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THE EFFECT OF THE POISSON RATIO ON THE YIELD FUNCTION IN PLANE STRAIN CONDITION

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Initial and subsequent yield surfaces are specified in the plane strain condition in terms of three stress components acting within the deformation plane and the state variable representing the lateral stress. The effect of the Poisson ratio on the yield condition is investigated. The rate constitutive relations are derived and applied to several simple cases.

1. Introduction

It is well known that the yield function expressed in terms of stress tensor components does not depend on the elastic properties of the material. However, in a particular case of plane strain state, the stress component normal to the plane of deformation depends on the Poisson ratio. As a result, this ratio will appear in the expression for the yield condition when expressed in terms of stress components acting within the deformation plane. During the elastic-plastic deformation, an evolution of the yield surface occurs in the space of these stress components. In the present work the initial yield surface will be specified and its evolution will be determined by regarding the stress component normal to the plane of deformation as an additional state variable. It is often assumed that the yield condition, such as Coulomb's or Mises', used to solve plane elasto-plastic problems is the same as that for a rigid-plastic body. This assumption, however, is not legitimate in general and, when selecting specific material parameters, it may lead to considerable errors. A general analysis will be illustrated by several examples. A similar problem has already been treated by Matos [2] and Griffith [3].

2. Initial and subsequent yield surfaces in plane strain state

2.1. Initial yield surfaces

Consider an isotropic, elastic-perfectly plastic body deforming within the plane x_1, x_3, and the x_2-axis of the orthogonal Cartesian system x_1, x_2,

x_3 is normal to this plane. The principal strain component ε_2 vanishes, so for the elastic state there is

$$E\varepsilon_2^e = \sigma_2^e - \nu(\sigma_1^e + \sigma_3^e) = 0, \tag{2.1}$$

where σ_1^e, σ_2^e, σ_3^e are the principal elastic stresses and E and ν are the elastic parameters. We assume that the x_1 and x_3 axes coincide with the principal stresses σ_1 and σ_3. It follows from Eq. (2.1) that σ_2 need not be an intermediate stress (let, for instance, $\sigma_1 = 50$ kPa, $\sigma_3 = 40$ kPa, $\nu = 1/3$, then one has $\sigma_2 = 30$ kPa, thus σ_2 is the minor stress). For some yield conditions, (such as the Coulomb or Tresca condition), the major and minor principal stresses specify the filure plane. If the stress σ_2 is a major or minor principal stress, then the failure plane will not coincide with the deformation plane, Fig. 1. There is an exception for an incompressible

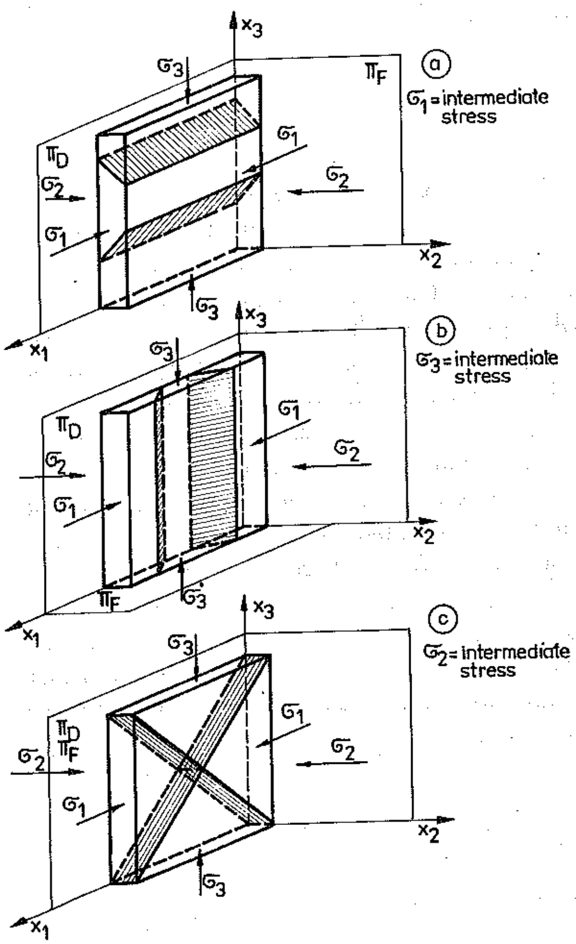

FIG. 1. Deformation and failure planes (Π_D, Π_F) for Tresca material.

material for which $\nu = 1/2$ and the stress $\sigma_2^e = (\sigma_1 + \sigma_3)/2$ is always intermediate. A formal proof of this statement also for an elastic-plastic regime was provided by ZIEGLER [1]. Consider first the Tresca, Coulomb and Drucker–Prager yield conditions. For simplicity, assume that

$$\sigma_1 \geqslant \sigma_3.$$

The Tresca yield condition has a general form

$$F_T = \sigma_{max} - \sigma_{min} - 2c = 0, \tag{2.2}$$

where c denotes the cohesive strength. In the plane strain state, particular cases can be distinguished:

$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ or $\sigma_1 \geqslant \nu\sigma_3/(1-\nu)$, $\sigma_1 \geqslant (1-\nu)\,\sigma_3/\nu$:

$$F_{Ta} = \sigma_1 - \sigma_3 - 2c = 0, \tag{2.2'}$$

$\sigma_1 > \sigma_3 > \sigma_2$ or $\sigma_3 > \nu\sigma_1/(1-\nu)$:

$$F_{Tb} = \sigma_1 - \nu\,(\sigma_1 + \sigma_3) - 2c = 0, \tag{2.2''}$$

$\sigma_2 > \sigma_1 > \sigma_3$ or $\sigma_3 > (1-\nu)\,\sigma_1/\nu$:

$$F_{Tc} = \nu\,(\sigma_1 + \sigma_3) - \sigma_3 - 2c = 0. \tag{2.2'''}$$

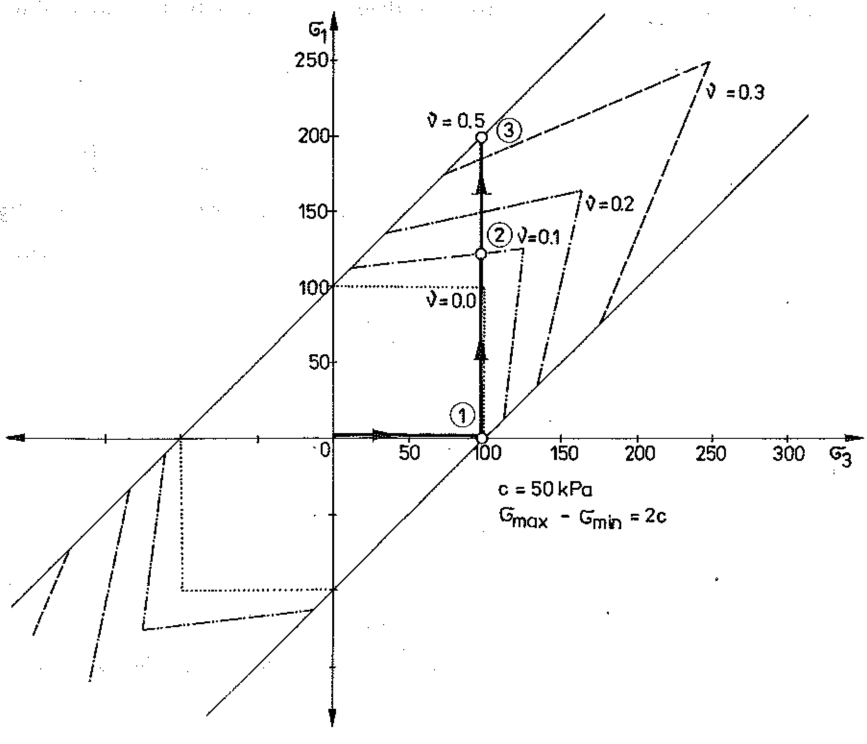

FIG. 2. Tresca yield surfaces and a stress path for further example.

The yield surface, when depicted in the plane σ_1, σ_3 for different values of $\nu < 1/2$, becomes a hexagon with its size increasing rapidly with ν. This hexagon corresponds to an intersection of the Tresca prism in the three-dimensional space σ_1, σ_2, σ_3 by the plane $\sigma_2 = \nu(\sigma_1+\sigma_3)$ with subsequent projection onto the plane $\sigma_2 = 0$. When $\nu = 0$, $\sigma_2 = 0$ and the yield surface (2.2) corresponds to the plane stress case (Fig. 2).

Consider now the Coulomb yield condition

$$F_C = (\sigma_{\max}-\sigma_{\min})+(\sigma_{\max}+\sigma_{\min}) \sin \Phi - 2c \cos \Phi = 0, \tag{2.3}$$

where Φ denotes the angle of internal friction and c is the cohesion. Similarly to (2.2′), we have for the respective three cases

$$\text{(2.3′)} \quad \begin{aligned} &\text{a)} \quad F_{Ca} = (\sigma_1-\sigma_3)+(\sigma_3+\sigma_1) \sin \Phi - 2c \cos \phi = 0, \\ &\text{b)} \quad F_{Cb} = [\sigma_1 - \nu(\sigma_1+\sigma_3)] + [\sigma_1+\nu(\sigma_1+\sigma_3)] \sin \Phi - 2c \cos \Phi = 0, \\ &\text{c)} \quad F_{Cc} = [\nu(\sigma_1+\sigma_3)-\sigma_3] + [\nu(\sigma_1+\sigma_3)+\sigma_3] \sin \Phi - 2c \cos \Phi = 0. \end{aligned}$$

These yield conditions are presented in the plane σ_1, σ_3 in Fig. 3. For the cohesionless material ($c = 0$), the plane $\sigma_2 = \nu(\sigma_1+\sigma_3)$ passes through the vertex of the Coulomb prism and the yield condition is specified by two lines in the (σ_1, σ_3) plane. It may happen, however, that the plane $\sigma_2 = \nu(\sigma_1+\sigma_3)$ does not intersect the Coulomb prism, which means that an

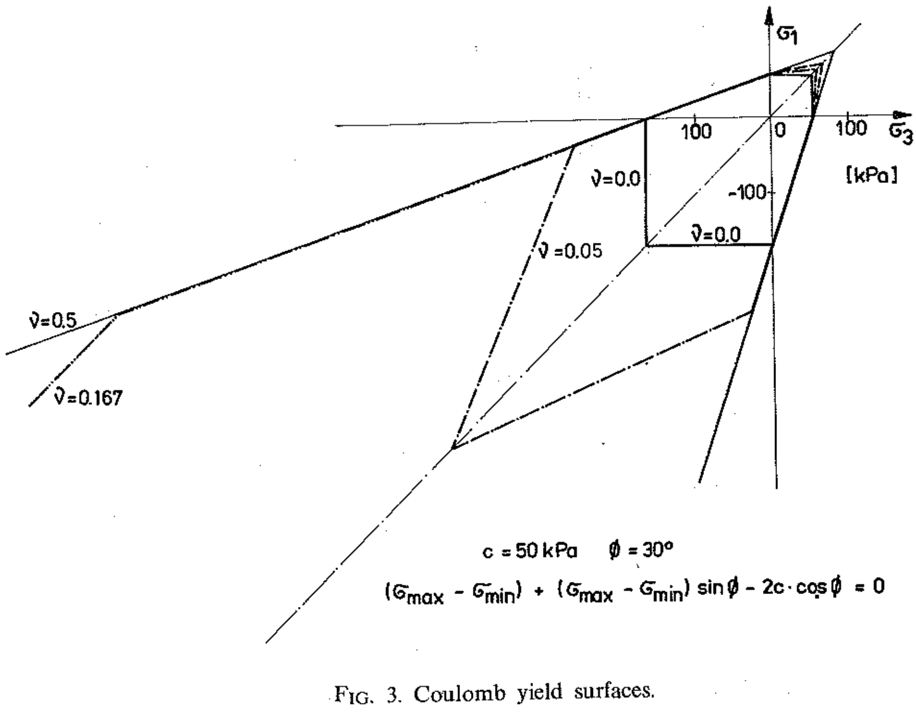

FIG. 3. Coulomb yield surfaces.

elastic state of such a material is impossible in the condition of plane strain. This situation occurs when

$$(2.4)\qquad 2\nu+(1+2\nu)\sin\Phi<1.$$

Let us now discuss the Drucker–Prager yield condition which, unlike the Coulomb or Tresca conditons, is also sensitive to the intermediate stress. In terms of stress invariants, this yield condition is expressed as follows:

$$(2.5)\qquad F_{Dp}=I_2'^{1/2}+\alpha I_1/3-k=0,$$

where

$$(2.6)\qquad \begin{aligned} I_1 &= \sigma_1+\sigma_2+\sigma_3,\\ I_2' &= \frac{1}{6}\,[(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+(\sigma_1-\sigma_2)^2],\\ \alpha &= 3\ \mathrm{tg}\,\phi/(9+12\ \mathrm{tg}^2\phi)^{1/2},\\ k &= 3c/(9+12\ \mathrm{tg}^2\phi)^{1/2}. \end{aligned}$$

In the space of principal stresses Eq. (2.5) is represented by a conical surface, Fig. 4. Substituting $\sigma_2=\nu(\sigma_1+\sigma_3)$ into Eq. (2.5), the initial yield condition is expressed as follows:

$$(2.7)\qquad F_{DPa}=[4\beta p^2-(p^2-q^2)]^{1/2}-\alpha(1+\nu)\,2p/3-k=0,$$

where

$$(2.8)\qquad \beta=(\nu^2-\nu+1)/3,\quad p=-(\sigma_1+\sigma_3)/2,\quad q=(\sigma_1-\sigma_3)/2.$$

Eq. (2.7) can be transformed to a quadratic form

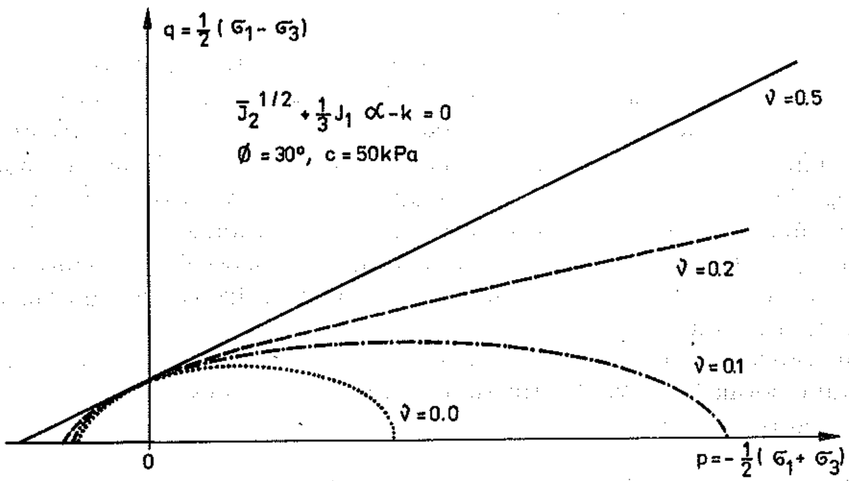

FIG. 4. Drucker–Prager yield surfaces.

$$C_{11}\,p^2+C_{22}\,q^2+2C_{12}\,pq+2C_{31}\,p+2C_{32}\,q+C_{33}=0, \tag{2.9}$$

and, depending on the sign of its determinant $D=C_{11}\,C_{22}-C_{12}^2$, can be represented by an ellipse ($D>0$), parabola ($D=0$), or hyperbola ($D<0$). As D can be expressed in terms of the friction angle and the Poisson ratio, these parameters affect the type of yield surface. Fig. 5. In the special case of a cohesionless material, $k=c=0$ and the yield surface in the σ_1, σ_3 plane is represented by two straight lines passing through the origin of the coordinate system. For more details on the initial yield surface of the Drucker–Prager cone see [3].

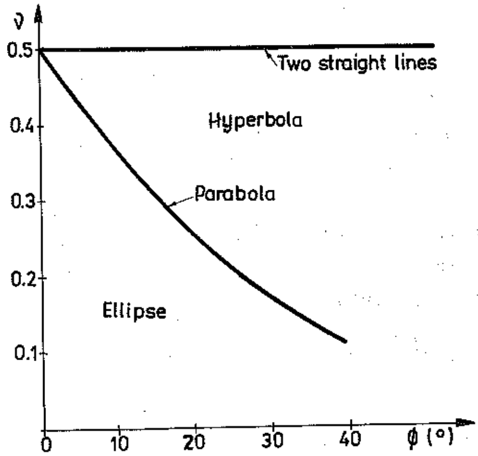

FIG. 5. Type of yield surface for ϕ and ν values.

2.2. *Subsequent yield surface*

When plastic flow occurs, the initial yield surface specified in terms of σ_1, σ_3 evolves, since the value of σ_2 varies during the deformation process and eventually reaches its limit value $\sigma_2=(\sigma_1+\sigma_3)/2$ corresponding to a limit state of unconfined plastic flow (or to a rigid plastic model). Noting that the limit surface is formally obtained by setting $\nu=0.5$, the initial and the limit yield surfaces can be denoted, respectively, by $F_\nu(\sigma_1,\sigma_3)=0$ and $F_{0.5}(\sigma_1,\sigma_3)=0$. During the elastic-plastic deformation, σ_2 need not be an intermediate stress though it takes the intermediate value in the limit state.

Introduce a plane coordinate system Oxy preserving the index 2 for the direction normal to the deformation plane. Let us present the stress and strain tensors as vectors:

$$\boldsymbol{\sigma}=[\sigma_x,\sigma_y,\tau_{xy},\sigma_2]^T, \quad \boldsymbol{\varepsilon}=[\varepsilon_x,\varepsilon_y,\gamma_{xy},\varepsilon_2]^T. \tag{2.10}$$

The yield condition can be represented in terms of the stress components

σ_x, σ_y, τ_{xy} and the stress σ_2 can be regarded as "state variable" similarly to the case of a hardening solid, thus

(2.11) $$F(\boldsymbol{\sigma}^*, \sigma_2) = 0, \quad \boldsymbol{\sigma}^*_{[3\times 1]} = [\sigma_x, \sigma_y, \tau_{xy}]^T.$$

The evolution rule for σ_2 is generated by requiring $\dot{\varepsilon}_2 = \dot{\varepsilon}_2^e + \dot{\varepsilon}_2^p = 0$, thus for the associated flow rule we have

(2.12) $$\dot{\varepsilon}_2 = \dot{\lambda}\delta F/\delta\sigma_2 + [\dot{\sigma}_2 - \nu(\dot{\sigma}_x + \dot{\sigma}_y)]/E = 0,$$

and

(2.13) $$\dot{\sigma}_2 = -E\dot{\lambda}(\delta F/\delta\sigma_2) + \nu(\dot{\sigma}_x + \dot{\sigma}_y).$$

The scalar multiplier $\dot{\lambda}$ is obtained from the consistency condition $\dot{F} = 0$,

(2.14) $$(\delta F/\delta\boldsymbol{\sigma}^*)\,\dot{\boldsymbol{\sigma}}^* + (\delta F/\delta\sigma_2)\,\dot{\sigma}_2 = 0.$$

Substituting Eq. (2.13) into Eq. (2.14), one obtains

(2.15) $$\dot{\lambda} = \frac{1}{E}\left[(\delta F/\delta\boldsymbol{\sigma}^*)\,\dot{\boldsymbol{\sigma}}^* + (\delta F/\delta\sigma_2)\,\nu(\dot{\sigma}_x + \dot{\sigma}_y)\right](\delta F/\delta\sigma_2)^{-2}.$$

It is seen that σ_2 varies both in elastic and plastic states. In order to generate a new state variable which evolves only because of plastic deformations, let us introduce the concept of "plastic stress" p_2 specified by the relation

(2.16) $$p_2 = \sigma_2 - \sigma_2^e = \sigma_2 - \nu(\sigma_x + \sigma_y),$$

and with the evolution rule associated only with the plastic strain rate [conf. (2.13)]:

(2.17) $$\dot{p}_2 = -E\dot{\lambda}(\delta F/\delta\sigma_2).$$

The "elastic" stress component $\sigma_2^e = \nu(\sigma_x + \sigma_y)$ can now be added to the plane stress vector $\boldsymbol{\sigma}^*$, thus generating a transformed stress vector $\boldsymbol{\sigma}^e_{[4\times 1]} = [\sigma_x, \sigma_2^e, \sigma_y, \tau_{xy}]^T$. The yield condition can now be expressed as follows:

(2.18) $$F(\boldsymbol{\sigma}^e, p_2) = 0,$$

and since

(2.19) $$\delta F/\delta\sigma_2 = \delta F/\delta p_2 = \delta F/\delta\sigma_2^e,$$

instead of Eq. (15) we have

(2.20) $$\dot{\lambda} = \frac{1}{E}\left[(\delta F/\delta\boldsymbol{\sigma}^e)\,\dot{\boldsymbol{\sigma}}^e\right](\delta F/\delta\sigma_2)^{-2}.$$

Before discussing the rate constitutive relations, let us define the vector $\mathbf{n}_{[4\times 1]}$ normal to the yield surface, with

(2.21) $$n_1 = \delta F/\delta\sigma_x, \quad n_2 = \delta F/\delta\sigma_2, \quad n_3 = \delta F/\delta\sigma_y, \quad n_4 = \delta F/\delta\tau_{xy}.$$

The flow rule now takes the form

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\,(\delta F/\delta\sigma) = \frac{1}{E\,(n_2)^2}\,(\mathbf{n}\times\mathbf{n})\,\dot{\boldsymbol{\sigma}}^e = \mathbf{C}^p_{[4\times4]}\,\dot{\boldsymbol{\sigma}}^e. \tag{2.22}$$

The elastic-plastic relation can now be expressed as follows:

$$\dot{\boldsymbol{\varepsilon}}_{[4\times1]} = \mathbf{C}^e_{[4\times4]}\,\dot{\boldsymbol{\sigma}}_{[4\times1]} + \mathbf{C}^p_{[4\times4]}\,\dot{\boldsymbol{\sigma}}^e_{[4\times1]}, \tag{2.23}$$

and it can be reduced to plane components:

$$\dot{\boldsymbol{\varepsilon}}_{[3\times1]} = \mathbf{C}^{ep}_{[3\times3]}\,\dot{\boldsymbol{\sigma}}^*_{[3\times1]}, \tag{2.24}$$

with two separate equations

$$\dot{\varepsilon}_2 = 0 \quad \text{and} \quad \dot{\sigma}_2 = \dot{\sigma}^e_2 + \dot{p}_2. \tag{2.25}$$

In the matrix form we obtain

$$\begin{bmatrix}\dot{\varepsilon}_x\\ \dot{\varepsilon}_y\\ \dot{\gamma}_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1+(N_1)^2+2\nu N_1 & N_1\,N_3+\nu\,(N_1+N_3-1) & N_1\,N_4+\nu N_4\\ \cdots & 1+(N_3)^2+2\nu N_3 & N_3\,N_4+\nu N_3\\ \text{symmetric} & \cdots & 2\,(1+\nu)+(N_4)^2\end{bmatrix}\begin{bmatrix}\dot{\sigma}_x\\ \dot{\sigma}_y\\ \dot{\tau}_{xy}\end{bmatrix}, \tag{2.26}$$

and

$$\begin{aligned}\dot{\sigma}_2 &= \nu\,(\dot{\sigma}_x+\dot{\sigma}_y) - [N_1\,\dot{\sigma}_x + \nu\,(\dot{\sigma}_x+\dot{\sigma}_y) + N_3\,\dot{\sigma}_y + N_4\,\dot{\tau}_{xy}],\\ \dot{p}_2 &= -[N_1\,\dot{\sigma}x + \nu\,(\dot{\sigma}_x+\dot{\sigma}_y) + N_3\,\dot{\sigma}_y + N_4\,\dot{\tau}_{xy}],\end{aligned} \tag{2.27}$$

where $N_i = n_i/n_2$.

The inverse relations between stress and strain rates can be obtained in a general form:

$$\dot{\boldsymbol{\sigma}}_{[4\times1]} = \mathbf{D}^e_{[4\times4]}\,\dot{\boldsymbol{\varepsilon}}_{[4\times1]} - \mathbf{D}^p_{[4\times4]}\,\dot{\boldsymbol{\varepsilon}}_{[4\times1]}, \tag{2.28}$$

where

$$\mathbf{D}^p_{[4\times4]} = \mathbf{D}^e\,\mathbf{n}\otimes\mathbf{n}^T\,\mathbf{D}^e/(K+\mathbf{n}\mathbf{D}^e\,\mathbf{n}^T), \tag{2.29}$$

and K denotes the hardening modulus (for the perfectly plastic material $K=0$). These relations can be expressed in terms of plane stress and strain rate components, namely

$$\begin{bmatrix}\dot{\sigma}_x\\ \dot{\sigma}_y\\ \dot{\tau}_{xy}\end{bmatrix} = \begin{bmatrix} A-d_1\,d_1/M & \dfrac{A\nu}{(1-\nu)}-d_1\,d_2/M & -d_1\,d_4/M\\ & A-d_3\,d_3/M & -d_1\,d_3/M\\ \text{symmetric} & & \dfrac{E}{2\,(1+\nu)}-d_3\,d_3/M\end{bmatrix}\begin{bmatrix}\dot{\varepsilon}_x\\ \dot{\varepsilon}_y\\ \dot{\gamma}_{xy}\end{bmatrix}, \tag{2.30}$$

where

$$(2.31)\quad \begin{aligned} d_1 &= A\,[n_1+\nu\,(n_2+n_3)/(1-\nu)],\\ d_2 &= A\,[n_2+\nu\,(n_1+n_3)/(1-\nu)],\\ d_3 &= A\,[n_3+\nu\,(n_1+n_2)/(1-\nu)],\\ d_4 &= A\,[n_4+(1-2\nu)/(2-2\nu)],\\ A &= \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)},\\ M &= A\left[n_1\,n_1+n_2\,n_2+n_3\,n_3+n_4\,n_4\,\frac{1-2\nu}{2\,(1-\nu)}+\right.\\ &\qquad \left.+\frac{2\nu}{1-\nu}\,(n_1\,n_2+n_2\,n_3+n_1\,n_3)\right], \end{aligned}$$

or briefly

$$(2.32)\qquad \mathbf{d} = \mathbf{D}^e\,\mathbf{n}, \qquad M = \mathbf{n}^T\,\mathbf{D}^e\,\mathbf{n}.$$

The compliance matrix $\mathbf{C}_{[3\times 3]} = \mathbf{D}^{-1}_{[3\times 3]}$ can be formed by the inversion of the matrix $\mathbf{D}$ if $\mathbf{D}$ is not singular, i.e. the material is not in the critical state when $n_2 = 0$ and $\mathbf{C}$ does not exist. Consider now a particular case, namely the uniaxial strain $\varepsilon_1 \neq 0$, $\varepsilon_2 = \varepsilon_3 = 0$. For a smooth yield surface the external normal vector $\mathbf{n}_{[3\times 1]} = \delta F/\delta \boldsymbol{\sigma}$ satisfies the condition $n_2 = n_3$ for an isotropic material. The constitutive relation has the form

$$(2.33)\qquad \dot{\sigma}_1 = \dot{\varepsilon}_1\,\frac{2E}{1-\nu}\,n_2\,n_2/[n_1\,n_1+(4\nu n_1\,n_2+2n_2\,n_2)/(1-\nu)]\,,$$

with the suplementary condition

$$(2.34)\qquad \dot{\sigma}_2 = \dot{\sigma}_3 = \dot{\sigma}_1\,\{\nu/(1-\nu)-[n_1+2n_2\,\nu/(1-\nu)]/(2n_2)\}.$$

For a singular flow rule corresponding to an edge of intersection of two regular surfaces $F_1 = 0$ and $F_2 = 0$ one has to specify two vectors $\mathbf{n}$ and $\mathbf{m}$ normal, respectively, to $F_1 = 0$ and $F_2 = 0$ at the edge. Consider a particular case when $m_1 = n_1$, $m_2 = n_3$ and $m_3 = n_2$ (the Tresca and Coulomb conditions satisfy these relations). Using Koiters' flow rule, one obtains

$$(2.35)\qquad \dot{\sigma}_1 = \dot{\varepsilon}_1\,\frac{E}{1-\nu}\,n_2\,n_2/[2n_1\,n_1+(4\nu n_1\,n_2+n_2\,n_2)/(1-\nu)],$$

and

$$(2.36)\qquad \dot{\sigma}_2 = \dot{\sigma}_3 = \dot{\sigma}_1\,\{\nu/(1-\nu)-[n_1+n_2\,\nu/(1-\nu)]/n_2\}.$$

3. Examples

Two examples will be presented, the first one concerning a stress-strain relation for Tresca material and the second dealing with a stress prediction in the Coulomb type material (influence of the overconsolidation ratio on the K_0 value in soils).

i) Application to Tresca material

In the first part let us take into account an isotropic elastic, perfectly plastic material in a homogeneous plane strain and assume the Tresca yield condition with the associated flow rule. For further simplicity, impose the condition

$$\sigma_1 > \sigma_3 > \sigma_2,$$

and consider the principal stresses and strains only. Under such circumstances the yield condition takes the form

$$F_T = \sigma_1^e - \sigma_2^e - p_2 - 2c = 0.$$

The normal vector $\mathbf{n}$ is constant and equals

$$\mathbf{n} = [1, -1, 0, 0]^T.$$

From Eq. (2.27) the rate of evolution parameter is obtained:

$$\dot{p}_2 = +1\dot{\sigma}_1^e - \nu(\dot{\sigma}_1 + \dot{\sigma}_3) = \dot{\sigma}_1 - \dot{\sigma}_2^e,$$

and the following equations can finally be written for the elastic regime:

$$\begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_3 \end{bmatrix} = \frac{1}{E} \left[\begin{array}{c|c} 1-\nu^2 & -\nu-\nu^2 \\ \hline -\nu-\nu^2 & 1-\nu^2 \end{array} \right] \begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_3 \end{bmatrix},$$

for the transitional regime, according to the form (2.26),

$$\begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_3 \end{bmatrix} = \frac{1}{E} \left[\begin{array}{c|c} 2(1-\nu) & -2\nu \\ \hline -2\nu & 1 \end{array} \right] \begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_3 \end{bmatrix}.$$

Accounting for the fact that the elastic-plastic matrix $\mathbf{C}^{ep}$ is constant (in general it can depend on $\boldsymbol{\sigma}$ and p_2), we shall use our incremental equations for increments of stress. The relation between stress and strain corresponding to the stress path of Fig. 2 is shown in Fig. 6. The stress path 1—2—3 was chosen and the following material parameters were assumed: $c = 50$ kPa, $\nu = 0.1$, $E = 100\,000$ kPa.

ii) Overconsolidated material

Our second example concerns horizontal stresses produced by a large prestress in soil. Let us assume uniaxial strain and an elastic-perfectly

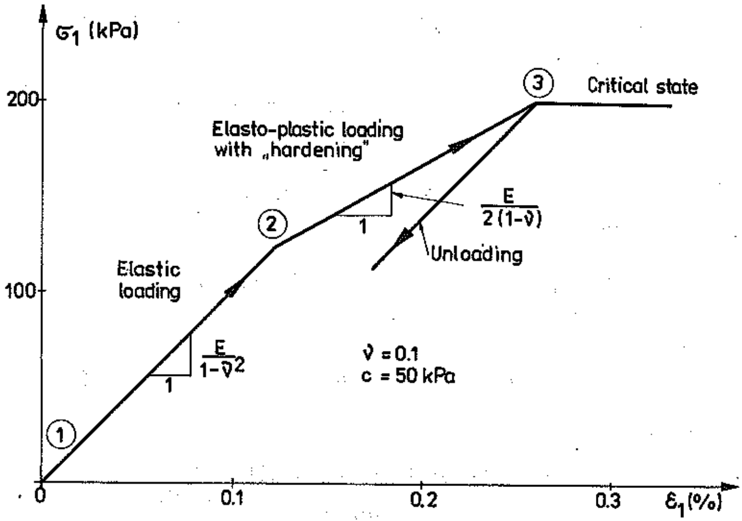

FIG. 6. Stress-strain relation for the stress path of example 1.

plastic material with the Coulomb yield criterion and the associated flow rule. For simplicity, assume the parameters c, Φ, ν, γ to be constant in the process of loading and unloading. Here γ denotes a specific weight of soil per unit volume. To be consistent with the geotechnical convention of signs, we assume the confining stress as positive. Let us divide the process into three stages:

initial state of stress (geostatic),
uniformly distributed surcharge,
unloading.

The initial state. The vertical stress σ_v increases with depth linearly $\sigma_v = \gamma z$ but the horizontal stress σ_h, in general, must be divided into an elastic part $\sigma_h^e = \sigma_v \nu/(1-\nu)$ (that results from Hooke's law) and a "plastic" part equal to the evolution parameter $p_h = \sigma_h - \sigma_h^e$. The latter occurs only when the initial yield condition is violated by the elastic stress, that is when

$$(\sigma_v - \sigma_h^e) - (\sigma_v + \sigma_h^e) \sin \Phi - 2c \cos \Phi > 0,$$

or, substituting the expressions for σ_v and σ_h,

$$\gamma z \, [(1 - \sin \Phi - 2\nu)/(1-\nu)] > 2c \cos \Phi.$$

If the term in brackets is negative, the material can be regarded as purely elastic in the uniaxial strain. This will be the case in nearly incompressible soils or soils with a high friction angle. The following discussion is conducted for "weak" soils which do not satisfy the condition $\sin \Phi + 2\nu > 1$. For cohesive materials, one can calculate a depth z_c beneath which some plastic deformations occur, thus

$$z_c = \frac{2c(1-\nu)\cos\Phi}{(1-\sin\Phi-2\nu)\gamma}.$$

As the total stress must satisfy the yield condition, the consistency condition is derived in the form

$$(\delta F/\delta\sigma_v)\,\dot{\sigma}_v+(\delta F/\delta\sigma_h^e)\,\dot{\sigma}_h^e+(\delta F/\delta p_h)\,\dot{p}_h = 0,$$

where

$$F = (\sigma_v-\sigma_h^e-p_h)-(\sigma_v+\sigma_h^e+p_h)\sin\Phi-2c\cos\Phi = 0.$$

Hence we obtain

$$\dot{p}_h = \frac{(1-\sin\Phi)-(1+\sin\Phi)\,\nu/(1-\nu)}{1+\sin\Phi}\,\sigma_v.$$

The equation of evolution for p_h is independent of the stress level so it can be integrated. The value of the evolution parameter is easy to calculate as a part of horizontal pressure:

$$p_h = \frac{(1-\sin\Phi)-(1+\sin\Phi)\,\nu/(1-\nu)}{1+\sin\Phi}(\sigma_v-\gamma z_c),$$

provided that $\sigma_v-\gamma z_c > 0$. A distribution of geostatic stresses in soil is shown in Fig. 7.

Loading. At any depth z the soil experiences the same vertical stress increment $\Delta\sigma_v = q$ and the same elastic horizontal stress increment $\Delta\sigma_h^e = q\nu/(1-\nu)$. The evolution parameter can be obtained from

$$p_h = \frac{(1-\sin\Phi)-(1+\sin\Phi)\,\nu/(1-\nu)}{1+\sin\Phi}(q+\gamma z-\gamma z_c) \quad (>0)$$

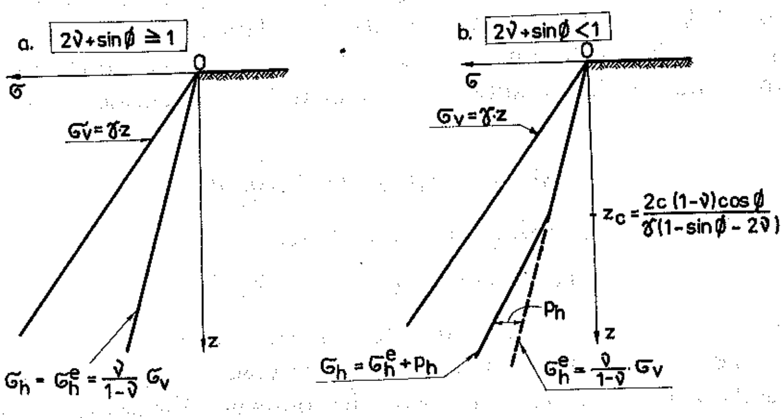

FIG. 7. Geostatic stresses for: a — strong and b — weak soils.

and the total stress is expressed as follows:

$$\sigma_v = \gamma z + q,$$
$$\sigma_h = \sigma_h^e + p_h = (\gamma z + q)\, \nu/(1-\nu) + p_h.$$

Let us notice that at a certain depth there can occur both purely elastic and elastic-plastic increments (Fig. 8, section 0—1 and 1—2, respectively).

Unoloading. The process will continue according to the relation

$$\sigma_h = \sigma_v\, \nu/(1-\nu),$$

until the horizontal stress is much bigger than the vertical one and the "passive state" of the Coulomb condition is reached, i.e.

$$(\sigma_h - \sigma_v) - (\sigma_v + \sigma_h) \sin \Phi - 2c \cos \Phi > 0.$$

Therefore, once again we have to distinguish two stages, purely elastic

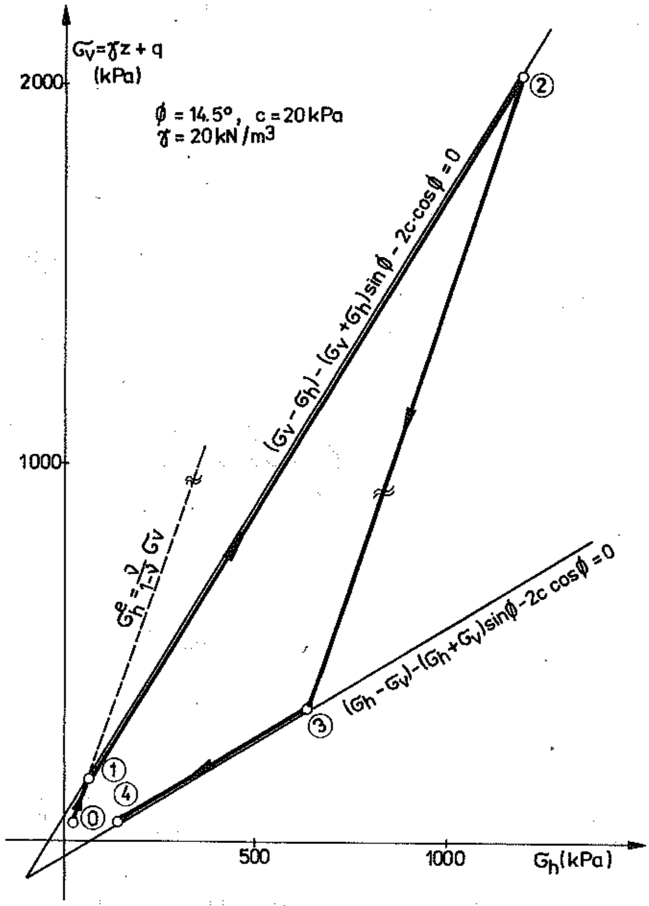

FIG. 8. The process of loading and unloading of soil in the uniaxial state of strain.

and elastic-plastic (Fig. 8, sections 2—3 and 3—4, respectively). After simple calculations we obtain the value of vertical elastic unloading:

$$\Delta r^e = 2\sigma_v^{(2)} \frac{[(1-2\nu)/(1-\nu)-p_h^{(2)}]\,(1-\nu)}{1-2\nu+\sin\Phi},$$

where the upper index (2) denotes the stress state at the point 2 of Fig. 8. During the subsequent process of unloading, our evolution parameter p_h decreases according to the consistency condition associated with the Coulomb condition (conf. Fig. 8). The following equation is obtained:

$$\dot{p}_h = \frac{1-\sin\Phi-2\nu}{(1-\nu)(1-\sin\Phi)}\,\dot{\sigma}_v,$$

and as σ_v is negative, the evolution parameter p_h decreases. Defining the overconsolidation ratio as $OCR = \sigma_v^{(2)}/\sigma_v^{(0)}$ and the coefficient of earth pressure at rest as $K_0 = \sigma_h^{(4)}/\sigma_v^{(4)}$, one can easily interrelate them as shown in Fig. 9.

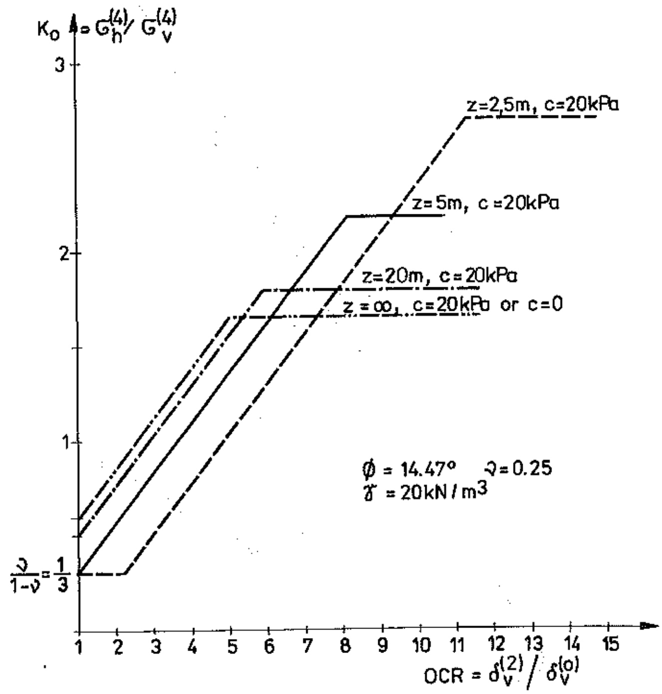

FIG. 9. Earth pressure coefficient at rest K_0 as a function of the overconsolidation ratio.

References

1. H. ZIEGLER, *Zum plastischen Potential in der Bodenmechanik*, Z. angew. Math. Phys., (ZAMP), **20**, pp. 659—675, 1969.
2. A. C. MATOS, *The numerical influence of the Poisson ratio on the safety factor*, Proc. Fourth International Conference on Numerical Methods in Geomechanics, Edmonton, 1982.
3. D. V. GRIFFITH, *Some theoretical observations on conical failure criteria in principal stress space*, Int. J. Solids Structures, **22**, 5, pp. 553—565, 1986.

Streszczenie

WPŁYW STAŁEJ POISSONA NA UPLASTYCZNIENIE W PŁASKIM STANIE ODKSZTAŁCENIA

Wyznaczono początkową i kolejne powierzchnie plastyczności w warunkach płaskiego stanu naprężenia, wyrażając je za pomocą trzech składowych naprężeń działających w płaszczyźnie odkształcenia oraz zmiennej stanu przedstawiającej naprężenia w kierunku poprzecznym. Zbadano wpływ stałej Poissona na warunek plastyczności. Wyprowadzono równania konstytutywne i zastosowano je do kilku prostych przypadków.

Резюме

ВЛИЯНИЕ ПОСТОЯННОЙ ПУАССОНА НА ПЕРЕХОД В ПЛАСТИЧЕСКОЕ СОСТОЯНИЕ В ПЛОСКОМ ДЕФОРМАЦИОННОМ СОСТАЯНИИ

Определены первая и последовательные пластические поверхности в условиях плоского напряженного состояния, выражая их при помощи трех составляющих напряжений, действующих в плоскости деформации и переменной состояния, представляющей напряжения в поперечном направлении. Исследовано влияние постоянной Пуассона на условие пластичности. Введены определяющие уравнения и они применены к нескольким простым случаям.

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