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NUMERICAL ANALYSIS OF THERMO-ELASTO-PLASTIC PROBLEMS

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The purpose of this paper is to present the theoretical and numerical background adopted in the course of the development of the computer program TEMAS (Thermo-Mechanical Analysis of Structures). The paper demonstrates the effectiveness of the finite element analysis in predicting the behaviour of inelastic materials subjected to mechanical and thermal loadings. A number of computational examples illustrate the paper.

1. Introduction

In this paper the computer-oriented approach to problems of isothermal elasto-plasticity is extended to cover thermo-elasto-plastic materials [1—5]. This is done by assuming that material properties are temperature-dependent. Every effort is exerted to preserve the classical structure of finite element equations describing inelastic materials, which assures efficient computer implementation of the theory. The resulting incremental equations contain additional terms attributable to nonisothermal conditions and can easily be used to modify any existing finite element program designed for the analysis of inelastic materials under isothermal constraints.

The decoupled thermo-mechanical formulation is considered only and it is based upon the assumption that no mechanical terms enter the heat conduction equation. The temperature distribution is assumed to be known in the region considered — otherwise the solution to the corresponding Fourier problem can be obtained by using computer programs described in [6], for instance. In Sect. 2 we briefly review the governing equations of the theory of thermo-elasto-plasticity. For the sake of generality a combined isotropic-kinematic hardening rule is employed. In Sect. 3 we discuss details of numerical implementation. The Newton-Raphson iteration scheme coupled with the radial return algorithm tailored to accommodate thermal effects are used as the basic numerical concepts. Section 4 contains a number of numerical illustrations.

The formulation presented in the paper is meant to be used for infinitesimal deformation analysis only. However, the generalization to cover finite deformation effects can easily be accomplished along the lines suggested in [5], for instance. In fact, the coding of the program is based on such a general theory, and the follow-up paper presenting details of our numerical experience with finite deformation thermo-plasticity is to be published shortly.

2. High-temperature inelastic analysis

The purpose of this section is, first, to give a short review of the equations governing the thermo-elastic-plastic flow so as to faciliate later both the presentation of a numerical algorithm and the discussion of examplary analyses of structures which will follow in Sects. 3 and 4.

The notation and derivation of successive equations will be commented upon in a more extensive way only at these places at which issues not already standardized in the recent literature will appear.

We assume the linear elastic constitutive law to hold for the elastic strain contribution $\varepsilon_{kl}^{(el)}$ to the total strain ε_{kl}, so that

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{(pl)} - \varepsilon_{kl}^{(\theta)}), \qquad i, j, k, l = 1, 2, 3, \tag{2.1}$$

where σ_{ij} — stress tensor, C_{ijkl} — tensor of elastic moduli, $\varepsilon_{kl}^{(pl)}$ — plastic strain tensor, $\varepsilon_{kl}^{(\theta)}$ — thermal strain tensor.

We further assume that the thermal strain $\varepsilon_{ij}^{(\theta)}$ may depend in any complex way on temperature θ.

$$\varepsilon_{ij}^{(\theta)} = a_{ij}(\theta). \tag{2.2}$$

The time derivative of Eq. (2.2) gives

$$\dot{\varepsilon}_{ij}^{(\theta)} = \overset{*}{a}_{ij}(\theta)\,\dot{\theta}, \tag{2.3}$$

with the coefficients $\overset{*}{a}_{ij} = \dfrac{da_{ij}}{d\theta}$ assumed to be known.

In order to characterize the plastic strain we assume the yield condition in the form

$$f(\sigma_{ij}, \sigma_y) = F(\sigma_{ij}) - \sigma_y(\bar{\varepsilon}^{(pl)}, \theta) = \sqrt{\frac{3}{2}\sigma_{ij}^D \sigma_{ij}^D} - \sigma_y = \bar{\sigma} - \sigma_y = 0, \tag{2.4}$$

in which the tensile yield stress σ_y is a known function of its two arguments $\bar{\sigma}$ is the stress intensity, $\sigma_{ij}^D = \sigma_{ij} - \frac{1}{3}\sigma_{nn}\delta_{ij}$ is the stress deviator and $\bar{\varepsilon}^{(pl)}$ is the equivalent (or effective) plastic strain. The associated flow rule reads

$$\dot{\varepsilon}_{ij}^{(pl)} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3}{2}\frac{\sigma_{ij}^D}{\sigma_y} = \dot{\lambda}\sqrt{\frac{3}{2}}\,n_{ij} = \dot{\bar{\varepsilon}}^{(pl)}\sqrt{\frac{3}{2}}\,n_{ij}, \tag{2.5}$$

where:

$n_{ij} = \sqrt{\frac{3}{2}}\frac{\sigma_{ij}^D}{\sigma_y}$ — unit normal to the yield surface (2.4) in the stress space,

$\dot{\bar{\varepsilon}}^{(pl)} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{(pl)}\dot{\varepsilon}_{ij}^{(pl)}}$ — plastic strain rate intensity (or rate of effective plastic strain).

In Eq. (2.5) the relation $\dot{\lambda} = \dot{\bar{\varepsilon}}^{(pl)}$ has been used, which directly results from substituting Eq. (2.5) into the definition of $\dot{\bar{\varepsilon}}^{(pl)}$.

The consistency condition which assures that the stress point remains on the yield surface during plastic flow reads

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - \left.\frac{\partial \sigma_y}{\partial \bar{\varepsilon}^{(pl)}}\right|_{\theta = \mathrm{const}}\dot{\bar{\varepsilon}}^{(pl)} - \left.\frac{\partial \sigma_y}{\partial \theta}\right|_{\bar{\varepsilon}^{(pl)} = \mathrm{const}}\dot{\theta} = 0. \tag{2.6}$$

The isothermic hardening parameter $\xi = \xi(\bar{\varepsilon}^{(pl)}, \theta)$ is defined as

$$\xi = \left.\frac{\partial \bar{\sigma}}{\partial \bar{\varepsilon}^{(pl)}}\right|_{\theta = \mathrm{const}}. \tag{2.7}$$

By using the identity $\frac{\partial f}{\partial \sigma_{ij}} = \frac{3}{2}\frac{\sigma_{ij}^D}{\sigma_y}$, Eq. (2.6) may be transformed to yield

$$\dot{\bar{\sigma}} - \xi\dot{\bar{\varepsilon}}^{(pl)} - \frac{\partial \sigma_y}{\partial \theta}\dot{\theta} = 0, \tag{2.8}$$

or

$$\xi = \frac{d\bar{\sigma}}{d\bar{\varepsilon}^{(pl)}} - \frac{\partial \sigma_y}{\partial \theta}\frac{d\theta}{d\bar{\varepsilon}^{(pl)}}. \tag{2.9}$$

Defining the non-isothermic hardening modulus as

$$\overset{*}{\xi} = \frac{d\bar{\sigma}}{d\bar{\varepsilon}^{(pl)}}, \tag{2.10}$$

we arrive at

$$\overset{*}{\xi} = \xi + \frac{\partial \sigma_y}{\partial \theta}\frac{d\theta}{d\bar{\varepsilon}^{(pl)}}. \tag{2.11}$$

It is often convenient to use other hardening moduli defined as

$$h = \frac{2}{3}\xi, \quad \overset{*}{h} = \frac{2}{3}\overset{*}{\xi}, \tag{2.12}$$

and then the flow rule (2.5) becomes

(2.13) $$\dot{\varepsilon}_{ij}^{(pl)} = \frac{1}{\overset{*}{h}} (n_{kl}\, \dot{\sigma}_{kl})\, n_{ij}.$$

Noting that

(2.14) $$\overset{*}{h} = h + \frac{2}{3} \frac{\partial \sigma_y}{\partial \theta} \frac{d\theta}{d\bar{\varepsilon}^{(pl)}},$$

Eq. (2.13) may be transformed to the form

(2.15) $$\dot{\varepsilon}_{ij}^{(pl)} = \frac{1}{h} (n_{kl}\, \dot{\sigma}_{kl})\, n_{ij} - \sqrt{\frac{2}{3}}\, \frac{1}{h} \frac{\partial \sigma_y}{\partial \theta}\, \dot{\theta}\, n_{ij},$$

The first part on the RHS of Eq. (2.15) corresponds to the isothermic process ($\dot{\theta} = 0$) whereas the second part is the correction due to thermal effects.

The rate form of the constitutive equation (2.1) is

(2.16) $$\dot{\sigma}_{ij} = C_{ijkl} \left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^{(pl)} - \overset{*}{a}_{kl}(\theta)\, \dot{\theta} \right) + \overset{*}{\sigma}_{ij},$$

with

(2.17) $$\overset{*}{\sigma}_{ij} = \dot{C}_{ijkl} \left(\varepsilon_{kl} - \varepsilon_{kl}^{(pl)} - a_{kl}(\theta) \right).$$

The term given by Eq. (2.17) indicates that we do not exclude the possibility of accounting for temperature variations of elastic material constants. By substituting Eq. (2.15) into Eq. (2.17), we obtain

(2.18) $$\dot{\sigma}_{ij} = C_{ijkl} \left[\dot{\varepsilon}_{kl} - \frac{1}{h} (n_{mn}\, \dot{\sigma}_{mn})\, n_{kl} + \sqrt{\frac{2}{3}}\, \frac{1}{h} \frac{\partial \sigma_y}{\partial \theta}\, \dot{\theta} n_{kl} - \overset{*}{a}_{kl}\, \dot{\theta} \right] + \overset{*}{\sigma}_{ij}.$$

The standard derivation of the incremental constitutive law leads to the final relationship

(2.19) $$\dot{\sigma}_{ij} = \left[C_{ijkl} - \frac{C_{ijst}\, n_{st}\, C_{prkl}\, n_{pr}}{h + n_{mn}\, C_{mntu}\, n_{tu}} (\dot{\varepsilon}_{kl} - \overset{*}{a}_{kl}\, \dot{\theta}) + \overset{*}{\sigma}_{ij} + \overset{**}{\sigma}_{ij} \right],$$

where

(2.20) $$\overset{**}{\sigma}_{ij} = C_{ijkl}\, n_{kl}\, \frac{-n_{pr}\, C_{prst}\, n_{st} \sqrt{\frac{2}{3}}\, \frac{1}{h} \frac{\partial \sigma_y}{\partial \theta}\, \dot{\theta} - \overset{*}{\sigma}_{pr}\, n_{pr}}{h + n_{mn}\, C_{mntu}\, n_{tu}} + \sqrt{\frac{2}{3}}\, \frac{1}{h} \frac{\partial \sigma_y}{\partial \theta} \dot{\theta}.$$

Using the explicit form of the elastic constitutive tensor, Eqs. (2.17), (2,19) and (2.20) simplify to the form

(2.21) $$\dot{\sigma}_{ij} = \left[C_{ijkl} - \frac{2G n_{ij}\, 2G n_{kl}}{h + 2G} \right] (\dot{\varepsilon}_{kl} - \overset{*}{a}_{kl}\, \dot{\theta}) + \overset{*}{\sigma}_{ij} + \overset{**}{\sigma}_{ij},$$

(2.22) $$\overset{*}{\sigma}_{ij} = [\dot{\lambda} \delta_{ij}\, \delta_{kl} + \dot{G}\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk})]\, \left(\varepsilon_{kl} - \varepsilon_{kl}^{(pl)} - a_{kl}(\theta) \right),$$

(2.23)
$$\overset{**}{\sigma}_{ij} = 2Gn_{ij}\frac{\sqrt{\frac{2}{3}}\frac{\partial\sigma_y}{\partial\theta}\dot{\theta}-\overset{*}{\sigma}_{kl}n_{kl}}{h+2G}.$$

In order to generalize the above results to the case of a mixed isotropic--kinematic hardening, we consider the following yield condition:

(2.24)
$$f = F(\sigma_{ij},\alpha_{ij})-\sigma_y(\bar{\varepsilon}^{(pl)},\theta) = \sqrt{\frac{3}{2}\bar{\sigma}^D_{ij}\bar{\sigma}^D_{ij}}-\sigma_y = \bar{\sigma}-\sigma_y = 0,$$

where now the stress intensity $\bar{\sigma}$ is formed from the components of $\bar{\sigma}^D_{ij}$ which is the deviator of the tensor $\bar{\sigma}_{ij}=\sigma_{ij}-\alpha_{ij}$, α_{ij} representing the translation of the yield surface in the stress space (the so-called back stress). Introducing the concept of a mixed hardening by considering the stress rate as being composed of two parts corresponding to the isotropic and kinematic hardening, respectively,

(2.25)
$$\dot{\sigma}_{ij} = \dot{\sigma}^{(i)}_{ij}+\dot{\sigma}^{(k)}_{ij} = \beta\dot{\sigma}_{ij}+(1-\beta)\dot{\sigma}_{ij}, \quad \beta\in[0,1],$$

β being the material parameter which determines the proportion of each particular hardening type, we mploy the consistency condition $\dot{f}=0$ in the form of the two following equations:

(2.26)
$$\frac{\partial f}{\partial\sigma_{ij}}\dot{\sigma}^{(k)}_{ij}-\frac{\partial f}{\partial\alpha_{ij}}\dot{\alpha}_{ij}=0,$$
$$\frac{\partial f}{\partial\sigma_{ij}}\dot{\sigma}^{(i)}_{ij}-\frac{\partial\bar{\sigma}}{\partial\bar{\varepsilon}^{(pl)}}\dot{\bar{\varepsilon}}^{(pl)}-\frac{\partial\sigma_y}{\partial\theta}\dot{\theta}=0.$$

The standard derivations lead now to the evolution equations for the back stress

(2.27)
$$\dot{\alpha}_{ij} = (1-\beta)\overset{*}{h}\dot{\varepsilon}^{(pl)}_{ij},$$

and for the yield limit, cf. Eq. (2.10)

(2.28)
$$\dot{\sigma}_y = \beta\overset{*}{\xi}\dot{\bar{\varepsilon}}^{(pl)}.$$

The replacing in Eqs. (2.18)—(2.20) of the stress σ_{ij} by $\sigma_{ij}-\alpha_{ij}$, together with Eqs. (2.27) and (2.28) completes the theory of associated non-isothermal plastic flow with isotropic-kinematic hardening provided appropriate loading--unloading conditions are employed. These have the form:

elastic process:

$$\bar{\sigma}<\sigma_y,$$

or

(2.29)
$$\bar{\sigma}=\sigma_y \quad\text{and}\quad n_{ij}\dot{\sigma}_{ij}-\sqrt{\frac{2}{3}}\frac{\partial\sigma_y}{\partial\theta}\dot{\theta}\leqslant 0,$$

plastic process:

(2.30) $$\bar{\sigma} = \sigma_y \quad \text{and} \quad n_{ij}\dot{\sigma}_{ij} - \sqrt{\frac{2}{3}}\frac{\partial \sigma_y}{\partial \theta}\dot{\theta} > 0.$$

The expression $n_{ij}\dot{\sigma}_{ij} - \sqrt{\frac{2}{3}}\frac{\partial \sigma_y}{\partial \theta}\dot{\theta}$ may be conveniently redefined by employing the so-called trial rate of stress given by

(2.31) $$\dot{\sigma}_{ij}^{(tr)} = C_{ijkl}(\dot{\varepsilon}_{kl} - \overset{*}{a}_{kl}\dot{\theta}).$$

Using the relation

(2.32) $$C_{ijkl}(\dot{\varepsilon}_{kl} - \overset{*}{a}_{kl}\dot{\theta}) = \dot{\sigma}_{ij} + \frac{1}{h}C_{ijkl}n_{kl}n_{pr}\dot{\sigma}_{pr},$$

which, when multiplied by C_{ijkl}^{-1} expresses simply the additivity of strain rate contributions, we obtain

(2.33) $$n_{ij}\dot{\sigma}_{ij} = n_{ij}C_{ijkl}(\dot{\varepsilon}_{kl} - \overset{*}{a}_{kl}\dot{\theta})\frac{h}{h + n_{pr}C_{prst}n_{st}},$$

or

(2.34) $$n_{ij}\dot{\sigma}_{ij} = n_{ij}\dot{\sigma}^{(tr)}\frac{h}{h + n_{pr}C_{prst}n_{st}}.$$

In view of Eq. (2.34), the condition of local plastic "loading" takes the form

(2.35) $$\frac{h}{h + n_{pr}C_{prst}n_{st}}n_{ij}\dot{\sigma}_{ij}^{(tr)} - \sqrt{\frac{2}{3}}\frac{\partial \sigma_y}{\partial \theta}\dot{\theta} > 0 \quad \text{for} \quad \bar{\sigma} = \sigma_y.$$

The reason for taking the definition of elastic and plastic processes in the form (2.29), (2.30) is that the theory becomes directly applicable to softening and no attention needs to be paid to the sign of the actual hardening parameters. We also note that for $\overset{*}{h} < 0$ and $\beta > 0$ a stopping criterion needs to be introduced to ensure that the actual yield limit does not become negative.

3. Finite element equations and integration of thermo-elastic-plastic constitutive law

Without going into details of already standardized derivation, [3, 5], we shall now consider the fundamental equations describing the continuing equilibrium of a discretized system in the form

(3.1) $$\mathbf{K}_T\dot{\mathbf{r}} = \dot{\mathbf{R}}.$$

Equation (3.1) represents a system of N simultaneous ordinary differential equations of the first order with N unknown functions $\mathbf{r}=\mathbf{r}(t)$, N being the total number of degrees of freedom. The structure is assumed to be subjected to a slowly varying load $\mathbf{R}=\mathbf{R}(t)$.

The build-up of the tangent stiffness matrix $\mathbf{K}_T$ is a standard issue in the finite element methodology and refers directly to the rate form of the material constitutive law (2.19).

The system (3.1) has to be solved with appropriate initial conditions of the form

$$\mathbf{r}(0)=\hat{\mathbf{r}}_0. \tag{3.2}$$

Any kind of numerical integration scheme known in the vast literature can be employed to solve Eqs. (3.1) and (3.2).

In this study for instance, the full Newton–Raphson iterative scheme was adopted as described in [3, 5].

The only aspect of the procedure which requires additional comments in this paper is the integration of the constitutive law which is the central issue in the algorithm.

To construct the integration algorithm for the material model at hand, we may make use of Eqs. (2.21)—(2.23), (2.27)—(2.30), (2.35). The widely accepted and effective algorithm for integration of isothermal elastic-plastic constitutive laws is known as the radial return scheme, [5, 7]. It is based on the idea that the normal vector at the subsequent iteration is approximated in terms of the stress $\bar{\sigma}^{(tr)D}$ which is defined as

$$\bar{\sigma}_{ij}^{(tr)D}=\sigma_{ij}^{(tr)D}-\alpha_{ij}. \tag{3.3}$$

The generalization of the algorithm valid in the thermo-elastic-plastic case goes as follows, (parameter $s=1,2,\ldots$, numbers iterations at a given load level).

STEP 1. Obtain the s-th approximation to incremental strain $\Delta\varepsilon_{ij}^{(s)}$ by solving Eq. (3.1) for the s-th correction to Δr and using the explicit relation between generalized nodal displacements r and elemental strains ε_{ij}.

STEP 2. Calculate the $(s+1)$-th approximation to the trial stress increment

$$\Delta\sigma_{ij}^{(tr)(s+1)}=C_{ijkl}\,(\Delta\varepsilon_{kl}^{(s)}-\overset{*}{a}_{kl}\,\dot{\theta}).$$

STEP 3. Calculate the trial stresses

$$\sigma_{ij}^{(tr)(s+1)}=\sigma_{ij}^{(tr)(s)}+\Delta\sigma_{ij}^{(tr)(s+1)},$$

$$\bar{\sigma}_{ij}^{(tr)(s+1)}=\sigma_{ij}^{(tr)(s+1)}-\alpha_{ij}^{(s)},$$

STEP 4. Calculate the deviatoric part of $\bar{\sigma}^{(tr)(s+1)}$

$$\bar{\sigma}_{ij}^{(tr)D(s+1)} = \bar{\sigma}_{ij}^{(tr)(s+1)} - \frac{1}{3}\,\bar{\sigma}_{kk}^{(tr)(s+1)}\,\delta_{ij}.$$

STEP 5. Calculate the product of $\bar{\sigma}_{ij}^{(tr)D(s+1)}$ by itself

$$A = \bar{\sigma}_{ij}^{(tr)D(s+1)}\,\bar{\sigma}_{ij}^{(tr)D(s+1)}.$$

STEP 6. Check the yield condition:

$$\text{if } \frac{3}{2}A \leqslant \sigma_y^{(s)2} \quad \text{then} \quad \sigma_{ij}^{(s+1)} = \sigma^{(tr)(s+1)},$$

$$\sigma_y^{(s+1)} = \sigma_y^{(s)},$$

$$\alpha^{(s+1)} = \alpha^{(s)} \quad \text{(elastic process)}$$

go to step 1

if $\frac{3}{2}A > \sigma_y^{(s)2}$ then go to step 7 (plastic process).

STEP 7. Calculate the approximation to the unit normal

$$n_{ij} = \sqrt{\frac{3}{2}}\,\frac{\sigma_{ij}^{(tr)D(s+1)}}{\sigma_y^{(s)}}.$$

STEP 8. Calculate the effective plastic strain increment:

$$\Delta\bar{\varepsilon}^{(pl)} = \frac{1}{3G+\xi}\left(\bar{\sigma}_{ij}^{(tr)D(s+1)} - \sigma_y^{(s)}\right).$$

STEP 9. Update

$$\sigma_{ij}^{(s+1)} = \sigma_{ij}^{(tr)(s+1)} - 2G\Delta\bar{\varepsilon}^{(pl)}\,n_{ij},$$

$$\sigma_y^{(a+1)} = \sigma_y^{(s)} + \beta\xi\Delta\bar{\varepsilon}^{(pl)},$$

$$\alpha_{ij}^{(s+1)} = \alpha_{ij}^{(s)} + \frac{2}{3}(1-\beta)\,\xi\Delta\bar{\varepsilon}^{(pl)}\,n_{ij},$$

$$\varepsilon_{ij}^{(pl)(s+1)} = \varepsilon_{ij}^{(pl)(s)} + \Delta\bar{\varepsilon}^{(pl)}\,n_{ij},$$

$$\bar{\varepsilon}^{(pl)(s+1)} = \bar{\varepsilon}^{(pl)(s)} + \Delta\bar{\varepsilon}^{(pl)}.$$

4. NUMERICAL ILLUSTRATIONS

As the first example we consider an axisymmetric cylinder of an infinite length loaded by a uniform internal pressure and a ring of concentrated forces Q, subjected to a temperature difference $\Delta\theta$ between the inside and outside cylinder surfaces (Fig. 1 and 2). The linear distribution of temperature across the cylinder thickness is assumed. The material is assumed to be elastic-ideally plastic with $E = 208000$ [MPa], $\nu = 0.3$, $\sigma_y = 235$ [MPa], and

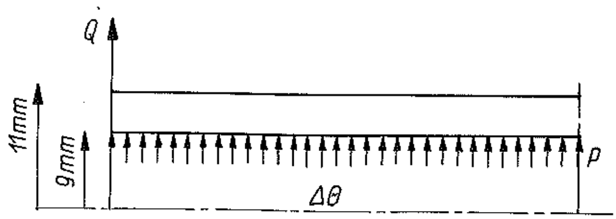

FIG. 1. Infinitely long cylinder subjected to concentrated force P, pressure p and temperature $\Delta\theta$.

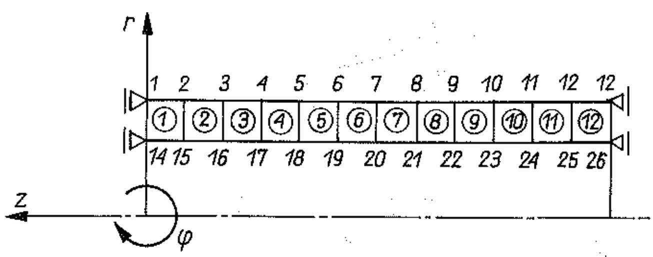

FIG. 2. Discretization mesh.

(4.1)
$$\varepsilon_{ij}^{(\theta)} = \alpha \Delta\theta \delta_{ij},$$

with the thermal expansion coefficient $\alpha = 0.00002$.

The yield limit σ_y is first assumed to be insensitive to temperature. The loading program consists of applying the forces Q in one step without exceeding the yield condition and a gradual proportional increase in p and $\Delta\theta$ (Fig. 3). Radial displacements at selected nodes are shown in Fig. 4 for load steps 6, 8 and 12.

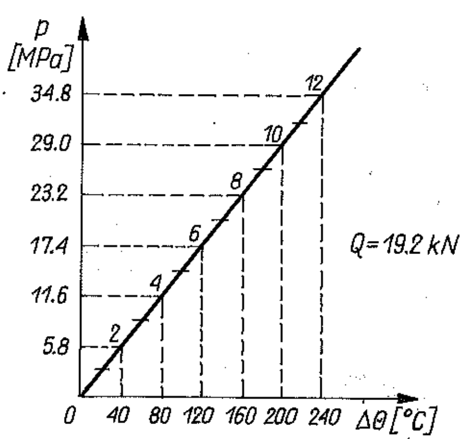

FIG. 3. History of loading.

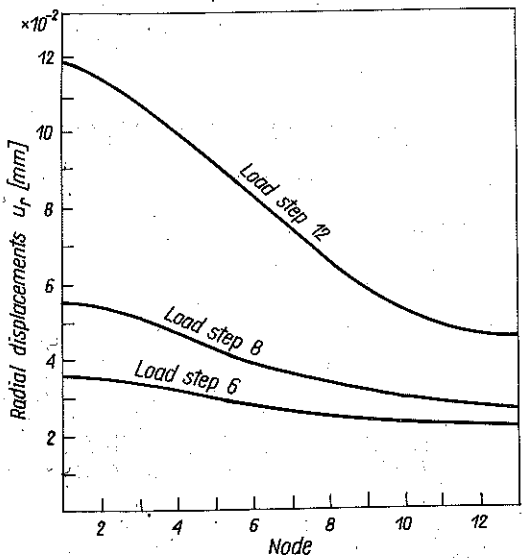

FIG. 4. Radial displacements at the boundary.

Figure 5 shows the circumferential strains: total, plastic and thermal, and accumulated effective plastic strain during the deformation process at element no. 1, integration point D.

Assuming the variation of the yield limit stress with temperature according to an approximate formula

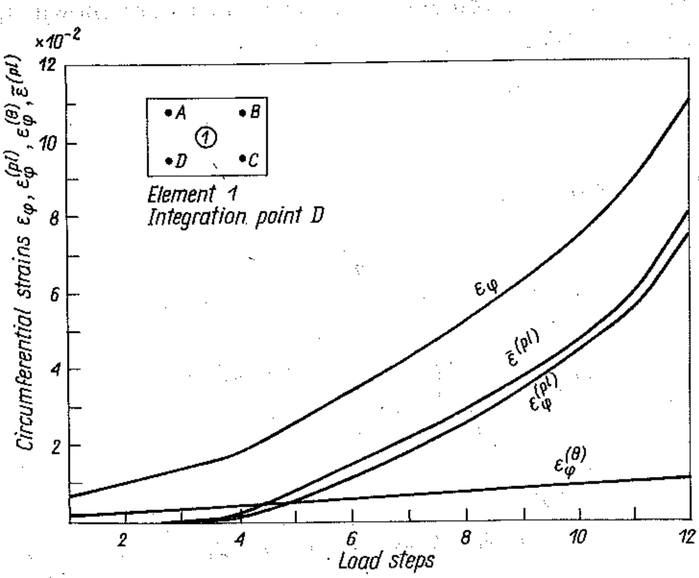

FIG. 5. Circumferential strains at elem. 1 for temperature-independent yield limit.

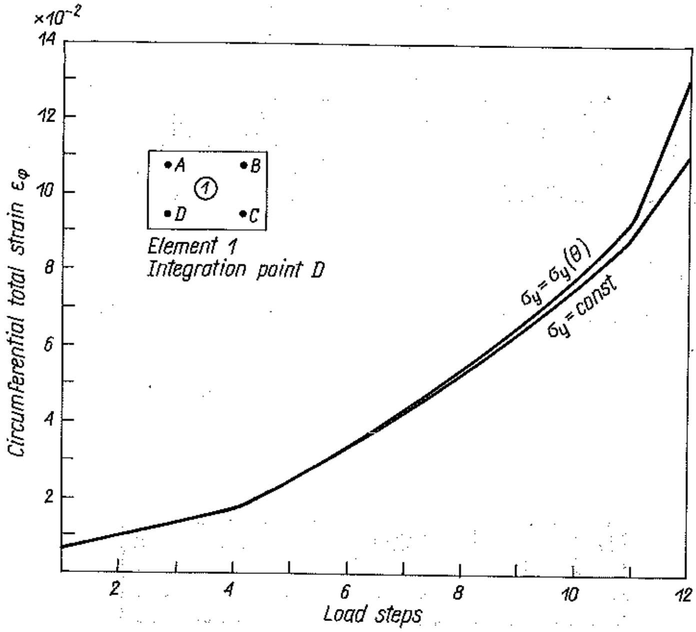

FIG. 6. Circumferential total strains at elem. 1 for temperature-dependend and independent yield limit.

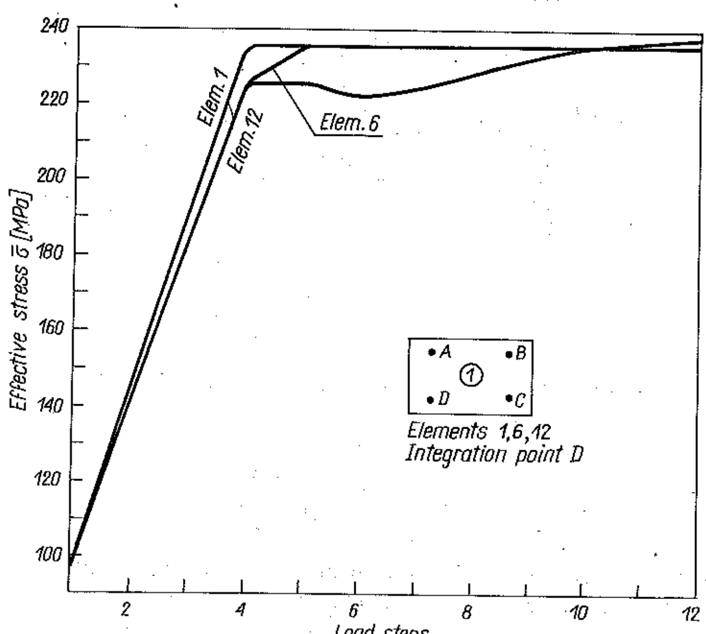

FIG. 7. Effective stresses at subsequent steps.

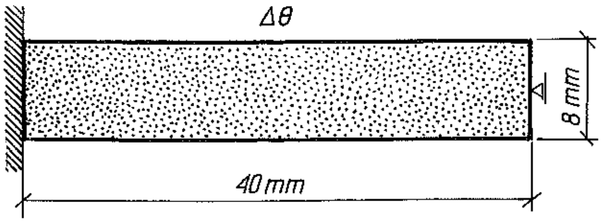

FIG. 8. Plane stress problem subjected to temperature variation.

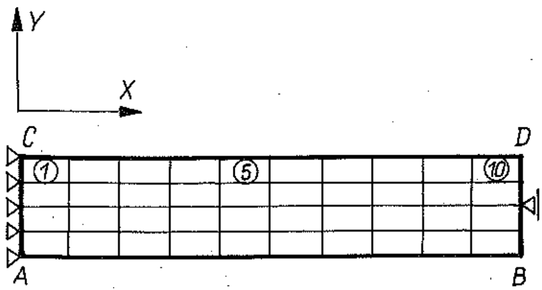

FIG. 9. Discretization mesh.

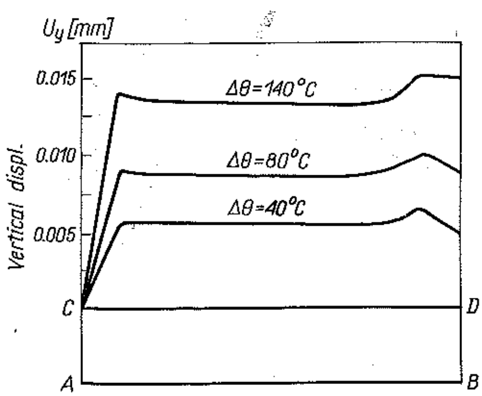

FIG. 10. Vertical displacement at the boundary after steps 2, 4 and 7.

(4.2) $$\sigma_y = \sigma_0 \left(1 - \left(\frac{\Delta\theta}{800}\right)^2\right),$$

we may observe changes in circumferential total strains at element 1 int. point D for $\sigma_y = \text{const}$ and $\sigma_y = \sigma_y(\theta)$, Fig. 6.

Changes in stress intensity $\bar{\sigma}$ at elements 1, 6 and 12 during the loading process are shown in Fig. 7. The yield limit is first achieved in the element 1 whereas the element 12 remains elastic up to the 10th load step.

As the second example, a plane stress problem shown in Figs. 8 and 9 is taken. The material is the same as in the previous case but two different moduli are considered: $E_T = 0$ and $E_T = 0.33\,E$. The loading is assumed to consist of a uniform temperature field with the nodal temperatures increasing from 20°C to 240°C with the step of 20°C ($\Delta\theta$ = load step number * 20°C).

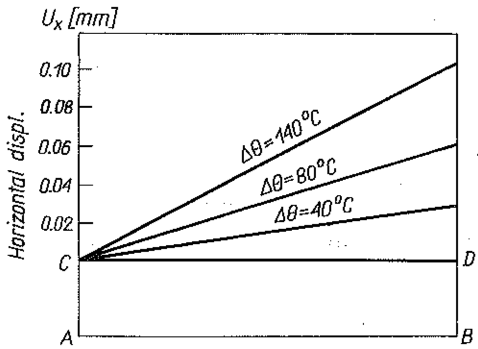

FIG. 11. Horizontal displacement at the boundary after steps 2, 4 and 7.

The vertical and horizontal displacements at the boundary C-D for the load 2, 4 and 7 are shown in Figs. 10 and 11, respectively.

Figures 12 and 13 illustrate changes in stress intensity for two plate elements (integration point A) located at the upper boundary. For the "no hardening" case (Fig. 12) the element 5 does not go plastic, whereas

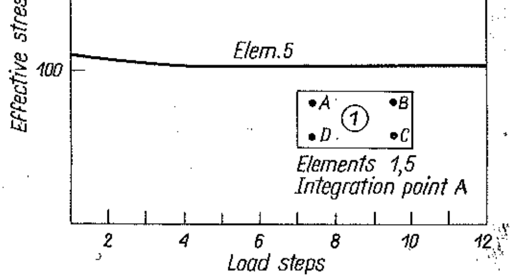

FIG. 12. Effective stresses for no hardening material.

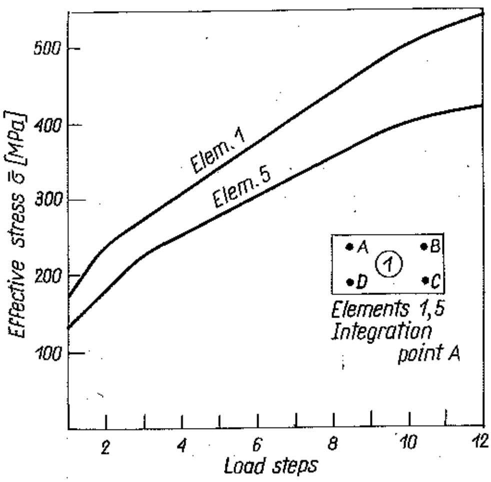

FIG. 13. Effective stresses for hardening material.

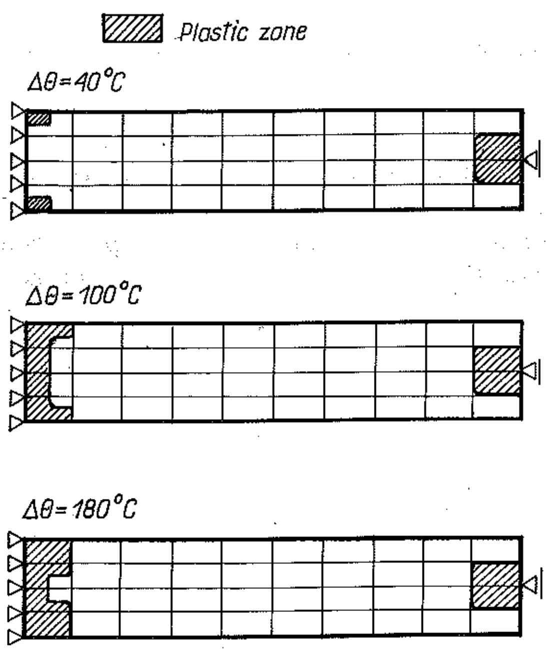

FIG. 14. Plastic zones development for no hardening material.

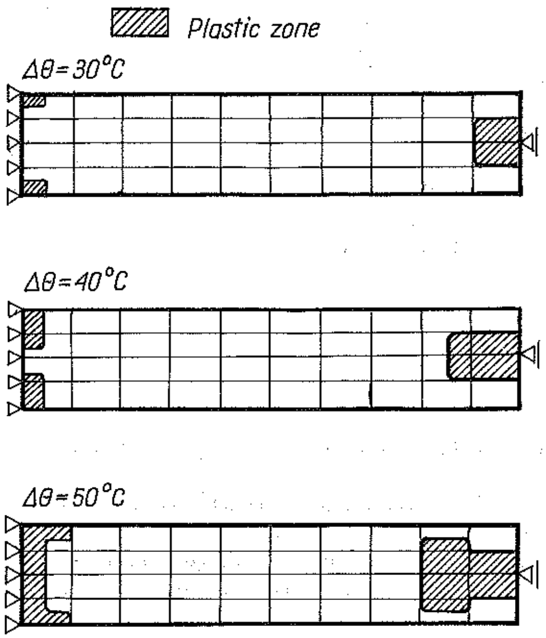

FIG. 15. Plastic zones development for hardening material.

the hardening material implies completely different stress distributions, (Fig. 13), with plasticity in the element 5 as well. The development of plastic zones is shown in Figs. 14 (no hardening) and 15 (hardening).

5. CONCLUSIONS

The paper describes an effective numerical algorithm for the analysis of a variety of complex thermo-elasto-plastic problems. The computer program based upon the algorithm has been shown to yield effective solutions to selected benchmark problems. Some simplifying assumptions have been introduced for the sake of presentation compactness — accounting for more general material characteristics is computationally straightforward. The work is being continued towards the analysis of viscoplastic materials under dynamic loadings.

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Streszczenie

NUMERYCZNA ANALIZA ZAGADNIEŃ TERMO-SPRĘŻYSTO-PLASTYCZNYCH

W pracy zaprezentowano teoretyczne i numeryczne podstawy programu termomechanicznej analizy konstrukcji TEMAS. Przedstawiono efektywną metodę analizy konstrukcji przy wykorzystaniu elementów skończonych z uwzględnieniem niesprężystego zachowania się materiału pod obciążeniem termicznym i mechanicznym. Zamieszczono również przykłady obliczeń.

Резюме

ЧИСЛЕННЫЙ АНАЛИЗ ТЕРМОУПРУГОПЛАСТИЧЕСКИХ ЗАДАЧ

В работе представлены теоретические и численные основы программы термомеханического анализа конструкций TEMAC. Представлен эффективный метод анализа конструкций, при использовании конечных элементов с учетом неупругого поведения материала под термическим и механическим нагружениюми. Помещены тоже примеры расчетов.

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