}{\partial p_j} \frac{p_j}{w_{i(k,l)}} = w_{i,j(k,l)} \frac{p_j}{w_{i(k,l)}}, \tag{2.12}$$

where $w_{i,j(k,l)}$ represents the k, l-th element of the second order sensitivity matrix $W_{i,j}$.

All elements of the considered sensitivity matrices are complex functions. It is difficult to understand what sensitivity means in terms of a complex function. To characterize the sensitivity, the real or imaginary part can be used. But most frequently the sensitivity function of the magnitude of the transfer matrix elements is utilized, which is considered over the whole frequency range. These magnitudes are well known as amplitude-frequency characteristics. The underlying idea is that the variations of magnitude are large if the values of the sensitivity function are large, and vice versa. Thus the sensitivity function of the amplitude-frequency characteristic carries very useful information for the researcher who designs or modifies the system.

3. Numerical examples

According to definition, the first order sensitivity function (first partial derivative) is valid for infinitesimal parameter variation, although the variation in practice is not infinitesimal. The first example shows how the amplitude-frequency characteristic changes when the value of the suitable parameter changes around the nominal value. To illustrate the problem, the two-degree-of-freedom system of simplified front vehicle suspension (Fig. 1) is considered.

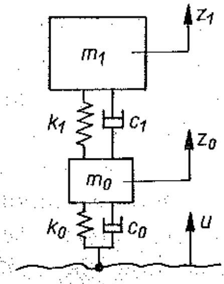


FIG. 1.

The equation of motion has the form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_0 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_0 \end{bmatrix}+\begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_0+c_1 \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_0 \end{bmatrix}+\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_0+k_1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_0 \end{bmatrix}=$$
$$=\begin{bmatrix} 0 \\ c_0 \end{bmatrix}\dot{u}+\begin{bmatrix} 0 \\ k_0 \end{bmatrix}u.$$

The components of the parameter vector $p = (m_0, m_1, c_0, c_1, k_0, k_1)$ have the nominal values: $m_0 = 70$ kg, $m_1 = 660$ kg, $c_0 = 100$ Ns/m, $c_1 = 2500$ Ns/m, $k_0 = 300000$ N/m, $k_1 = 45000$ N/m.

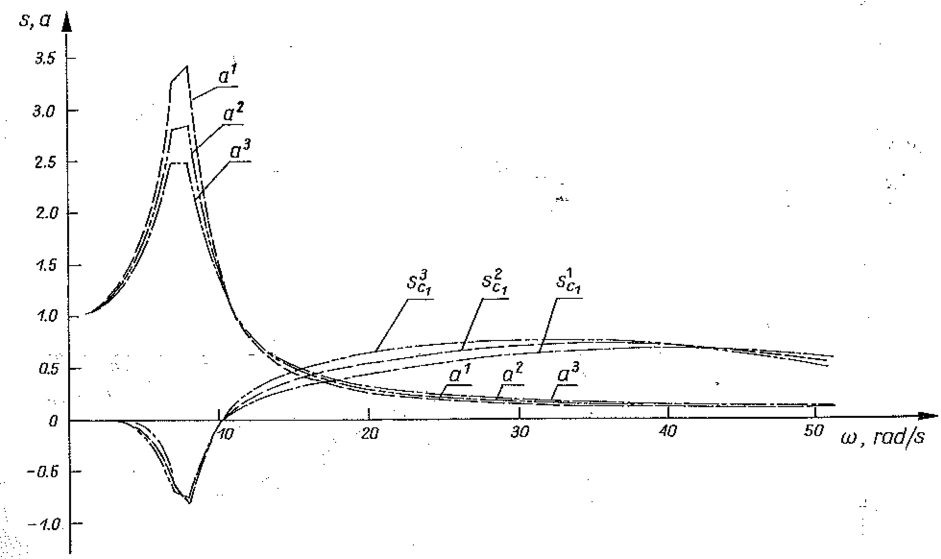


FIG. 2.

Displacement amplitude-frequency characteristics a^1, a^2, a^3 of sprung mass m_1 (input — u, output — z_1) and appropriate logarithmic sensitivity functions $s^1_{c_1}$, $s^2_{c_1}$, $s^3_{c_1}$ for different values of the damping coefficient: $c^1_1 = 2000$ Ns/m, $c^2_1 = 2500$ Ns/m, $c^3_1 = 3000$ Ns/m are plotted in Fig. 2. The largest changes of the amplitude-freguency characteristic occur for the frequency range $\omega = (5-10)$ rad/s (the first resonance of the system). In the same range sensitivity functions have negative values. It means that amplitude of mass m_1 decreases ($a^3 < a^2 < a^1$) when the damping coefficient increases ($c^3_1 > c^2_1 > c^1_1$). The opposite influence may be observed for $\omega > 10$ rad/s where sensitivity functions have positive values. It means that the increase of the damping coefficient causes an increase of vibration amplitude of mass m_1.

A similar conclusion may be formulated when the curves in Fig. 3 are analysed. The sensitivity functions show the influence of the damping parameter c_1 on the acceleration amplitude-frequency characteristic of the unsprung mass m_0 (u — input, z_0 — output). The values of c_1 are the same as in the sample above. The largest changes of the acceleration amplitude-frequency characteristic occur about the second resonance range $\omega = (60-80)$ rad/s.

Figure 4 shows the influence of the parameters m_1, c_1, and k_1 on the displacement amplitude-frequency characteristic of sprung mass m_1 (coordinate z_1; logarithmic sensitivity functions $s^{z_1}_{m_1}$, $s^{z_1}_{c_1}$, $s^{z_1}_{k_1}$) and on the displacement of the amplitude-frequency characteristic of unsprung mass m_0 (coordinate z_0; logarithmic sensitivity functions $s^{z_0}_{m_1}$, $s^{z_0}_{c_1}$, $s^{z_0}_{k_1}$). The curves presented lead to the following conclusions: the value of reduced mass m_1, suspension damping coefficient c_1 and the suspension stiffness coefficient k_1

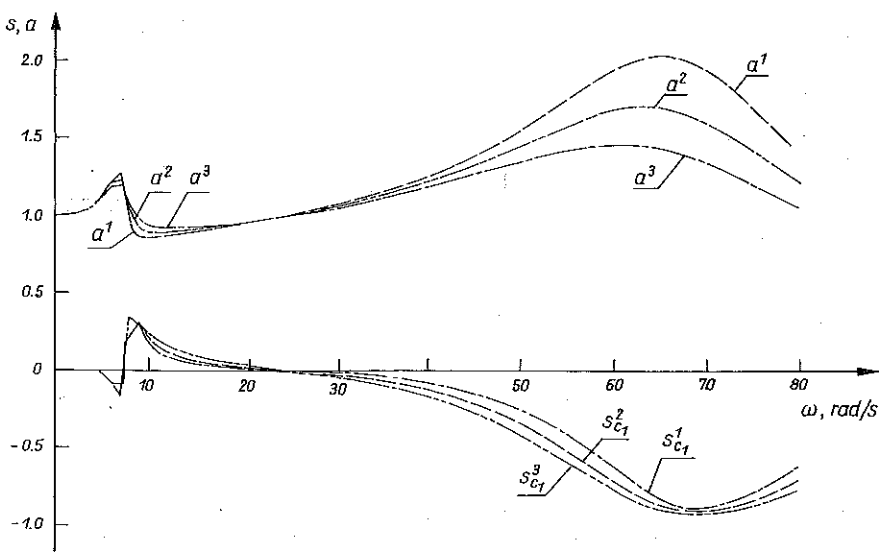
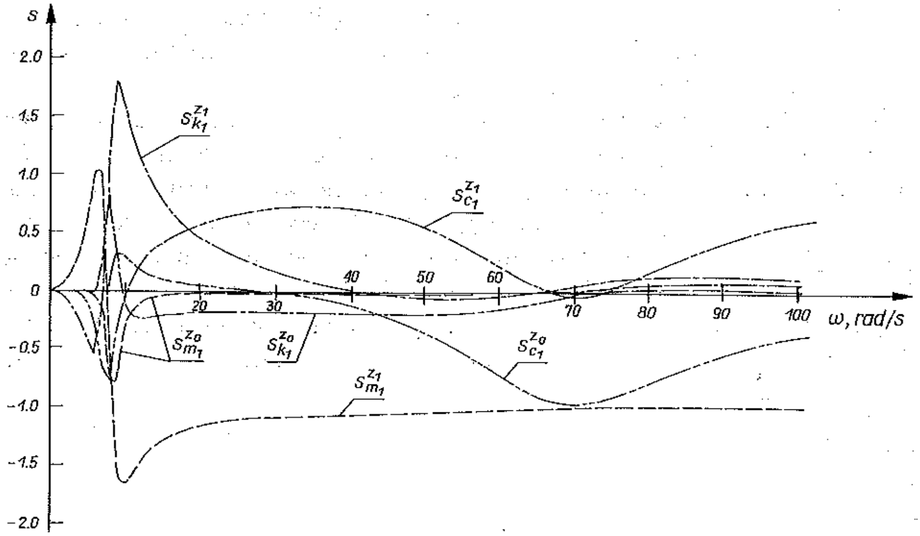
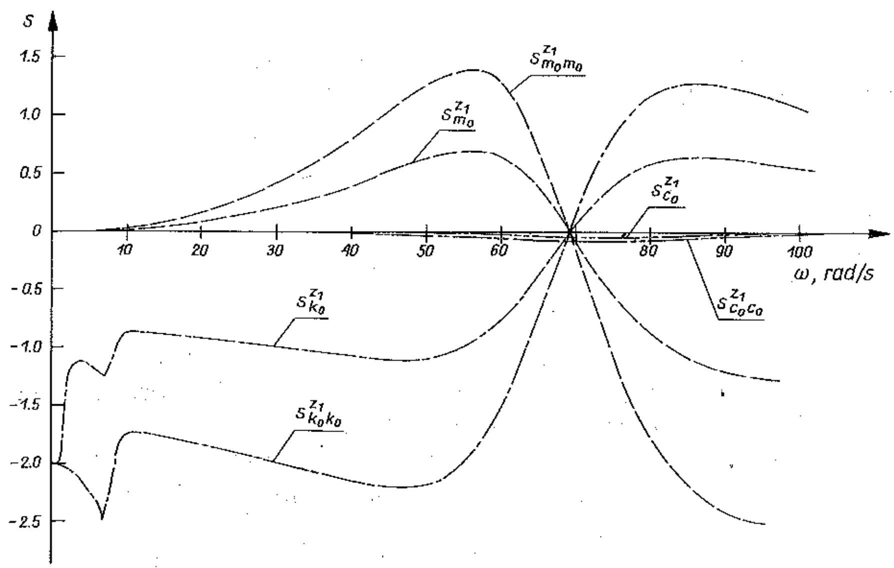


FIG. 3.



s
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0
$s_{k_1}^{z_1}$
$s_{c_1}^{z_1}$
$s_{m_1}^{z_0}$
$s_{k_1}^{z_0}$
$s_{c_1}^{z_0}$
$s_{m_1}^{z_1}$
20
30
40
50
60
70
80
90
100
ω, rad/s

FIG. 4.



s
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0
-2.5
$s_{m_0m_0}^{z_1}$
$s_{m_0}^{z_1}$
$s_{c_0}^{z_1}$
$s_{c_0c_0}^{z_1}$
$s_{k_0}^{z_1}$
$s_{k_0k_0}^{z_1}$
10
20
30
40
50
60
80
90
100
ω, rad/s

FIG. 5.

have the largest influence on the vibration amplitude of both masses m_1 and m_0 in a neighbourhood of the first resonance ($\omega < 10$ rad/s). In the frequency range around the second resonance ($\omega = (60-80)$ rad/s) the suspension damping coefficient c_1 is the only important parameter for the vibration amplitude of mass m_0 (see the sensitivity function $s_{c_1}^{z_0}$). Also the parameter m_1 has a significant influence on the vibration amplitude of sprung mass m_1 (see the sensitivity function $s_{m_1}^{z_1}$).

The influence of the parameters m_0, c_0, k_0 on the amplitude-frequency characteristic of sprung mass m_1 is shown in Fig. 5 where the first order logarithmic sensitivity functions $s_{m_0}^{z_1}$, $s_{c_0}^{z_1}$, $s_{k_0}^{z_1}$ and the second order logarithmic sensitivity-functions $s_{m_0m_0}^{z_1}$, $s_{c_0c_0}^{z_1}$ and $s_{k_0k_0}^{z_1}$ and plotted. From the comparison of both types of the sensitivity function it can be seen that the second order sensitivity functions are more "sensitive" than the first order. The example shown also indicates that the proposed definition (2.12) of the second order sensitivity function preserves the qualitative character of the first order logarithmic sensitivity function.

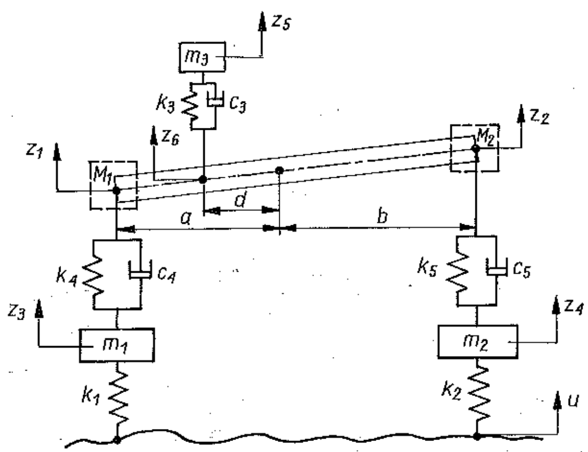


FIG. 6.

It can be noticed that the influence of tire damping c_0 on the vibration amplitude of sprung mass m_1 is small, what confirms that tire damping may be neglected in vehicle vibration modelling.

The second example of sensitivity analysis is a five-degree-of-freedom model of vehicle shown in Fig. 6. The equation of motion has matrix form:

$$M\ddot{z}+C\dot{z}+Kz = K'u.$$

Matrices M, C, K, and K' are given in Table 1.

Computation has been carried out for the frequency range $\omega=(0-50)$ rad/s which includes the resonances of the reduced masses M_1 and M_2 and the

Table 1

$$M = \begin{bmatrix} M_1 & M_3 & 0 & 0 & 0 \\ M_3 & M_2 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & m_3 \end{bmatrix} \qquad C = \begin{bmatrix} c_4+\alpha & \beta c_3 & -c_4 & 0 & -\alpha c_3 \\ \beta c_3 & c_5+\beta c_3 & 0 & -c_5 & -\beta c_3 \\ -c_4 & 0 & c_4 & 0 & 0 \\ 0 & -c_5 & 0 & c_5 & 0 \\ -\alpha c_3 & -\beta c_3 & 0 & 0 & c_3 \end{bmatrix}$$

$$K = \begin{bmatrix} k_4+\alpha^2 k_3 & \gamma k_3 & -k_4 & 0 & -\alpha k_3 \\ \gamma k_3 & k_5+\beta^2 k_3 & 0 & -k_5 & -\beta k_3 \\ -k_4 & 0 & k_1+k_4 & 0 & 0 \\ 0 & -k_5 & 0 & k_2+k_5 & 0 \\ -\alpha k_3 & -\beta k_3 & 0 & 0 & k_3 \end{bmatrix} \qquad K' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_1 & 0 \\ 0 & k_2 e^{-j\omega \frac{l}{v}} \\ 0 & 0 \end{bmatrix}$$

$$M_1 = \frac{b^2+\varrho^2}{l^2} M$$

$$M_3 = \frac{ab-\varrho^2}{l^2} M$$

$$M_2 = \frac{a^2+\varrho^2}{l^2} M$$

$$\alpha = \frac{b+d}{l}, \quad \beta = \frac{a-d}{l}$$

$$\gamma = \frac{(b+d)(a-d)}{l^2}, \quad l = a+b$$

ϱ — radius of gyration of mass $M = M_1 + M_2$ with respect to the axis passing through the center of gravity of mass M and perpendicular to the longitudinal plane.

range when the effect of acceleration is most significant for the car driver body represented by mass m_3.

The following values have been assumed into computation:

$$M = 1150\ \text{kg},\quad M_1 = 660\ \text{kg},\quad M_2 = 490\ \text{kg},\quad m_1 = 70\ \text{kg},$$
$$m_2 = 110\ \text{kg},\quad m_3 = 80\ \text{kg},\quad c_3 = 5000\ \text{Ns/m},\quad c_4 = 2500\ \text{Ns/m},$$
$$c_5 = 3700\ \text{Ns/m},\quad k_1 = 300000\ \text{N/m},\quad k_2 = 350000\ \text{N/m},$$
$$k_3 = 20000\ \text{N/m},\quad k_4 = 45000\ \text{N/m},\quad k_5 = 35000\ \text{N/m},$$
$$a = 1.15\ \text{m},\quad b = 1.55\ \text{m},\quad d = 0.4\ \text{m},\quad \varrho^2 = 1.82\ \text{m}^2.$$

The influence of the stiffness and damping coefficients of the seat driver (k_3 and c_3) and front and rear suspensions (k_4, k_5, c_4, c_5) have been investigated when the vehicle is moving at a spped $v = 20$ m/s (72 km/h).

The logarithmic sensitivity functions plotted in Fig. 7 yield the following conclusions:

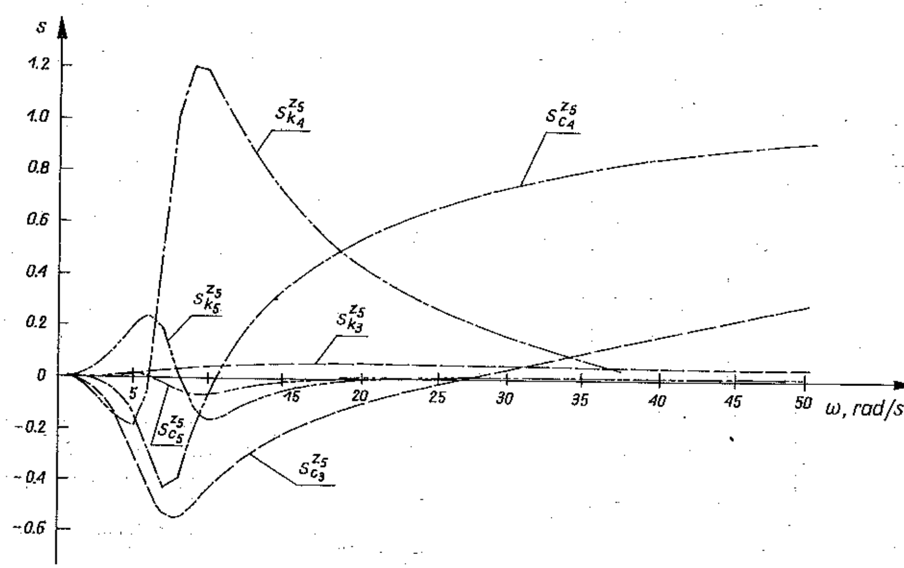


FIG. 7.

1. The front suspension stiffness coefficient k_4 and damping coefficient c_4 as well as the damping coefficient c_3 of the seat driver have significant influence in the frequency range $\omega = (3-10)$ rad/s, i.e. in the neighbourhood of the reduced masses M_1, M_2 resonances. The sensitivity functions $s_{c_3}^{z_5}$ and $s_{c_4}^{z_5}$ of the damping coefficients c_3 and c_4 have negative values, what means that the increase of the damping parameters decrease the vibration amplitude in the considered frequency range.

2. The only damping coefficient c_4 of the seat driver has a significant influence in the frequency range $\omega = (25-50)$ rad/s (this is the frequency

range where the human body is most sensitive to vertical acceleration). The sensitivity function $s_{c4}^{z_5}$ has positive values. This means that the increase of this parameter increases the vibration amplitude of mass m_3 which represents the driver body.

3. It can be seen that the stiffness coefficient k_3 of the seat driver, and the coefficients k_5 and c_5 of the rear suspension have a very small influence on the vibration amplitude of mass m_3 in the whole considered frequency range.

4. Concluding remarks

The method presented in the paper for sensitivity analysis in the frequency domain provides useful design information to the designers of mechanical systems in order to achieve the most efficient modification. Particularly it is useful for reducing the vibration amplitude of the system when the effect of many parameters have to be examined.

The sensitivity procedure has been brought to simple matrix operations which can easily be handled numerically.

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STRESZCZENIE

WRAŻLIWOŚĆ PIERWSZEGO I DRUGIEGO RZĘDU MACIERZY TRANSMITANCJI UKŁADU MECHANICZNEGO

W artykule przedstawiono analizę wrażliwości w przestrzeni częstotliwości. Macierze wrażliwości pierwszego i drugiego rzędu otrzymano dla dyskretnego układu mechanicznego wykorzystując pochodne cząstkowe macierzy transmitancji. Zaprezentowane przykłady pokazują wpływ zmian parametrów na charakterystyki amplitudowo-częstotliwościowe.

РЕЗЮМЕ

ЧУВСТВИТЕЛЬНОСТЬ ПЕРВОГО И ВТОРОГО ПОРЯДКОВ МАТРИЦЫ ПЕРЕДАТОЧНОЙ ФУНКЦИИ МЕХАНИЧЕСКОЙ СИСТЕМЫ

В статье представлен анализ чувствительности в пространстве частот. Матрицы чувствительности первого и второго порядков получены для дискретной механической системы, используя частные производные матрицы передаточной функции. Представленные примеры показывают влияние изменений параметров на амплитудно-частотные характеристики.

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