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REVIEW ON SPECTRAL DECOMPOSITION OF HOOKE'S TENSOR FOR ALL SYMMETRY GROUPS OF LINEAR ELASTIC MATERIAL

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The spectral decomposition of elasticity tensor for all symmetry groups of a linearly elastic material is reviewed. In the paper it has been derived in non-standard way by imposing the symmetry conditions upon the orthogonal projectors instead of the stiffness tensor itself. The numbers of independent Kelvin moduli and stiffness distributors are provided. The corresponding representation of the elasticity tensor is specified.

Key words: linear elasticity, anisotropy, symmetry group, spectral decomposition.

1. Introduction

This work is devoted to the review on the spectral decomposition of the elasticity tensor (Hooke's tensor). Possibility of application of the spectral theorem within this context was first noticed by Lord Kelvin (W. Thompson) in 1856 [10]. Then the idea was forgotten and rediscovered by Rychlewski in 1983 [22] and independently by Mehrabadi and Cowin in 1990 [7]. The consequences of the theorem have been thoroughly explored by the above researchers and their co-workers, leading to many inspiring results, i.e. the spectral form of elasticity tensor was derived for all elastic symmetry classes [2, 6, 25], the role of pure shears was analyzed [3], the extremum of elastic energy was found for the selected sets of stress states [19], the properties of biological materials were identified [7]. After that the idea has found numerous applications, especially when dealing with anisotropic materials. Now, this invariant decomposition of the elasticity tensors is widely known, though, still some aspects of it remain not fully understood. The main goal of this paper is to clarify the issue of invariance of the decomposition, mainly the crucial notion of orthogonal projector introduced by Rychlewski [22] with respect to the notion of an eigen-state which is preferably used in the papers by Cowin and co-workers, e.g. [6]. Furthermore the spectral theorem is applied for elastic material of each symmetry class. The novelty of the present work is the derivation of the form of the stiffness tensor for the subsequent elastic symmetry groups by imposing the symmetry conditions upon the orthogonal projectors instead of the stiffness tensor itself. We think that the present paper will be useful for all who would like to apply the spectral theorem in their fields of research.

Linear elastic material (classical elastic body) is considered for which the small strain tensor $\boldsymbol{\varepsilon}$ depends on the stress tensor $\boldsymbol{\sigma}$ according to Hooke's law:

$$\boldsymbol{\varepsilon} = \mathbf{M} \cdot \boldsymbol{\sigma} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{L} \cdot \boldsymbol{\varepsilon}, \quad \mathbf{M} \circ \mathbf{L} = \mathbb{I}^S, \tag{1.1}$$

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl} \quad \text{or} \quad \sigma_{ij} = L_{ijkl}\varepsilon_{kl}, \quad M_{ijmn}L_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $\mathbf{M}$ is a compliance tensor and $\mathbf{L}$ is a stiffness tensor. The above law is valid for the stress states restricted by the limit Mises condition

$$\sigma \cdot \mathbf{H} \cdot \boldsymbol{\sigma} \leq 1, \qquad \sigma_{ij}H_{ijkl}\sigma_{kl} \leq 1,$$

where $\mathbf{H}$ is the limit tensor. Theory of elasticity of anisotropic bodies is presented in detail e.g. in [9, 16]. In this paper we deal only with the properties of the stiffness tensor resulting from its spectral decomposition, without referring to any boundary value problem.

Hooke's tensors $\mathbf{M}$, $\mathbf{L}$ are linear operators which project the space $\mathcal{S}$ of symmetric II-nd order tensors into itself. Hooke's tensors are defined as positive-definite IV-th order Euclidean tensors with the following internal symmetries, namely

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij} \qquad (\mathbf{A} \rightarrow \mathbf{L}, \mathbf{M}).$$

Because of the above internal symmetries, in any Cartesian basis Hooke's tensor is, in general, specified by 21 independent components M_{ijkl} and L_{mnrs}. These components change when the basis in physical space is transformed, therefore they are not material constants. The compliance and stiffness tensors are also used in quadratic forms specifying the energy functional

$$2\Phi = \boldsymbol{\sigma} \cdot \mathbf{M} \cdot \boldsymbol{\sigma} = \boldsymbol{\varepsilon} \cdot \mathbf{L} \cdot \boldsymbol{\varepsilon}\,. \tag{1.2}$$

Unfortunately, the complete set of the invariants for Hooke's tensor, which uniquely describe such tensor with an accuracy to the rigid rotation of the considered body, is not known. Because there are 21 independent components of Hooke's tensor, while the orientation of a sample with respect to the laboratory is specified by 3 parameters (i.e. Euler angles), an irreducible functional basis of orthogonal invariants for $\mathbf{L}$ $(\mathbf{M}, \mathbf{H})$ consists of $21-3 = 18$ invariants. Conventional approach does not provide the form of such basis for the whole set of elastic continua. However, some results can be derived when the material enjoys some external symmetries.

Note that for the tensor of even order, the following eigen-problem is well-posed. Using the general theory of linear operators one finds that the conditions

$$\mathbf{L} \cdot \boldsymbol{\omega} = \lambda \boldsymbol{\omega}, \qquad \mathbf{M} \cdot \boldsymbol{\omega} = \frac{1}{\lambda} \boldsymbol{\omega} \tag{1.3}$$

specify eigenvalues and eigen-elements of these operators. Eigen-elements corresponding to different eigenvalues are always pairwise orthogonal. The condition (1.3) is also the necessary condition for the elastic energy (1.2) to reach an extremum value over the unit sphere (that is for $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$).

In general, the tensor $\mathbf{L}$ has no more than six real different eigenvalues λ_I, λ_{II}, ..., λ_{VI} to which one can relate six mutually orthogonal unit eigen-elements $\boldsymbol{\omega}_I$, $\boldsymbol{\omega}_{II}$, ..., $\boldsymbol{\omega}_{VI}$. These normalized eigen-elements are called *elastic eigen-states.* They are specified with accuracy to a sign and constitute an orthonormal basis in the space $\mathcal{S}$ of the II-nd order tensors

$$\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}, \qquad K, L = I, \ldots, VI. \tag{1.4}$$

Eigenvalues λ_I, λ_{II}, ..., λ_{VI} specify the material stiffness in response to the deformations $\boldsymbol{\varepsilon} = e\boldsymbol{\omega}_K$ of direction of $\boldsymbol{\omega}_K$, where $\boldsymbol{\omega}_K$ are the elastic eigen-states. λ_K are called *stiffness moduli* or Kelvin moduli [22, 25], and they are non-negative. This is the only constraint imposed on elastic constants by thermodynamics. For any deformation $\boldsymbol{\varepsilon} = e\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is an eigen-state, Hooke's law takes the form of the proportionality rule

$$\boldsymbol{\sigma} = \lambda \boldsymbol{\varepsilon}\,,$$

where λ is the Kelvin modulus corresponding to $\boldsymbol{\omega}$. The resulting form of the elastic energy for the elastic eigen-states has been specified already by Kelvin [10] as follows:

$$\boldsymbol{\varepsilon} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon} = \lambda_I e_I^2 + \lambda_{II} e_I^2 + \ldots + \lambda_{VI} e_{VI}^2, \qquad e_K = \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega}_K$$

and

$$\boldsymbol{\varepsilon} = e_I \boldsymbol{\omega}_I + e_{II} \boldsymbol{\omega}_{II} + \ldots + e_{VI} \boldsymbol{\omega}_{VI}\,.$$

Each sequence

$$(\lambda_I, \ldots, \lambda_{VI}; \boldsymbol{\omega}_I, \ldots, \boldsymbol{\omega}_{VI}), \tag{1.5}$$

consisting of six Kelvin moduli $\lambda_K \geq 0$ and six elastic eigen-states $\boldsymbol{\omega}_K$ specifies an elastic material which is theoretically admissible.

In order to derive 6 eigen-states $\boldsymbol{\omega}_K$ (symmetric second-order tensors) it is sufficient to specify 15 quantities. Conditions (1.4) of orthonormality of the eigen-states provide 21 additional conditions

$$\binom{6}{1} + \binom{6}{2} = 6 + 15 = 21,$$

which reduce the number of independent quantities from 36 (6×6) to 15 ($36-21 = 15$). Consequently, the variety of elastic continua is locally described in a continuous way by a set of $6+15 = 21$ parameters.

Out of the 15 parameters describing eigen-states one can separate three which are not invariants. They orient the stiffness tensor $\mathbf{L}$ with respect to a reference frame (a laboratory). These three parameters can be defined as three Euler angles ϕ_1, ϕ_2, ϕ_3. Remaining 12 parameters are dimensionless material constants – invariants and common invariants of eigen-states (eigen-tensors) $\aleph_k$ [15]. They are common for the stiffness tensor $\mathbf{L}$ and compliance tensor $\mathbf{M}$ and they are called *stiffness distributors* as far as they characterize the distribution of stiffness between the material fibers and the material planes. Stiffness distributors specify the orthonormal basis of eigen-states $\boldsymbol{\omega}_K$ with accuracy to the rotation in a physical space [25].

In conclusion, parameters describing some elastic continua can be subdivided into three groups

$$(6 + 12) + 3 = 21.$$

1. The first group consists of 6 **Kelvin moduli** $\lambda_I, \ldots, \lambda_{VI}$ which have a dimension of the stress tensor.
2. The second group consists of dimensionless 12 **stiffness distributors** $\aleph_1, \ldots, \aleph_{12}$,
3. The third group consists of three **Euler angles** ϕ_1, ϕ_2, ϕ_3.

Therefore, one has

$$\langle \lambda_I, \ldots, \lambda_{VI}; \quad \aleph_1, \ldots, \aleph_{12}; \quad \phi_1, \phi_2, \phi_3 \rangle. \tag{1.6}$$

Two elastic bodies are made of the same material if values of 18 invariants, that is $\lambda_I, \ldots, \lambda_{VI}$ and $\aleph_1, \ldots, \aleph_{12}$, are equal for both of them.

Knowing the Kelvin moduli λ_K and the corresponding elastic eigen-states $\boldsymbol{\omega}_K$, the tensors $\mathbf{L}$ and $\mathbf{M}$ can be represented in the form of their spectral decompositions [17, 22, 28]:

$$\mathbf{L} = \lambda_I \boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \ldots + \lambda_{VI} \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}, \tag{1.7}$$

$$\mathbf{M} = \frac{1}{\lambda_I} \boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \ldots + \frac{1}{\lambda_{VI}} \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}. \tag{1.8}$$

Note that the following relations result from the above equations:

$$\mathrm{Tr}\mathbf{L} = L_{ijij} = \lambda_I + \lambda_{II} + \ldots + \lambda_{VI},$$

$$\mathbf{L} \cdot \mathbf{L} = L_{ijkl} L_{ijkl} = \lambda_I^2 + \lambda_{II}^2 + \ldots + \lambda_{VI}^2,$$

in view of which $1/6\mathrm{Tr}\mathbf{L}$ is the average stiffness modulus, while $\sqrt{\mathbf{L} \cdot \mathbf{L}}$ is the total stiffness (the norm of $\mathbf{L}$). Moreover, as for any other basis in $\mathcal{S}$, the identity tensor $\mathbb{I}^S$ is

$$\mathbb{I}^S = \boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \ldots + \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}\,. \tag{1.9}$$

As a consequence of the spectral theorem, the space of symmetric second-order tensors $\mathcal{S}$ has been decomposed into the sum of six one-dimensional pairwise orthogonal subspaces $\mathcal{P}_K$ of eigen-states

$$\mathcal{S} = \mathcal{P}_I \oplus \mathcal{P}_{II} \oplus \ldots \oplus \mathcal{P}_{VI}.$$

Let us introduce the notion of *projector*. Projector is defined as a identity operator for the subspace $\mathcal{P}$ of second-order tensors, that is, it is the IV-th order tensor $\mathbf{P}$ which specifies the linear operation defined as follows:

$$\mathbf{P} \cdot \boldsymbol{\omega} = \begin{cases} \boldsymbol{\omega} & \text{if} \quad \boldsymbol{\omega} \in \mathcal{P}, \\ \mathbf{0} & \text{if} \quad \text{otherwise.} \end{cases}$$

Consider the identity operation for the subspace $\mathcal{P}_K$ of eigen-states and find the corresponding projector $\mathbf{P}_K$, called now the eigen-projector. Using (1.9) we find (no summation over repeated indices!)

$$\begin{aligned} \mathbf{P}_K = \mathbf{P}_K \circ \mathbb{I}^S = \mathbf{P}_K \circ (\boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \ldots + \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}) \\ = (\mathbf{P}_K \cdot \boldsymbol{\omega}_K) \otimes \boldsymbol{\omega}_K = \boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_K. \end{aligned} \tag{1.10}$$

Accordingly for any II-nd order tensor $\boldsymbol{\varpi} \in \mathcal{S}$ the following relation is true

$$\mathbf{P}_K \cdot \boldsymbol{\varpi} = \alpha \boldsymbol{\omega}_K \in \mathcal{P}_K.$$

Projectors $\mathbf{P}_K$ and $\mathbf{P}_L$ corresponding to two eigen-subspaces are orthogonal, that is

$$\mathbf{P}_K \circ \mathbf{P}_L = \begin{cases} \mathbb{O} & \text{if} \quad K \neq L, \\ \mathbf{P}_K & \text{if} \quad K = L, \end{cases}$$

and

$$\mathbf{P}_I + \ldots + \mathbf{P}_{VI} = \mathbb{I}^S.$$

The above conditions of orthogonality of projectors result from the orthogonality of corresponding eigen-subspaces. Decompositions (1.7), (1.8) and orthogonal projectors $\mathbf{P}_K$ (1.10) have the above diadic form if the corresponding Kelvin moduli are single, that is if $\lambda_K \neq \lambda_L$ for all $K \neq L$. Only in such a case the spectral decompositions (1.7), (1.8) are unique.

If the material enjoys some symmetry then the number of parameters describing this material decreases. The sequence of parameters (1.6) can be then presented as follows:

$$\langle \lambda_1, \ldots, \lambda_\rho;\ \aleph_1, \ldots, \aleph_t;\ \phi_1, \ldots, \phi_n \rangle \,, \tag{1.11}$$

where $\rho \leq 6$, $t \leq 12$ and $n \leq 3$. Kelvin moduli can be then multiple and the spectral theorem takes the form

$$\mathbf{L} = \lambda_1 \mathbf{P}_1 + \ldots + \lambda_\rho \mathbf{P}_\rho, \qquad \rho \leq 6 \tag{1.12}$$

and

$$\mathcal{S} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \ldots \oplus \mathcal{P}_\rho, \qquad \mathbb{I}^S = \mathbf{P}_1 + \ldots + \mathbf{P}_\rho \,.$$

The dimension of the subspace $\mathcal{P}_k$ is equal to the multiplicity of the corresponding Kelvin modulus λ_k. The decomposition (1.12) is unique. In order to show how the orthogonal eigen-projector looks like in the case of multiple Kelvin moduli, let us assume that $\lambda_V = \lambda_{VI}$ in (1.7). In such a case, the subspace $\mathcal{P}_{V,VI}$ is two-dimensional and one can define in this subspace the basis $\{\boldsymbol{\omega}_V, \boldsymbol{\omega}_{VI}\}$. Using (1.9) we find

$$\begin{aligned} \mathbf{P}_{V,VI} &= \mathbf{P}_{V,VI} \circ \mathbb{I}^S = \mathbf{P}_{V,VI} \circ (\boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \ldots + \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}) \\ &= (\mathbf{P}_{V,VI} \cdot \boldsymbol{\omega}_V) \otimes \boldsymbol{\omega}_V + (\mathbf{P}_{V,VI} \cdot \boldsymbol{\omega}_{VI}) \otimes \boldsymbol{\omega}_{VI} = \boldsymbol{\omega}_V \otimes \boldsymbol{\omega}_V + \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}. \end{aligned}$$

It can be easily verified that the form of eigen-projector does not depend on the basis of eigen-states selected in the subspace $\mathcal{P}_{V,VI}$.

If one denotes the dimensions of eigen-subspaces $\mathcal{P}_1, \ldots, \mathcal{P}_\rho$ by $q_1, \ldots, q_\rho$, correspondingly then according to [22], the expression

$$\langle q_1 + q_2 + \ldots + q_\rho \rangle, \qquad q_1 + q_2 + \ldots + q_\rho = 6 \tag{1.13}$$

is called the **I-st structural index** of material, while the expression

$$[\rho + t + n] \tag{1.14}$$

is the **II-nd structural index**. These expressions are material characteristics.

It should be noted that the symmetry of the tensor $\mathbf{L}$, which is equivalent to the symmetry of a linear elastic continuum, results from the properties of the IV-th order symmetric Euclidean tensors or, to be more specific, from the linearity of Hooke's law and the properties of 3-dimensional Euclidean space. Therefore, the classification of the linear elastic materials in view of their symmetry has, in general, nothing to do with the crystallography. Elastic anisotropy of crystals is classified in the same way as elastic anisotropy of other bodies without crystal structure. Consequently, some of the crystal structures have their counterparts within the elastic symmetry classes, while some of them have not [11]. An example of the latter case are crystals of hexagonal lattice symmetry. As far as they have a 6-fold axis of symmetry, in view of Hermann-German theorem [25], in order to account for all present symmetries, they must be described as elastically transversely isotropic.

2. Kelvin moduli $\lambda_I, \ldots, \lambda_{VI}$

The Kelvin moduli $\lambda_I, \ldots, \lambda_{VI}$ are obtained as roots of characteristic polynomial, which has the form

$$\det(\mathbf{L} - \lambda \mathbb{I}^S) = \lambda^6 + a_1(\mathbf{L})\lambda^5 + \ldots + a_5(\mathbf{L})\lambda + a_6(\mathbf{L}) = 0. \tag{2.1}$$

Determinant of a IV-th order tensor $\mathbf{A}$ is defined as follows:

$$\det \mathbf{A} \equiv \det(A_{KL}) = \det(\boldsymbol{\nu}_K \cdot \mathbf{A} \cdot \boldsymbol{\nu}_L), \tag{2.2}$$

where $\boldsymbol{\nu}_K$, $(K = I, \ldots, VI)$ is any orthonormal basis in $\mathcal{S}$, while A_{KL} is the 6×6 matrix of representation of the tensor $\mathbf{A}$ in this basis (see Appendix). The choice of a basis $\boldsymbol{\nu}_K$ has no influence on the value of the coefficients $a_i(\mathbf{L})$ in the Eq. (2.1); therefore, they are the invariants of $\mathbf{L}$.

For the considered $\lambda^\star$ the corresponding eigen-state $\boldsymbol{\omega}^\star$ is derived from the homogeneous system of 6 linear equations:

$$\mathbf{L} \cdot \boldsymbol{\omega}^\star = \lambda^\star \Longrightarrow (\mathbf{L} - \lambda \mathbb{I}^S) \cdot \boldsymbol{\omega}^\star = \mathbf{0} \tag{2.3}$$

with constraint $\boldsymbol{\omega}^\star \cdot \boldsymbol{\omega}^\star = \mathrm{tr}(\boldsymbol{\omega}^\star)^2 = 1$. If the basis $\boldsymbol{\nu}_K = \boldsymbol{\omega}_K$, that is it coincides with the basis of eigen-states, then the matrix $L_{KL} = \boldsymbol{\omega}_K \cdot \mathbf{L} \cdot \boldsymbol{\omega}_L$ is diagonal.

3. Orthogonal projectors $\mathbf{P}_1, \ldots, \mathbf{P}_\rho$

Knowing Kelvin's moduli λ_K, number ρ of which is different, one can introduce some rule which orders them $\lambda_1, \ldots, \lambda_\rho$. For example, one can number the moduli by increasing (decreasing) values. After unique numbering of moduli, the corresponding orthogonal projectors $\mathbf{P}_k$ can be derived using the following system of ρ tensorial equations of fourth-order [22]:

$$
\begin{aligned}
\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_\rho &= \mathbb{I}^S, \\
\lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \ldots + \lambda_\rho \mathbf{P}_\rho &= \mathbf{L}, \\
\vdots \quad \ddots \quad & \quad \vdots \\
\lambda_1^{\rho-1} \mathbf{P}_1 + \lambda_2^{\rho-1} \mathbf{P}_2 + \ldots + \lambda_\rho^{\rho-1} \mathbf{P}_\rho &= \mathbf{L}^{\rho-1},
\end{aligned}
\tag{3.1}
$$

where

$$\mathbf{L}^k = \underbrace{\mathbf{L} \circ \mathbf{L} \circ \ldots \circ \mathbf{L}}_{k \quad \text{times}}.$$

Consequently, one obtains

$$
\begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_\rho \end{bmatrix}
=
\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_\rho \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{\rho-1} & \lambda_2^{\rho-1} & \ldots & \lambda_\rho^{\rho-1} \end{bmatrix}^{-1}
\begin{bmatrix} \mathbb{I}^S \\ \mathbf{L} \\ \vdots \\ \mathbf{L}^{\rho-1} \end{bmatrix}.
$$

Inversion of the above matrix exists because its determinant (the Vandermonde determinant) is equal to

$$\Delta = \prod_{\rho \geq k \neq l \geq 1} (\lambda_k - \lambda_l)$$

and by definition $\lambda_k \neq \lambda_l$. One finds

$$\mathbf{P}_k = \frac{(\mathbf{L} - \lambda_1 \mathbb{I}^S) \circ \ldots \circ (\mathbf{L} - \lambda_{k-1} \mathbb{I}^S) \circ (\mathbf{L} - \lambda_{k+1} \mathbb{I}^S) \circ \ldots \circ (\mathbf{L} - \lambda_\rho \mathbb{I}^S)}{(\lambda_k - \lambda_1) \ldots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \ldots (\lambda_k - \lambda_\rho)}.$$

Distributors $\aleph_1, \ldots, \aleph_{12}$ are parameters which enable one to specify, in a unique way, the orthogonal projectors $\mathbf{P}_k$ in the selected basis. The form of these functions, which would enable one to specify the projectors for all material symmetries, has not been proposed yet. Some proposal for orthotropic symmetry has been derived in [15]. To this end the harmonic decomposition discussed in [8, 26, 27] was utilized.

Using the relation $(3.1)_1$ it can be shown that the following identity is true:

$$\mathbf{1} \cdot \mathbf{P}_1 \cdot \mathbf{1} + \mathbf{1} \cdot \mathbf{P}_2 \cdot \mathbf{1} + \ldots + \mathbf{1} \cdot \mathbf{P}_\rho \cdot \mathbf{1} = \mathbf{1} \cdot \mathbb{I}^S \cdot \mathbf{1} = 3.$$

The above identity provides the following relation between the traces of the eigen-states $\boldsymbol{\omega}_K$, if $\rho = 6$:

$$(\mathrm{tr}\boldsymbol{\omega}_I)^2 + (\mathrm{tr}\boldsymbol{\omega}_{II})^2 + \ldots + (\mathrm{tr}\boldsymbol{\omega}_{VI})^2 = 3. \tag{3.2}$$

4. Symmetries of an anisotropic linear elastic material

4.1. Notation and symmetry conditions

In what follows, the following notation is used:

- $\mathcal{Q}$ – orthogonal group in 3-dimensional Euclidean space E^3, the set of all orthogonal tensors,
- $\mathcal{Q}^+$ – the group of rotations in E^3, the set of all orthogonal tensors for which $\det \mathbf{Q} = 1$, where $\mathcal{Q}^+ \subset \mathcal{Q}$,
- $\mathbf{R}_{\mathbf{a}}^{\phi}$ – the orthogonal tensor describing the right-hand rotation around the axis of direction $\mathbf{a}$ about the angle ϕ. For the rotation presented in Fig. 1 one obtains the following representation of $\mathbf{R}_{\mathbf{a}}^{\phi}$ in the basis $\{\mathbf{e}_i\}$

$$\mathbf{R}_{\mathbf{a}}^{\phi} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

  while the corresponding orthogonal tensor in 6-dimensional space has the following representation in poly-basis $\{\mathbf{a}_K\}$ (see Appendix):

$$\mathbb{R}_{\mathbf{a}}^{\phi} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\phi^2 & \sin\phi^2 & -\sqrt{2}\sin\phi\cos\phi & 0 & 0 \\ 0 & \sin\phi^2 & \cos\phi^2 & \sqrt{2}\sin\phi\cos\phi & 0 & 0 \\ 0 & \sqrt{2}\sin\phi\cos\phi & -\sqrt{2}\sin\phi\cos\phi & \cos 2\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix}.$$

  Hooke's tensors are of even order, therefore one can restrict analysis only to the rotation tensors because symmetry resulting from the mirror reflection will be equivalent to the symmetry resulting from the rotation around the appropriate axis through the angle π. Note that the representation of the orthogonal tensor in six-dimensional space looses the information about the determinant of the corresponding 3×3 orthogonal matrix.
- $\mathbf{I}_{\mathbf{a}}$ – the orthogonal tensor which describes the mirror reflection with respect to the plane with the unit normal $\mathbf{a} = \mathbf{e}_1$. For the mirror reflection presented in Fig. 1 one obtains the following representation of $\mathbf{I}_{\mathbf{a}}$ in $\{\mathbf{e}_i\}$:

$$\mathbf{I}_{\mathbf{a}} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the corresponding representation in 6-dimensional space is the same as that for the rotation through the angle π around $\mathbf{a} = \mathbf{e}_1$:

$$\mathbb{I}_{\mathbf{a}} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

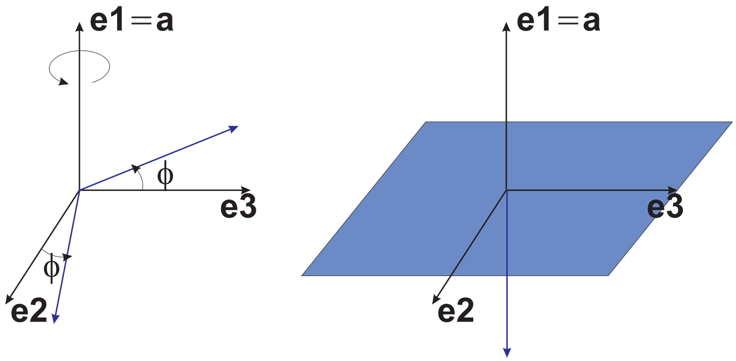

FIG. 1. Rotation and mirror reflection specified by the direction **a**.

Below, we explain the relation between the spectral decomposition of stiffness (compliance) tensor and the well-known classification of linear elastic bodies according to their material symmetry. As it was already discussed, if the material enjoys some symmetry properties then the number of Kelvin moduli and stiffness distributors decreases. The symmetry group $\mathcal{Q}_{\mathbf{L}}$ of a stiffness tensor $\mathbf{L}$ (a compliance tensor $\mathbf{M}$) is defined as follows:

$$\mathcal{Q}_{\mathbf{L}} = \mathcal{Q}_{\mathbf{M}} = \{\mathbf{Q} \in \mathcal{Q}; \mathbf{Q} \star \mathbf{L} = \mathbf{L}\}, \tag{4.1}$$

where $\mathbf{Q}$ is the orthogonal II-nd order tensor in 3-dimensional physical space. It should be recalled that one has for $\mathbf{Q}$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1}. \tag{4.2}$$

Symbol $\star$ denotes the rotation operation for the IV-th order tensor defined in the following way. Let $\{\mathbf{e}_i\}$ be the selected orthonormal basis in E^3, consequently

$$\mathbf{L} = L_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

and then

$$\mathbf{Q} \star \mathbf{L} = L_{ijkl}(\mathbf{Q}\mathbf{e}_i) \otimes (\mathbf{Q}\mathbf{e}_j) \otimes (\mathbf{Q}\mathbf{e}_k) \otimes (\mathbf{Q}\mathbf{e}_l)$$
$$= L_{mnpq} Q_{im} Q_{jn} Q_{kp} Q_{lq} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

where

$$\mathbf{Q} = Q_{ij} \mathbf{e}_i \otimes \mathbf{e}_j.$$

The orthogonal tensor **Q** belongs to the symmetry group of **L** if the following condition is true:

$$\mathbf{Q} \star \mathbf{L} = \mathbf{L} \Leftrightarrow L_{mnpq} Q_{im} Q_{jn} Q_{kp} Q_{lq} = L_{ijkl}. \tag{4.3}$$

Therefore, we have in general 21 scalar equations which impose some constraints on the components of **L** for the considered **Q**. The classification of the linearly elastic materials according to their symmetry includes the classical eight classes of elastic symmetry [4, 6]. The full anisotropy ($\mathcal{Q}_{\mathbf{L}} = \{\mathbf{1}, -\mathbf{1}\}$) and the full isotropy ($\mathcal{Q}_{\mathbf{L}} = \mathcal{Q}$) are two extreme cases. Symmetry groups for some classes of symmetry are contained within the symmetry group of other class. Corresponding inclusion relations are schematically shown in Fig. 2.

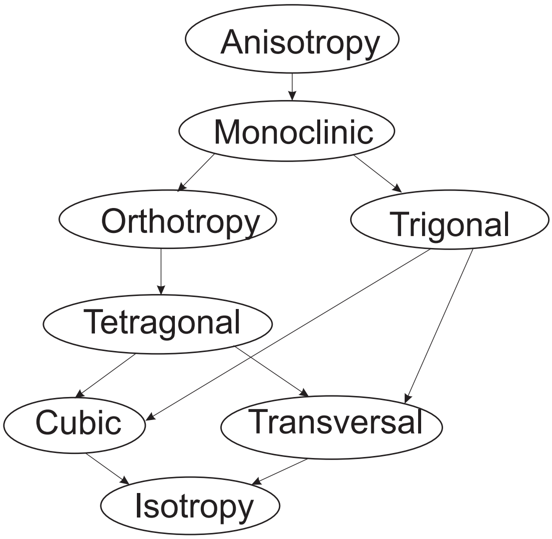

Fig. 2. Scheme of relation between eight classes of elastic symmetry. Each arrow corresponds to the additional symmetry conditions imposed on Hooke's tensor.

Usually, the reduced form of the stiffness (compliance) tensor for the subsequent symmetry groups is derived using the relations (4.3). Then the spectral decomposition of this reduced form is performed to specify the structural indices valid for the considered symmetry group. Below, we derive the form of a stiffness tensor and the structural indices for the subsequent symmetry groups in a non-standard way. Consider the external symmetry of the eigen-projector of $\mathbf{L}$. Any orthogonal tensor belonging to the symmetry group of $\mathbf{P}_k$ fulfills the condition

$$\bigwedge_{\mathbf{Q}\in\mathcal{Q}_{\mathbf{P}k}} \quad \mathbf{Q}\star\mathbf{P}_k = \mathbf{P}_k. \tag{4.4}$$

It can be shown that the symmetry group of the tensor $\mathbf{L}$, $\mathcal{Q}_{\mathbf{L}}$ is the common set of symmetry groups $\mathcal{Q}_{\mathbf{P}k}$ of all its projectors, namely

$$\mathcal{Q}_{\mathbf{L}} = \mathcal{Q}_{\mathbf{P}1} \cap \mathcal{Q}_{\mathbf{P}2} \cap \ldots \cap \mathcal{Q}_{\mathbf{P}\rho}. \tag{4.5}$$

In the components in the selected basis $\{\mathbf{e}_i\}$, relation (4.4) has the form

$$P^{(k)}_{mnpq} Q_{im} Q_{jn} Q_{kp} Q_{lq} = P^{(k)}_{ijkl}.$$

If the subspace $\mathcal{P}_K$ is one-dimensional then the symmetry condition (4.4), together with (4.5), is equivalent to

$$\bigwedge_{\mathbf{Q}\in\mathcal{Q}_{\mathbf{L}}} \quad \mathbf{Q}\star(\boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_K) = (\mathbf{Q}\boldsymbol{\omega}_K\mathbf{Q}^T)\otimes(\mathbf{Q}\boldsymbol{\omega}_K\mathbf{Q}^T) = \boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_K.$$

Consequently

$$\bigwedge_{\mathbf{Q}\in\mathcal{Q}_{\mathbf{L}}} \quad \mathbf{Q}\boldsymbol{\omega}_K\mathbf{Q}^T = \pm\boldsymbol{\omega}_K. \tag{4.6}$$

In components of $\boldsymbol{\omega}_K$ in the basis $\{\mathbf{e}_i\}$, the above equation is specified as

$$\omega^K_{mn} Q_{im} Q_{jn} = \pm\omega^K_{ij}.$$

If the representation of a IV-th order tensor as a II-nd order tensor in 6-dimensional space is used (see Appendix), then the orthogonal tensor in the 3-dimensional space can be replaced by a corresponding orthogonal tensor $\mathbb{Q}$ in the 6-dimensional space, such that

$$\mathbf{Q}\star\mathbf{L} \Leftrightarrow \mathbb{Q}\widehat{\star}^6\mathbf{L} = L_{KL}(\mathbb{Q}\mathbf{a}_K)\otimes(\mathbb{Q}\mathbf{a}_L)$$

and in components, for $\mathbb{Q} = \mathrm{Q}_{KL}\mathbf{a}_K\otimes\mathbf{a}_L$, one has

$$L_{KL} = L_{MN}\mathrm{Q}_{KM}\mathrm{Q}_{LN}. \tag{4.7}$$

In this paper, using the above conditions imposed on $\mathbf{P}_k$ or $\boldsymbol{\omega}_K$, the specific form of eigen-states and eigen-projectors, two structural indices, as well as the stiffness tensor $\mathbf{L}$ will be derived for all 8 symmetry groups of linear elastic material.

4.2. Fully anisotropic material

The symmetry group of Hooke's tensor is never empty. For **full anisotropy**, that is for totally anisotropic material, a symmetry group is defined as

$$\mathcal{Q}^a_{\mathbf{L}} = \{\mathbf{1},\ -\mathbf{1}\}.$$

The symmetry conditions are fulfilled by any normalized set of six mutually orthogonal symmetric II-nd order tensors

$$\boldsymbol{\omega}_K \sim \begin{bmatrix} \omega^K_{11} & \omega^K_{12} & \omega^K_{13} \\ \omega^K_{12} & \omega^K_{22} & \omega^K_{23} \\ \omega^K_{13} & \omega^K_{23} & \omega^K_{33} \end{bmatrix}, \qquad K = I, \dots, VI. \tag{4.8}$$

The specific form of them, that is the value of 12 stiffness distributors, depends on the specific properties of the considered anisotropic material which have to be established in experiments. If one of the eigenstates is purely hydrostatic, namely

$$\boldsymbol{\omega} = \pm\frac{1}{\sqrt{3}}\mathbf{1},$$

then material is called volumetrically isotropic [15]. Note that although the number of independent components is then reduced to 16, in general the material may remain fully anisotropic.

Any material, which is not totally anisotropic is called *a symmetric elastic material* [23]. Such material has at least one symmetry plane.

4.3. Material of monoclinic symmetry

For **monoclinic symmetry**, symmetry of a prism with irregular basis, there exists a single symmetry plane (see Fig. 3) and a symmetry group is the following:

$$\mathcal{Q}^m_{\mathbf{L}} = \{\mathbf{1},\ -\mathbf{1},\ \mathbf{I}_{\mathbf{e}_1}\}, \tag{4.9}$$

where $\mathbf{I}_{\mathbf{e}_1}$ denotes the tensor describing the mirror reflection with respect to the plane with unit normal $\mathbf{e}_1$. In the basis, in which $\mathbf{e}_1$ is specified, two angles ϕ_1 and ϕ_2 are specified. Using the symmetry conditions (4.6) one obtains two following matrix equations

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \omega^K_{11} & \omega^K_{12} & \omega^K_{13} \\ \omega^K_{12} & \omega^K_{22} & \omega^K_{23} \\ \omega^K_{13} & \omega^K_{23} & \omega^K_{33} \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \pm\begin{bmatrix} \omega^K_{11} & \omega^K_{12} & \omega^K_{13} \\ \omega^K_{12} & \omega^K_{22} & \omega^K_{23} \\ \omega^K_{13} & \omega^K_{23} & \omega^K_{33} \end{bmatrix}$$

which after performing the multiplications take the form

$$\begin{bmatrix} \omega_{11}^K & -\omega_{12}^K & -\omega_{13}^K \\ -\omega_{12}^K & \omega_{22}^K & \omega_{23}^K \\ -\omega_{13}^K & \omega_{23}^K & \omega_{33}^K \end{bmatrix} = \begin{bmatrix} -\omega_{11}^K & -\omega_{12}^K & -\omega_{13}^K \\ -\omega_{12}^K & -\omega_{22}^K & -\omega_{23}^K \\ -\omega_{13}^K & -\omega_{23}^K & -\omega_{33}^K \end{bmatrix}$$

and

$$\begin{bmatrix} \omega_{11}^K & -\omega_{12}^K & -\omega_{13}^K \\ -\omega_{12}^K & \omega_{22}^K & \omega_{23}^K \\ -\omega_{13}^K & \omega_{23}^K & \omega_{33}^K \end{bmatrix} = \begin{bmatrix} \omega_{11}^K & \omega_{12}^K & \omega_{13}^K \\ \omega_{12}^K & \omega_{22}^K & \omega_{23}^K \\ \omega_{13}^K & \omega_{23}^K & \omega_{33}^K \end{bmatrix}.$$

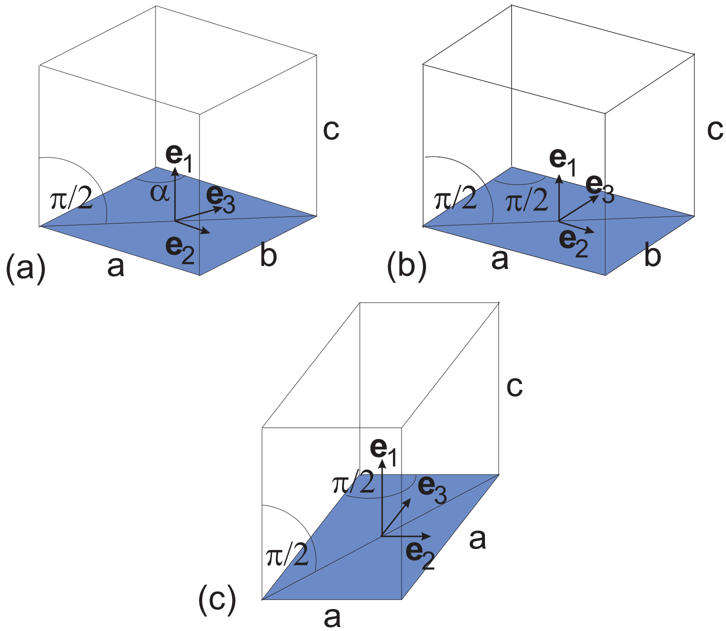

FIG. 3. Schematic representation of monoclinic symmetry (a), orthotropy (b) and tetragonal symmetry (c).

Eigen-states fulfilling the above relations are as follows ($K = III, \ldots, VI$):

$$\text{(4.10)} \qquad \boldsymbol{\omega}_{I,II} \sim \begin{bmatrix} 0 & \omega_{12}^{I,II} & \omega_{13}^{I,II} \\ \omega_{12}^{I,II} & 0 & 0 \\ \omega_{13}^{I,II} & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\omega}_K \sim \begin{bmatrix} \omega_{11}^K & 0 & 0 \\ 0 & \omega_{22}^K & \omega_{23}^K \\ 0 & \omega_{23}^K & \omega_{33}^K \end{bmatrix}.$$

Using orthonormality conditions of eigen-states $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_{II}$, the following form of them is obtained

$$\boldsymbol{\omega}_I \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ \sin\phi & 0 & 0 \\ \cos\phi & 0 & 0 \end{bmatrix},$$

(4.11)

$$\boldsymbol{\omega}_{II} \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \cos\phi & -\sin\phi \\ \cos\phi & 0 & 0 \\ -\sin\phi & 0 & 0 \end{bmatrix}.$$

It can be noted that after changing the basis by proper rotation around $\mathbf{e}_1$ about $\phi_3 = \phi$ (that way we specify the third Euler angle), one arrives at

(4.12) $$\boldsymbol{\omega}_I \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\omega}_{II} \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigen-states $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_{II}$, in the form of pure shears, are identical for any material of monoclinic symmetry, provided a proper frame in the physical space is used. This frame is defined by the unit normal $\mathbf{e}_1$ to the symmetry plane, being the common direction of shearing for the above pure shears in the sense discussed in [3], and two directions: $\mathbf{e}_2$, $\mathbf{e}_3$ which specify the unit normal to the corresponding shearing planes as they were defined in [3]. The derived form of eigen-states complies with the theorem formulated in [3] according to which for any symmetric material, at least two eigen-states of the stiffness tensor are pure shears. The specific form of remaining eigenstates $\boldsymbol{\omega}_K$, $(K = III, \ldots, VI)$, defined by 6 stiffness distributors, depends on the properties of the considered material of monoclinic symmetry. Using (1.7) the representation of $\mathbf{L}$ in the poly-basis $\{\mathbf{a}_K\}$ composed of diads of the above selected unit vectors $\mathbf{e}_i$ is derived as

(4.13) $$\mathbf{L} \sim \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & 0 \\ L_{12} & L_{22} & L_{23} & L_{24} & 0 & 0 \\ L_{13} & L_{23} & L_{33} & L_{34} & 0 & 0 \\ L_{14} & L_{24} & L_{34} & L_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{55} = \lambda_{II} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{66} = \lambda_I \end{bmatrix}$$

therefore it is specified by 12 independent components. The Kelvin moduli $\lambda_{III}, \ldots, \lambda_{VI}$ are obtained as eigenvalues of 4×4 upper left sub-matrix of (4.13). An example of material of elastic monoclinic symmetry is the martensite phase, the lower symmetry phase in CuZnAl shape memory alloy.

4.4. Orthotropic material

In the case of **orthotropic** material, that is the material possessing symmetry of a prism with rectangular basis (see Fig. 3), the symmetry group includes the elements

$$\mathcal{Q}^o_{\mathbf{L}} = \{\mathbf{1},\ \ -\mathbf{1},\ \ \mathbf{I}_{\mathbf{e}_1},\ \ \mathbf{I}_{\mathbf{e}_2}\}\,. \tag{4.14}$$

The symmetry conditions (4.6) can be imposed on the derived form of eigen-states for the material of monoclinic symmetry as far as the symmetry group of the latter material is included in the symmetry group of orthotropic material ($\mathcal{Q}^m_{\mathbf{L}} \subset \mathcal{Q}^o_{\mathbf{L}}$, Fig. 2). Thus, any orthotropic material is the material of monoclinic symmetry. Let us consider two groups of eigen-states obtained for material of monoclinic symmetry. Imposing additional condition (related to the orthogonal tensor $\mathbf{I}_{\mathbf{e}_2}$) on the first group in (4.10), we find

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \omega_{12}^{I,II} & \omega_{13}^{I,II} \\ \omega_{12}^{I,II} & 0 & 0 \\ \omega_{13}^{I,II} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \pm \begin{bmatrix} 0 & \omega_{12}^{I,II} & \omega_{13}^{I,II} \\ \omega_{12}^{I,II} & 0 & 0 \\ \omega_{13}^{I,II} & 0 & 0 \end{bmatrix}$$

which after multiplications simplifies to the relations

$$\begin{bmatrix} 0 & -\omega_{12}^{I,II} & \omega_{13}^{I,II} \\ -\omega_{12}^{I,II} & 0 & 0 \\ \omega_{13}^{I,II} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pm\omega_{12}^{I,II} & \pm\omega_{13}^{I,II} \\ \pm\omega_{12}^{I,II} & 0 & 0 \\ \pm\omega_{13}^{I,II} & 0 & 0 \end{bmatrix}.$$

They are identically true for the eigen-states (4.12), where the direction $\mathbf{e}_2$ agrees with the unit normal to the shearing plane for one of these eigenstates.

Imposing additional condition on the second group of eigen-states in (4.10), it is obtained

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_{11}^{K} & 0 & 0 \\ 0 & \omega_{22}^{K} & \omega_{23}^{K} \\ 0 & \omega_{23}^{K} & \omega_{33}^{K} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \pm \begin{bmatrix} \omega_{11}^{K} & 0 & 0 \\ 0 & \omega_{22}^{K} & \omega_{23}^{K} \\ 0 & \omega_{23}^{K} & \omega_{33}^{K} \end{bmatrix}$$

and after multiplications, the following constraints are found

$$\begin{bmatrix} \omega_{11}^{K} & 0 & 0 \\ 0 & \omega_{22}^{K} & -\omega_{23}^{K} \\ 0 & -\omega_{23}^{K} & \omega_{33}^{K} \end{bmatrix} = \begin{bmatrix} \pm\omega_{11}^{K} & 0 & 0 \\ 0 & \pm\omega_{22}^{K} & \pm\omega_{23}^{K} \\ 0 & \pm\omega_{23}^{K} & \pm\omega_{33}^{K} \end{bmatrix}$$

which are true for the following forms of $\boldsymbol{\omega}$ $(K = IV, V, VI)$

$$\boldsymbol{\omega}_{III} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_{23}^{III} \\ 0 & \omega_{23}^{III} & 0 \end{bmatrix}, \qquad \boldsymbol{\omega}_{K} \sim \begin{bmatrix} \omega_{11}^{K} & 0 & 0 \\ 0 & \omega_{22}^{K} & 0 \\ 0 & 0 & \omega_{33}^{K} \end{bmatrix}. \tag{4.15}$$

After normalization we obtain the following eigen-states in the form of pure shears [2] in the basis $\{\mathbf{e}_i\}$ specified by three directions of orthotropy (this way three Euler angles are specified):

$$\begin{gathered} \boldsymbol{\omega}_{I} \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \boldsymbol{\omega}_{II} \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ \boldsymbol{\omega}_{III} \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \end{gathered} \tag{4.16}$$

and three subsequent eigen-states in the diagonal form in this basis, which after utilizing orthonormality conditions we can present in the form [2]

$$\boldsymbol{\omega}_{IV} \sim \begin{bmatrix} \cos\theta_1 & 0 & 0 \\ 0 & \sin\theta_1 \cos\theta_2 & 0 \\ 0 & 0 & \sin\theta_1 \sin\theta_2 \end{bmatrix},$$

$$\boldsymbol{\omega}_{V} \sim \begin{bmatrix} -\cos\theta_3 \sin\theta_1 & 0 & 0 \\ 0 & \begin{matrix} \cos\theta_1 \cos\theta_2 \cos\theta_3 + \\ -\sin\theta_2 \sin\theta_3 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} \cos\theta_1 \sin\theta_2 \cos\theta_3 + \\ + \sin\theta_3 \cos\theta_2 \end{matrix} \end{bmatrix},$$

$$\boldsymbol{\omega}_{VI} \sim \begin{bmatrix} \sin\theta_1 \sin\theta_3 & 0 & 0 \\ 0 & \begin{matrix} -\sin\theta_3 \cos\theta_1 \cos\theta_2 + \\ -\cos\theta_3 \sin\theta_2 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} -\sin\theta_3 \cos\theta_1 \sin\theta_2 + \\ + \cos\theta_3 \cos\theta_2 \end{matrix} \end{bmatrix}.$$

For any orthotropic material there exist three uniquely defined (within a sign) eigen-states in the form of pure shears, while the form of eigen-states $\boldsymbol{\omega}_{IV,V,VI}$ is specified by three angles θ_1, θ_2, θ_3 which themselves are the functions of three stiffness distributors. They depend on the properties of the considered material of orthotropic symmetry. In the paper [15] it was proposed to define these distributors in the following way[1)]

$$(4.17)\qquad \eta_1 = \mathrm{tr}\mathbf{h}_{VI}^2, \qquad \eta_2 = \frac{\det \mathbf{h}_{VI}}{(\mathrm{tr}\boldsymbol{\omega}_{VI})^3}, \qquad \eta_3 = \frac{\mathrm{tr}(\boldsymbol{\omega}_{VI}^2\boldsymbol{\omega}_V)}{\mathrm{tr}\boldsymbol{\omega}_V},$$

where $\mathbf{h}_K$ are deviators of $\boldsymbol{\omega}_K$. The above definition must be modified in the case when $\eta_1 = 0$ or two eigenvalues of $\boldsymbol{\omega}_{VI}$ are equal to each other correspondingly in the form

$$(4.18)\qquad \eta_3^* = (\det \mathbf{h}_V)^2, \qquad \eta_3^{**} = \frac{\det \mathbf{h}_V}{(\mathrm{tr}\mathbf{w}_V)^3}.$$

The representation of $\mathbf{L}$ in poly-basis $\{\mathbf{a}_K\}$ constructed with use of orthotropy directions $\{\mathbf{e}_k\}$ is

$$(4.19)\qquad \mathbf{L} \sim \begin{bmatrix} L_{11} & L_{12} & L_{13} & 0 & 0 & 0 \\ L_{12} & L_{22} & L_{23} & 0 & 0 & 0 \\ L_{13} & L_{23} & L_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{44} = \lambda_{III} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{55} = \lambda_{II} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{66} = \lambda_I \end{bmatrix}$$

therefore it is specified by 9 independent components. The Kelvin moduli $\lambda_{IV}, \ldots, \lambda_{VI}$ are obtained as eigenvalues of 3×3 upper left sub-matrix of (4.19). The orthotropic symmetry is characteristic for metal sheets with texture resulting from rolling process.

For the above two classes of symmetry one obtains one-dimensional eigen-subspaces $\mathcal{P}_K$.

4.5. Material of trigonal symmetry

Material of **trigonal symmetry** (symmetry of a cube uniformly elongated along one of its main diagonals, see Fig. 5, where the diagonal is coaxial with the main symmetry axis $\mathbf{e}_1$) has the following symmetry group:

[1)] In [15] it was assumed that Kelvin moduli λ_K are ordered in view of increasing value of the corresponding $(\mathrm{tr}\boldsymbol{\omega}_K)^2$.

$$\mathcal{Q}^{3t}_{\mathbf{L}} = \left\{ \mathbf{1}, \ -\mathbf{1}, \ \mathbf{R}^{2\pi/3}_{\mathbf{e}_1}, \ \mathbf{I}_{\mathbf{e}_2} \right\}, \tag{4.20}$$

where $\mathbf{R}^{2\pi/3}_{\mathbf{e}_1}$ denotes the rotation around the axis $\mathbf{e}_1$ through the angle $2\pi/3$.

It should be noted that the monoclinic symmetry group $\mathcal{Q}^{m}_{\mathbf{L}} \subset \mathcal{Q}^{3t}_{\mathbf{L}}$ if the direction $\mathbf{e}_1$ is replaced by $\mathbf{e}_2$. For the symmetric direction specified in this way with respect to the basis $\{\mathbf{e}_i\}$, two groups of eigen-states in (4.10) have the representations $(K = III, \ldots, VI)$

$$\boldsymbol{\omega}_{I,II} \sim \begin{bmatrix} 0 & \omega^{I,II}_{12} & 0 \\ \omega^{I,II}_{12} & 0 & \omega^{I,II}_{23} \\ 0 & \omega^{I,II}_{23} & 0 \end{bmatrix}, \qquad \boldsymbol{\omega}_{K} \sim \begin{bmatrix} \omega^{K}_{11} & 0 & \omega^{K}_{13} \\ 0 & \omega^{K}_{22} & 0 \\ \omega^{K}_{13} & 0 & \omega^{K}_{33} \end{bmatrix}. \tag{4.21}$$

Fulfilling the additional symmetry condition (4.6) related to the orthogonal tensor $\mathbf{R}^{2\pi/3}_{\mathbf{e}_1}$ for the second group of eigen-states (4.21), we derive the constraints

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \omega^{K}_{11} & 0 & \omega^{K}_{13} \\ 0 & \omega^{K}_{22} & 0 \\ \omega^{K}_{13} & 0 & \omega^{K}_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \pm \begin{bmatrix} \omega^{K}_{11} & 0 & \omega^{K}_{13} \\ 0 & \omega^{K}_{22} & 0 \\ \omega^{K}_{13} & 0 & \omega^{K}_{33} \end{bmatrix},$$

which after multiplications take the form

$$\begin{bmatrix} \omega^{K}_{11} & -\frac{\sqrt{3}}{2}\omega^{K}_{13} & -\frac{1}{2}\omega^{K}_{13} \\ -\frac{\sqrt{3}}{2}\omega^{K}_{13} & \frac{1}{4}(\omega^{K}_{22} + 3\omega^{K}_{33}) & \frac{\sqrt{3}}{4}(\omega^{K}_{33} - \omega^{K}_{22}) \\ -\frac{1}{2}\omega^{K}_{13} & \frac{\sqrt{3}}{4}(\omega^{K}_{33} - \omega^{K}_{22}) & \frac{1}{4}(3\omega^{K}_{22} + \omega^{K}_{33}) \end{bmatrix} = \pm \begin{bmatrix} \omega^{K}_{11} & 0 & \omega^{K}_{13} \\ 0 & \omega^{K}_{22} & 0 \\ \omega^{K}_{13} & 0 & \omega^{K}_{33} \end{bmatrix}.$$

The above relations can be fulfilled only by two linearly independent unit eigen-states with the below representation in the basis $\{\mathbf{e}_i\}$[2)]

$$\boldsymbol{\omega}_{V,VI} \sim \begin{bmatrix} \omega^{V,VI}_{11} & 0 & 0 \\ 0 & \omega^{V,VI}_{22} & 0 \\ 0 & 0 & \omega^{V,VI}_{22} \end{bmatrix}. \tag{4.22}$$

[2)] As it can be noticed in Fig. 5, the direction $\mathbf{e}_2$ can be specified with the accuracy to the rotation about $2\pi/3$ around $\mathbf{e}_1$.

They define two one-dimensional eigen-subspaces. After normalization and application of orthogonality conditions, they take the form

$$
(4.23)\qquad
\begin{aligned}
\boldsymbol{\omega}_V &\sim \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}\sin\phi & 0 & 0 \\ 0 & -\cos\phi & 0 \\ 0 & 0 & -\cos\phi \end{bmatrix},\\
\boldsymbol{\omega}_{VI} &\sim \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}\cos\phi & 0 & 0 \\ 0 & \sin\phi & 0 \\ 0 & 0 & \sin\phi \end{bmatrix}.
\end{aligned}
$$

In general, the above eigen-states are not pure shears.

Imposing the symmetry condition (4.6) on the first group of eigenstates (4.21) we find only trivial solution $\boldsymbol{\omega} = \mathbf{0}$, which of course is unacceptable. Consequently, the remaining eigen-subspaces must be more than one-dimensional and their form will be found using the symmetry condition (4.4). Any IV-th order tensor orthogonal to the eigen-projectors composed of eigen-states (4.23) has the representation

$$
\mathbf{P} \sim \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -P_{23} & P_{23} & P_{24} & P_{25} & P_{26} \\
0 & P_{23} & -P_{23} & -P_{24} & -P_{25} & -P_{26} \\
0 & P_{24} & -P_{24} & P_{44} & P_{45} & P_{46} \\
0 & P_{25} & -P_{25} & P_{45} & P_{55} & P_{56} \\
0 & P_{26} & -P_{26} & P_{46} & P_{56} & P_{66}
\end{bmatrix}.
$$

The representation of a orthogonal tensor $\mathbf{R}^{2\pi/3}_{\mathbf{e}_1}$ in the six-dimensional space is the following one

$$
\mathbb{R}^{2\pi/3}_{\mathbf{e}_1} \sim \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{3}{4} & \frac{\sqrt{6}}{4} & 0 & 0 \\
0 & \frac{3}{4} & \frac{1}{4} & -\frac{\sqrt{6}}{4} & 0 & 0 \\
0 & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{bmatrix}.
$$

If we perform the rotation operation for the tensor $\mathbf{P}$ using the relation (4.7), then we find the following non-zero components of the rotated tensor $\mathbb{R}_{\mathbf{e}_1}^{2\pi/3} \widehat{\star}^6 \mathbf{P}$:

$$\widetilde{P}_{22} = \frac{1}{8}(-2P_{23} - 2\sqrt{6}P_{24} + 3P_{44}), \tag{4.24}$$

$$\widetilde{P}_{23} = \frac{1}{8}(2P_{23} + 2\sqrt{6}P_{24} - 3P_{44}), \tag{4.25}$$

$$\widetilde{P}_{24} = \frac{1}{8}(-2\sqrt{6}P_{23} - 4P_{24} - \sqrt{6}P_{44}), \tag{4.26}$$

$$\widetilde{P}_{25} = \frac{1}{8}(2P_{25} - 2\sqrt{3}P_{26} - \sqrt{6}P_{45} + 3\sqrt{2}P_{46}), \tag{4.27}$$

$$\widetilde{P}_{26} = \frac{1}{8}(2\sqrt{3}P_{25} + 2P_{26} - 3\sqrt{2}P_{45} - \sqrt{6}P_{46}), \tag{4.28}$$

$$\widetilde{P}_{33} = \frac{1}{8}(-2P_{23} - 2\sqrt{6}P_{24} + 3P_{44}), \tag{4.29}$$

$$\widetilde{P}_{34} = \frac{1}{8}(2\sqrt{6}P_{23} + 4P_{24} + \sqrt{6}P_{44}), \tag{4.30}$$

$$\widetilde{P}_{35} = \frac{1}{8}(-2P_{25} + 2\sqrt{3}P_{26} + \sqrt{6}P_{45} - 3\sqrt{2}P_{46}), \tag{4.31}$$

$$\widetilde{P}_{36} = \frac{1}{8}(-2\sqrt{3}P_{25} - 2P_{26} + 3\sqrt{2}P_{45} + \sqrt{6}P_{46}), \tag{4.32}$$

$$\widetilde{P}_{44} = \frac{1}{4}(-6P_{23} + 2\sqrt{6}P_{24} + P_{44}), \tag{4.33}$$

$$\widetilde{P}_{45} = \frac{1}{4}(\sqrt{6}P_{25} - 3\sqrt{2}P_{26} + P_{45} - \sqrt{3}P_{46}), \tag{4.34}$$

$$\widetilde{P}_{46} = \frac{1}{4}(3\sqrt{2}P_{25} + \sqrt{6}P_{26} + \sqrt{3}P_{45} + P_{46}), \tag{4.35}$$

$$\widetilde{P}_{55} = \frac{1}{4}(P_{55} - 2\sqrt{3}P_{56} + 3P_{66}), \tag{4.36}$$

$$\widetilde{P}_{56} = \frac{1}{4}(\sqrt{3}P_{55} - 2P_{56} - \sqrt{3}P_{66}), \tag{4.37}$$

$$\widetilde{P}_{66} = \frac{1}{4}(3P_{55} + 2\sqrt{3}P_{56} + P_{66}). \tag{4.38}$$

After algebraic manipulations, setting $\hat{P}_{KL} = P_{KL}$, the representations of two projectors in the poly-basis $\{\mathbf{a}_I\}$ are found, namely

$$\mathbf{P}_K \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\omega_{23}^K)^2 & -(\omega_{23}^K)^2 & 0 & \sqrt{2}\omega_{12}^K\omega_{23}^K & 0 \\ 0 & -(\omega_{23}^K)^2 & (\omega_{23}^K)^2 & 0 & -\sqrt{2}\omega_{12}^K\omega_{23}^K & 0 \\ 0 & 0 & 0 & 2(\omega_{23}^K)^2 & 0 & 2\omega_{12}^K\omega_{23}^K \\ 0 & \sqrt{2}\omega_{12}^K\omega_{23}^K & -\sqrt{2}\omega_{12}^K\omega_{23}^K & 0 & 2(\omega_{12}^K)^2 & 0 \\ 0 & 0 & 0 & 2\omega_{12}^K\omega_{23}^K & 0 & 2(\omega_{12}^K)^2 \end{bmatrix}.$$

They project into two two-dimensional subspaces $\mathcal{P}_{I,II}$ and $\mathcal{P}_{III,IV}$ of deviatoric tensors. Using the orthogonality and after normalization of the elements, we arrive at the following representations of these projectors

$$\mathbf{P}_{I,II} \sim \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\sin\rho)^2 & -(\sin\rho)^2 & 0 & -\frac{\sqrt{2}}{2}\sin 2\rho & 0 \\ 0 & -(\sin\rho)^2 & (\sin\rho)^2 & 0 & \frac{\sqrt{2}}{2}\sin 2\rho & 0 \\ 0 & 0 & 0 & 2(\sin\rho)^2 & 0 & -\sin 2\rho \\ 0 & -\frac{\sqrt{2}}{2}\sin 2\rho & \frac{\sqrt{2}}{2}\sin 2\rho & 0 & 2(\cos\rho)^2 & 0 \\ 0 & 0 & 0 & -\sin 2\rho & 0 & 2(\cos\rho)^2 \end{bmatrix}, \tag{4.39}$$

$$\mathbf{P}_{III,IV} \sim \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\cos\rho)^2 & -(\cos\rho)^2 & 0 & \frac{\sqrt{2}}{2}\sin 2\rho & 0 \\ 0 & -(\cos\rho)^2 & (\cos\rho)^2 & 0 & -\frac{\sqrt{2}}{2}\sin 2\rho & 0 \\ 0 & 0 & 0 & 2(\cos\rho)^2 & 0 & \sin 2\rho \\ 0 & \frac{\sqrt{2}}{2}\sin 2\rho & -\frac{\sqrt{2}}{2}\sin 2\rho & 0 & 2(\sin\rho)^2 & 0 \\ 0 & 0 & 0 & \sin 2\rho & 0 & 2(\sin\rho)^2 \end{bmatrix}. \tag{4.40}$$

Any second-order tensor belonging to $\mathcal{P}_{I,II}$ and $\mathcal{P}_{III,IV}$, respectively, is deviatoric and has the following representation in the basis $\{\mathbf{e}_i\}$ ($\boldsymbol{\omega}$ any second order tensor):

$$\boldsymbol{\omega}_{I,II} = \frac{\mathbf{P}_{I,II}\cdot\boldsymbol{\omega}}{|\mathbf{P}_{I,II}\cdot\boldsymbol{\omega}|} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \cos\varphi\cos\rho & \sin\varphi\cos\rho \\ \cos\varphi\cos\rho & -\sin\varphi\sin\rho & -\cos\varphi\sin\rho \\ \sin\varphi\cos\rho & -\cos\varphi\sin\rho & \sin\varphi\sin\rho \end{bmatrix} \tag{4.41}$$

and

$$\boldsymbol{\omega}_{III,IV} = \frac{\mathbf{P}_{III,IV}\cdot\boldsymbol{\omega}}{|\mathbf{P}_{III,IV}\cdot\boldsymbol{\omega}|} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \cos\varphi\sin\rho & \sin\varphi\sin\rho \\ \cos\varphi\sin\rho & \sin\varphi\cos\rho & \cos\varphi\cos\rho \\ \sin\varphi\sin\rho & \cos\varphi\cos\rho & -\sin\varphi\cos\rho \end{bmatrix}, \tag{4.42}$$

where $\varphi \in \langle 0, 2\pi\rangle$. The bases in those sub-spaces can be composed of two elements: $\boldsymbol{\omega}_K(\varphi_1)$ and $\boldsymbol{\omega}_K(\varphi_2)$, where $\varphi_2 = \varphi_1 + \pi/2$. The simplest bases in $\mathcal{P}_{I,II}$ and $\mathcal{P}_{III,IV}$ is obtained setting $\phi_1 = 0$ and $\phi_2 = \pi/2$. Note that among infinite number of elements (4.41) and (4.42), one can indicate in both cases three which are pure shears. They are specified by angles φ being the solutions of two trigonometric equations

$$\det\boldsymbol{\omega}_{I,II} = 0 \Leftrightarrow \cos^2\rho\sin\varphi(\sin\rho - \cos^2\varphi(3\sin\rho + \cos\rho)) = 0,$$
$$\det\boldsymbol{\omega}_{III,IV} = 0 \Leftrightarrow \sin^2\rho\sin\varphi(\cos^2\varphi(3\cos\rho + \sin\rho) - \cos\rho) = 0.$$

Therefore, for any elastic material of trigonal symmetry at least six of its eigen-states are the pure shears [3]. Of course, not all of them are pairwise orthogonal as far as some of them correspond to the same eigen-value (the same Kelvin modulus).

The specific form of $\mathbf{P}_{I,II}$ and $\mathbf{P}_{III,IV}$ depends on the angle ρ being the function of one stiffness distributor. The value of this distributor is material characteristic for trigonal symmetry. Similarly, the specific form of eigen-states $\boldsymbol{\omega}_V$ and $\boldsymbol{\omega}_{VI}$ depends on the angle ϕ which is the function of the second stiffness distributor (compare [25]). One can define this distributor as follows:

$$\eta_2 = \frac{\det\mathbf{h}_{VI}}{(\mathrm{tr}\boldsymbol{\omega}_{VI})^3},$$

where $\mathbf{h}_{VI}$ is deviator of $\boldsymbol{\omega}_{VI}$.

The considered **material of trigonal symmetry** is defined by

1. 4 Kelvin moduli: $\lambda_1 = \lambda_{I,II}$, $\lambda_2 = \lambda_{III,IV}$, both of multiplicity 2, and $\lambda_3 = \lambda_V$, $\lambda_4 = \lambda_{VI}$ of multiplicity 1.
2. Two stiffness distributors which specify angles ρ and ϕ.
3. 3 Euler angles which orient symmetry axis $\mathbf{e}_1$ and the symmetry plane $\mathbf{e}_2$ with respect to laboratory.

The unique spectral decomposition takes the form

$$\mathbf{L} = \lambda_1\mathbf{P}_1(\rho) + \lambda_2\mathbf{P}_2(\rho) + \lambda_3\mathbf{P}_3(\phi) + \lambda_4\mathbf{P}_4(\phi), \tag{4.43}$$

where

$$\mathbf{P}_1(\rho) = \mathbf{P}_{I,II}(\rho), \qquad \mathbf{P}_2(\rho) = \mathbf{P}_{III,IV}(\rho),$$
$$\mathbf{P}_3(\phi) = \boldsymbol{\omega}_V(\phi)\otimes\boldsymbol{\omega}_V(\phi), \qquad \mathbf{P}_4(\phi) = \boldsymbol{\omega}_{VI}(\phi)\otimes\boldsymbol{\omega}_{VI}(\phi).$$

Using (4.43) the stiffness tensor $\mathbf{L}$ for the material of trigonal symmetry in the poly-basis $\mathbf{a}_K$ composed of diads of the basis $\mathbf{e}_i$, has the representation

$$\mathbf{L} \sim \begin{bmatrix} L_{11} & L_{12} & L_{12} & 0 & 0 & 0 \\ L_{12} & L_{22} & L_{23} & 0 & L_{25} & 0 \\ L_{12} & L_{23} & L_{22} & 0 & -L_{25} & 0 \\ 0 & 0 & 0 & L_{22}-L_{23} & 0 & \sqrt{2}L_{25} \\ 0 & L_{25} & -L_{25} & 0 & L_{55} & 0 \\ 0 & 0 & 0 & \sqrt{2}L_{25} & 0 & L_{55} \end{bmatrix} \tag{4.44}$$

therefore it is specified by 6 independent components. It can be shown that the Kelvin moduli $\lambda_V = \lambda_3$ and $\lambda_{VI} = \lambda_4$ are obtained as eigenvalues of the following 2×2 matrix

$$\frac{1}{3}\begin{bmatrix} L_{11}+2(2L_{12}+L_{23}+L_{33}) & \sqrt{2}(L_{11}+L_{12}-(L_{22}+L_{23})) \\ \sqrt{2}(L_{11}+L_{12}-(L_{22}+L_{23})) & 2L_{11}-4L_{12}+L_{23}+L_{22} \end{bmatrix}, \tag{4.45}$$

while the Kelvin $\lambda_{I,II} = \lambda_1$ and $\lambda_{III,IV} = \lambda_2$ of multiplicity 2 can be derived as eigenvalues of the following 2×2 matrix:

$$\begin{bmatrix} L_{22}-L_{23} & \sqrt{2}L_{25} \\ \sqrt{2}L_{25} & L_{55} \end{bmatrix}. \tag{4.46}$$

Single crystal of aluminum oxide Al_2O_3, ceramic material, has trigonal symmetry.

4.6. Material of tetragonal symmetry

Material of **tetragonal symmetry** (symmetry of a prism of square basis, see Fig. 3) is characterized by the following symmetry group

$$\mathcal{Q}^{4t}_{\mathbf{L}} = \left\{ \mathbf{1},\ -\mathbf{1},\ \mathbf{I}_{\mathbf{e}_1},\ \mathbf{I}_{\mathbf{e}_2},\ \mathbf{R}^{\pi/2}_{\mathbf{e}_1} \right\}. \tag{4.47}$$

Similarly like in the case of trigonal symmetry it is impossible to fulfill the symmetry conditions (4.6) by 6 mutually orthogonal eigen-states. Using the results for otrhotropic material, it can be checked that the additional condition of symmetry imposed by $\mathbf{R}^{\pi/2}_{\mathbf{e}_1}$ is fulfilled[3)] only by four eigen-tensors. Two of them are pure shears which have the following representations in basis $\{\mathbf{e}_i\}$:

[3)] The eigen-states of the material of tetragonal symmetry can be derived by imposing the additional symmetry condition on the eigen-states of orthotropic material because $\mathcal{Q}^{o}_{\mathbf{L}} \subset \mathcal{Q}^{4t}_{\mathbf{L}}$, Fig. 2.

(4.48) $$\boldsymbol{\omega}_{III} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\omega}_{IV} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

while other two eigen-states have the form

(4.49) $$\boldsymbol{\omega}_{V,VI} \sim \begin{bmatrix} \omega_{11}^{V,VI} & 0 & 0 \\ 0 & \omega_{22}^{V,VI} & 0 \\ 0 & 0 & \omega_{22}^{V,VI} \end{bmatrix}.$$

They define four one-dimensional subspaces $\mathcal{P}_K, K = III, IV, V, VI$. Moreover, from the symmetry conditions (4.4) we obtain the following projector $\mathbf{P}_{I,II}$ which projects into two-dimensional subspace $\mathcal{P}_{I,II}$ of pure shears with common shear direction. Its representation in the orthonormal poly-basis $\{\mathbf{a}_I\}$ composed of $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ (see Appendix) is as follows:

(4.50) $$\mathbf{P}_{I,II} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Any unit element of this two-dimensional subspace can be written in the form

(4.51) $$\boldsymbol{\omega}_{I,II} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \sin\varphi & \cos\varphi \\ \sin\varphi & 0 & 0 \\ \cos\varphi & 0 & 0 \end{bmatrix}, \qquad \varphi \in \langle 0, 2\pi\rangle.$$

An orthonormal basis in this subspace is composed of two tensors $\boldsymbol{\omega}_{I,II}(\varphi_1)$ and $\boldsymbol{\omega}_{I,II}(\varphi_2)$, such that $\varphi_2 = \varphi_1 + \pi/2$.

For any material of tetragonal symmetry we have obtained two uniquely specified (within a sign) eigen-states $\boldsymbol{\omega}_{III}$ and $\boldsymbol{\omega}_{IV}$ as well as the uniquely defined projector $\mathbf{P}_{I,II}$. The specific form of $\boldsymbol{\omega}_V$ and $\boldsymbol{\omega}_{VI}$ depends on the value of one stiffness distributor which is the material characteristic for the considered material. Using the result of [15], this distributor can be defined as

(4.52) $$\eta = \eta_2 = \frac{\det \mathbf{h}_{VI}}{(\mathrm{tr}\boldsymbol{\omega}_{VI})^3}.$$

It should be noted that for the material of tetragonal symmetry the direction $\mathbf{e}_1$ is uniquely defined, while the direction $\mathbf{e}_2$ can be specified only with accuracy to the angle $\pi/4$.

The considered elastic **material of tetragonal symmetry** is specified by

1. 5 Kelvin moduli: $\lambda_1 = \lambda_{I,II}$ of multiplicity 2, $\lambda_2 = \lambda_{III}$, $\lambda_3 = \lambda_{IV}$ $\lambda_4 = \lambda_V$ and $\lambda_5 = \lambda_{VI}$ of multiplicity 1.
2. One stiffness distributor η which specifies angle ϕ.
3. 3 Euler angles which orient symmetry axis $\mathbf{e}_1$ and the symmetry plane $\mathbf{e}_2$ with respect to laboratory.

The unique spectral decomposition takes the form

$$\mathbf{L} = \lambda_1\mathbf{P}_1 + \lambda_2\mathbf{P}_2 + \lambda_3\mathbf{P}_3 + \lambda_4\mathbf{P}_4(\phi) + \lambda_5\mathbf{P}_5(\phi) \tag{4.53}$$

where

$$\mathbf{P}_1 = \mathbf{P}_{I,II}\,, \qquad \mathbf{P}_2 = \boldsymbol{\omega}_{III} \otimes \boldsymbol{\omega}_{III}\,, \qquad \mathbf{P}_3 = \boldsymbol{\omega}_{IV} \otimes \boldsymbol{\omega}_{IV}$$

and

$$\mathbf{P}_4(\phi) = \boldsymbol{\omega}_V(\phi) \otimes \boldsymbol{\omega}_V(\phi)\,, \qquad \mathbf{P}_5(\phi) = \boldsymbol{\omega}_{VI}(\phi) \otimes \boldsymbol{\omega}_{VI}(\phi).$$

The representation of the stiffness tensor in poly-basis $\mathbf{a}_K$ for the material of tetragonal symmetry has the form similar to orthotropic material with additional relations

$$L_{13} = L_{12}, \qquad L_{33} = L_{22}, \qquad L_{66} = L_{55}; \tag{4.54}$$

therefore, it is specified by 6 independent components. The Kelvin moduli depend on L_{KL} as follows:

$$\lambda_{I,II} = \lambda_1 = L_{55}, \qquad \lambda_{III} = \lambda_2 = \lambda_{44}, \qquad \lambda_{IV} = \lambda_3 = L_{22} - L_{23} \tag{4.55}$$

and $\lambda_V = \lambda_4$ and $\lambda_{VI} = \lambda_5$ are found as eigenvalues of matrix (4.45). The stiffness distributor η is specified by components of $\mathbf{L}$ as follows

$$\eta = \frac{1}{27\sqrt{2}}\,\frac{\overline{L}_{12}}{\overline{L}_{11} - \lambda_V}, \tag{4.56}$$

where $\overline{L}_{KL}$ denote components of matrix (4.45), while λ_V is taken as a minimum (a maximum) of its eigenvalues if $\overline{L}_{11} > \overline{L}_{22}$ ($\overline{L}_{11} < \overline{L}_{22}$). The latter specification ensures that $(\mathrm{tr}\boldsymbol{\omega}_{VI})^2 > (\mathrm{tr}\boldsymbol{\omega}_V)^2$.

As an example of material of tetragonal symmetry, the γ-TiAl intermetallic is analyzed in Subsec. 4.10. Tetragonal symmetry has also a single crystal of martensitic phase of ferromagnetic shape memory alloy NiMnGa.

4.7. Transversely isotropic material

Material of **transversal isotropy** (cylindrical symmetry presented in Fig. 4) has the following symmetry group (note that $\mathcal{Q}^{4t}_{\mathbf{L}} \subset \mathcal{Q}^{t}_{\mathbf{L}}$):

$$\mathcal{Q}^{t}_{\mathbf{L}} = \left\{\mathbf{1},\ -\mathbf{1},\ \mathbf{I}_{\mathbf{e}_1},\ \mathbf{I}_{\mathbf{e}_2},\ \mathbf{R}^{\phi}_{\mathbf{e}_1}\right\}, \tag{4.57}$$

where the orthogonal tensor $\mathbf{R}^{\phi}_{\mathbf{e}_1}$ describes the rotation around the axis $\mathbf{e}_1$ through any angle ϕ. The symmetry condition (4.6) for this rotation tensor is fullfilled by two eigen-states (4.49) valid for tetragonal symmetry, which describe two one-dimensional subspaces $\mathcal{P}_V$ and $\mathcal{P}_{VI}$. Furthermore, the symmetry condition (4.4) is fullfilled for projector $\mathcal{P}_{I,II}$ specified by (4.50) and another projector $\mathcal{P}_{III,IV}$, both projecting into two 2-dimensional subspaces of pure shears. The projector $\mathcal{P}_{III,IV}$ has the representation

$$\mathbf{P}_{III,IV} \sim \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.58}$$

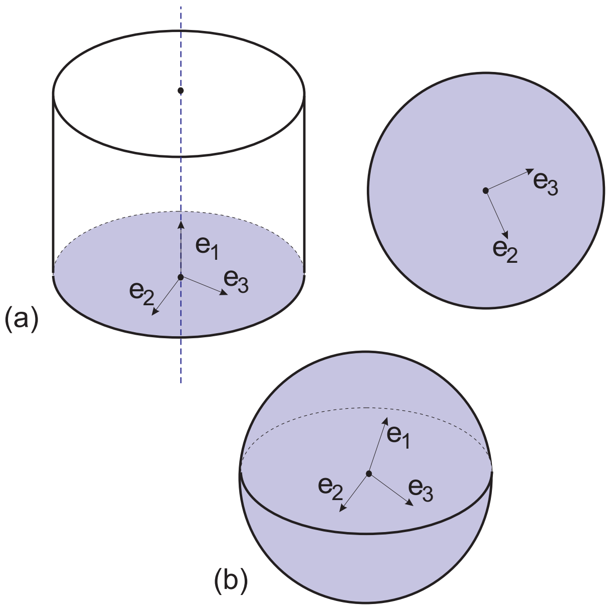

Fig. 4. Schematic representation of transversely isotropic material (a) and isotropic material (b).

written in the poly-basis $\{\mathbf{a}_I\}$. Any unit element of the two-dimensional subspace $\mathcal{P}_{III,IV}$ can be specified in the form

$$\boldsymbol{\omega}_{III,IV} \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & \sin\psi & -\cos\psi \end{bmatrix}, \qquad \psi \in \langle 0,\ 2\pi \rangle. \tag{4.59}$$

Orthonormal basis in this subspace is composed of two tensors $\boldsymbol{\omega}_{III,IV}(\psi_1)$ and $\boldsymbol{\omega}_{III,IV}(\psi_2)$, such that $\psi_2 = \psi_1 + \pi/2$.

It should be underlined that the representation of the eigen-states $\boldsymbol{\omega}_{V,VI}$ and the projectors $\mathbf{P}_{I,II}$ and $\mathbf{P}_{III,IV}$ is the same in any basis in which the direction $\mathbf{e}_1$ is coaxial with the material symmetry direction, therefore in order to specify the orientation of material sample with respect to the laboratory it is sufficient to specify two Euler angles ϕ_1 and ϕ_2.

For any transversely isotropic material one obtains two uniquely specified eigen-projectors $\mathbf{P}_{I,II}$ and $\mathbf{P}_{III,IV}$. The specific form of two eigen-states $\boldsymbol{\omega}_V$ and $\boldsymbol{\omega}_{VI}$, similarly as for the material of tetragonal symmetry depends on the stiffness distributor (4.52), the value of which is the material characteristic for the analyzed material (compare [12]).

Note that we can obtain transversely isotropic material considering also the material of trigonal symmetry if we set the angle $\rho = 0$. In such a case the projector $\mathbf{P}_1 = \mathbf{P}_{I,II}$ project into the space plane deviators (4.59) (they are the pure shears with common shearing plane $\mathbf{e}_1$) while the projector $\mathbf{P}_2 = \mathbf{P}_{III,IV}$ project into the space of pure shears (4.51) with common shearing direction $\mathbf{e}_1$.

The considered **transversely isotropic material** is defined by

1. 4 Kelvin moduli: $\lambda_1 = \lambda_{I,II}$, $\lambda_2 = \lambda_{III,IV}$, both of multiplicity 2, and $\lambda_3 = \lambda_V$, $\lambda_4 = \lambda_{VI}$ of multiplicity 1.
2. One stiffness distributor η which specifies angle ϕ.
3. 2 Euler angles which orient symmetry axis $\mathbf{e}_1$ with respect to laboratory.

The unique spectral decomposition takes the form

$$\mathbf{L} = \lambda_1\mathbf{P}_1 + \lambda_2\mathbf{P}_2 + \lambda_3\mathbf{P}_3(\phi) + \lambda_4\mathbf{P}_4(\phi) \tag{4.60}$$

where $\mathbf{P}_1 = \mathbf{P}_1^{\rm trig}(0)$ and $\mathbf{P}_2 = \mathbf{P}_2^{\rm trig}(0)$ while

$$\mathbf{P}_3(\phi) = \boldsymbol{\omega}_V(\phi) \otimes \boldsymbol{\omega}_V(\phi), \qquad \mathbf{P}_4(\phi) = \boldsymbol{\omega}_{VI}(\phi) \otimes \boldsymbol{\omega}_{VI}(\phi)$$

The representation of the stiffness tensor for the material of transversal isotropy in poly-basis $\mathbf{a}_K$ has the representation similar to orthotropic material with relations (4.54), valid for the tetragonal symmetry and additionally

$$L_{44} = L_{22} - L_{12}, \tag{4.61}$$

therefore, it is specified by 5 independent components. Kelvin moduli λ_K and stiffness distributor η are found as for the material of tetragonal symmetry, Eqs. (4.55)–(4.56), where in view of relation (4.61) one has $\lambda_{III} = \lambda_{IV}$.

There are many engineering materials which can be modelled as transversely isotropic. The classical example is the composite with the reinforcement in the form of elongated aligned fibers [5]. Moreover, as it was already signalled in the introduction, all materials for which the single crystal has the hexagonal symmetry, in view of their elastic anisotropy are transversely isotropic. Examples of such metals are analyzed in Subsec. 4.10.

4.8. Material of cubic symmetry

Material of **cubic symmetry** (symmetry of a cube, Fig. 5) has the following symmetry group ($\mathcal{Q}^{4t}_{\mathbf{L}} \subset \mathcal{Q}^{c}_{\mathbf{L}}$):

$$\mathcal{Q}^{k}_{\mathbf{L}} = \left\{\mathbf{1},\ -\mathbf{1},\ \mathbf{I}_{\mathbf{e}_1},\ \mathbf{R}^{k\pi/2}_{\mathbf{e}_1},\ \mathbf{R}^{k\pi/2}_{\mathbf{e}_2}\right\}. \tag{4.62}$$

The group of trigonal symmetry is also the subset of the cubic symmetry group, however, the symmetry axis is then coaxial with one of the main diagonals of a cube span by the vectors $\mathbf{e}_i$.

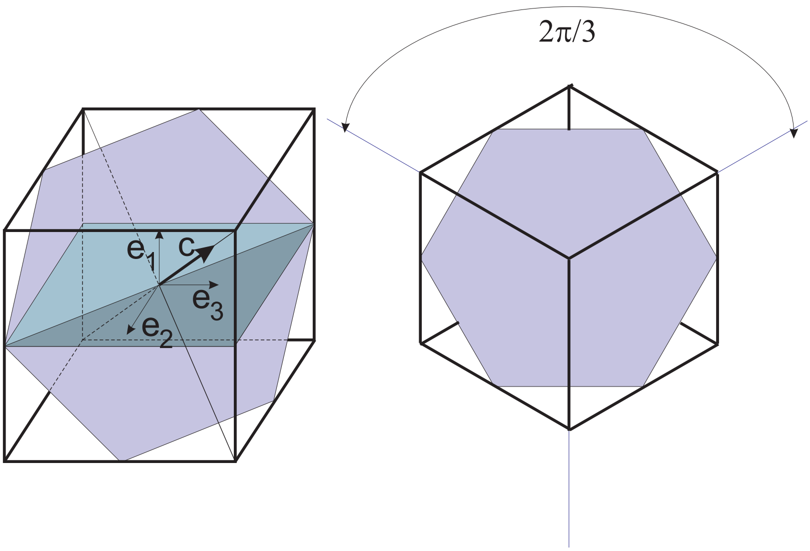

Fig. 5. Schematic representation of a material of cubic symmetry. Note that a crystal elongated along the main diagonal **c** would have trigonal symmetry with the main axis of symmetry coaxial with **c**.

From the symmetry condition (4.6) one eigen-state is obtained

$$\boldsymbol{\omega}_{VI} \sim \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.63}$$

which, as it is easy to note, describes one-dimensional subspace of hydrostatic tensors. From the symmetry conditions (4.4) we obtain two eigen-projectors (compare [19]). A projector $\mathbf{P}_{I,II,III}$ (again in poly-basis $\{\mathbf{a}_I\}$)

$$\mathbf{P}_{I,II,III} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.64}$$

projects into the 3-dimensional deviatoric subspace $\mathcal{P}_{I,II,III}$. Any unit element (not necessarily pure shear) of this subspace can be represented as follows:

$$\boldsymbol{\omega}_{I,II,III} \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \sin\varphi\cos\psi & \sin\varphi\sin\psi \\ \sin\varphi\cos\psi & 0 & \cos\varphi \\ \sin\varphi\sin\psi & \cos\varphi & 0 \end{bmatrix}, \tag{4.65}$$

where $\psi \in \langle 0, 2\pi\rangle$ and $\varphi \in \langle 0, \pi\rangle$. The second projector $\mathbf{P}_{IV,V}$ has the form

$$\mathbf{P}_{IV,V} \sim \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.66}$$

and projects into two-dimensional deviatoric subspace $\mathcal{P}_{IV,V}$. Any unit element of this subspace can be represented as follows:

$$\boldsymbol{\omega}_{IV,V} \sim \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\left(\theta + \dfrac{2\pi}{3}\right) & 0 \\ 0 & 0 & \cos\left(\theta - \dfrac{2\pi}{3}\right) \end{bmatrix}, \tag{4.67}$$

where $\theta \in \langle 0, 2\pi \rangle$. Orthonormal basis in this subspace can be composed of any two tensors $\boldsymbol{\omega}_{IV,V}(\theta_1)$ and $\boldsymbol{\omega}_{IV,V}(\theta_2)$ for which $\theta_2 = \theta_1 + \pi/2$.

For any material of cubic symmetry one obtains two uniquely defined projectors $\mathbf{P}_{I,II,III}$ and $\mathbf{P}_{IV,V}$ as well as one uniquely specified (within a sign) eigen-state $\boldsymbol{\omega}_{VI}$. The decomposition of the space $\mathcal{S}$ into three mutually orthogonal eigen-subspaces is identical for any material of cubic symmetry (there are no stiffness distributors). Material of trigonal symmetry reduces to the material of cubic symmetry if we set

$$\phi = \phi^0, \qquad \rho = \rho^0, \qquad \lambda_1^{trig} = \lambda_3^{trig} = \lambda_1^{cube}$$

and $\tan \phi^0 = \tan \rho^0 = \sqrt{2}$. Note that in this case the stiffness distributor $\eta_2 = 0$.

The considered **material of cubic symmetry** is specified by

1. 3 Kelvin moduli: $\lambda_1 = \lambda_{I,II,III}$ of multiplicity 3, $\lambda_2 = \lambda_{IV,V}$ of multiplicity 2 and $\lambda_3 = \lambda_{VI}$ of multiplicity 1.
2. 0 stiffness distributors.
3. 3 Euler angles which orient symmetry axes $\mathbf{e}_i$ with respect to laboratory.

The unique spectral decomposition takes the form

$$\mathbf{L} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3, \tag{4.68}$$

where

$$\mathbf{P}_1 = \mathbb{I}^S - \mathbb{K}, \qquad \mathbf{P}_2 = \mathbb{K} - \mathbb{I}_P, \qquad \mathbf{P}_3 = \mathbb{I}_P = \frac{1}{3}\mathbf{1} \otimes \mathbf{1}$$

and $\mathbb{K} = \sum_{i=1}^{3} \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_i$.

The representation of the stiffness tensor for the material of cubic symmetry has the form similar to an orthotropic material with additional relations between components, namely

$$L_{23} = L_{13} = L_{12}, \qquad L_{33} = L_{22} = L_{11}, \qquad L_{66} = L_{55} = L_{44}, \tag{4.69}$$

where

$$\lambda_{VI} = \lambda_3 = L_{11} + 2L_{12}, \qquad \lambda_{IV,V} = \lambda_2 = L_{11} - L_{12}$$

and

$$\lambda_{I,II,III} = \lambda_1 = L_{44},$$

therefore, it is specified by 3 independent components. Single crystals of Cu or Al are of cubic symmetry. Austenite phase, high-symmetry phase in shape memory alloys, e.g. NiTi, CuZnAl, NiMnGa, usually exhibit cubic symmetry.

4.9. Isotropic material

As it was already stated in Subsec. 4.1, the symmetry group of such material is the whole orthogonal group $\mathcal{Q}$. For **isotropic material** (Fig. 4) fulfillment of condition (4.6) leads to the hydrostatic eigen-state (4.63), while symmetry condition (4.4) leads to the projector being the sum of projectors (4.64) and (4.66) derived for the cubic symmetry, namely

$$\mathbf{P}_d = \mathbf{P}_{I,II,III} + \mathbf{P}_{IV,V} = \mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}. \tag{4.70}$$

This projector projects the II-nd order tensor into the 5-dimensional subspace of deviators. Its representation in the poly-basis $\{\mathbf{a}_I\}$ composed of diads of basis vectors of any orthonormal basis $\{\mathbf{e}_i\}$ is the same and has the form

$$\mathbf{P}_d \sim \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{4.71}$$

The considered **isotropic material** is specified by

1. 2 Kelvin moduli: $\lambda_1 = \lambda_{I,II,III,IV,V}$ of multiplicity 5 and $\lambda_2 = \lambda_{VI}$ of multiplicity 1.
2. 0 stiffness distributors.
3. 0 Euler angles (they are not needed because all material directions are equivalent).

The unique spectral decomposition takes the form

$$\mathbf{L} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2, \tag{4.72}$$

where

$$\mathbf{P}_1 = \mathbf{P}_d = \mathbb{I}^S - \mathbb{I}_P, \qquad \mathbf{P}_2 = \mathbb{I}_P = \frac{1}{3}\mathbf{1} \otimes \mathbf{1}.$$

The representation of the stiffness tensor for the isotropic material is obtained from the stiffness tensor for cubic symmetry with additional relation

$$L_{44} = L_{11} - L_{12} = \lambda_{I,II,III,IV,V} = \lambda_1; \tag{4.73}$$

therefore, it is specified by 2 independent components.

In Table 1 I and II structural index is provided for all 8 symmetry groups. In [14] the structural indices have been derived for the volumetrically isotropic materials (that is with the so-called Burzyński constraint) for all elastic symmetry groups. For such materials, the hydrostatic tensor (4.63) is one of the eigen-states. It results in reduction of the number of stiffness distributors. Note that for such materials the elasticity tensor is coaxial with the isotropic elasticity tensor.

Table 1. I and II-structural index for all symmetry classes of linear elastic materials.

Symmetry group	I structural index	II structural index	Number of parameters
full anisotropy	$\langle 1+1+1+1+1+1\rangle$	$[6+12+3]$	21
monoclinic symmetry (symmetry of a prism with irregular basis)	$\langle (1+1+1+1)+1+1\rangle$	$[6+6+3]$	15
orthotropy (symmetry of a prism with a rectangular basis)	$\langle (1+1+1)+1+1+1\rangle$	$[6+3+3]$	12
trigonal symmetry (symmetry of an elongated cube)	$\langle (1+1)+(2+2)\rangle$	$[4+2+3]$	9
tetragonal symmetry (symmetry of a prism with a square basis)	$\langle (1+1)+1+1+2\rangle$	$[5+1+3]$	9
transversal symmetry (cylindrical)	$\langle (1+1)+2+2\rangle$	$[4+1+2]$	7
cubic symmetry (symmetry of a cube)	$\langle 1+2+3\rangle$	$[3+0+3]$	6
isotropy	$\langle 1+5\rangle$	$[2+0+0]$	2

4.10. Examples

We apply the derived formulae for assessment of intensity of an elastic anisotropy of single crystals of selected metals and alloys. The intensity of anisotropy is here intuitively meant as a departure of the material behaviour from the isotropic one, i.e. strong variation of elastic properties depending on the direction in which they are measured. More information concerning this issue can be found e.g. in [18, 20, 24]. It should be underlined that in general, the intensity of an anisotropy is not equivalent to the notion of low or high symmetry of material. Material of high symmetry (e.g. cubic) can exhibit strong anisotropy, e.g. strong variation of directional Young modulus [19] and vice versa: the anisotropy of material of low symmetry can be weak.

In Table 2 the independent components of the elasticity tensor for single crystals of selected materials are collected. The hcp materials (Mg, Zn, Zr, Ti metals and α_2-Ti_3Al intermetallic) exhibit the hexagonal lattice symmetry, therefore, the stiffness and compliance tensors have the form equivalent to the transversal isotropy case with 5 independent components in anisotropy axes, Subsec. 4.7. In the case of crystal of tetragonal symmetry (γ-TiAl intermetallic) one has 6 independent components, Subsec. 4.6. High symmetry metals such as copper and aluminum are fcc materials of cubic symmetry with three independent components of $\mathbf{L}$.

Table 2. Elastic constants [GPa] of single crystals for selected metals and alloys of high specific stiffness and some fcc materials (axis 1 is the main symmetry axis).

Material	L_{2222}	L_{2233}	L_{1122}	L_{1111}	L_{1212}	L_{3232}
Mg [1]	59.3	25.7	21.4	61.5	16.4	
Zn [1]	163.7	36.4	53.0	63.5	38.8	
Zr [30]	143.5	72.5	65.4	164.9	32.1	
Ti [29, 31]	163.9	91.3	68.9	181.6	47.2	
α_2-Ti_3Al [31, 21]	175	88.7	62.3	220	62.6	
γ-TiAl[21]	183	74.1	74.4	178	105	78.4
Cu [1]	171.0	122.0			69.1	
Al [1]	186	157			42	

In Table 3 we provide the invariants resulting from spectral decomposition of the corresponding elasticity tensors for these materials [13] (relation between L_{ijkl} and L_{KL} components is specified in the Appendix by (A.2)). The following conclusions result from the analysis of this table:

- All analyzed metals and alloys, with exception of Zn, are close to be a volumetrically isotropic materials (ξ is close to zero). Note that Cu and Al, being cubic materials, are volumetrically isotropic exactly.
- In view of above property, the intensity of elastic anisotropy[4)] can be assessed comparing the Kelvin moduli λ_I, λ_{II}, ..., λ_V, or more specifically their properly defined ratios, e.g $\lambda_K/\lambda_{\max}$ where $\lambda_{\max} = \max\{\lambda_I, \ldots, \lambda_V\}$. For example, one observes that elastic anisotropy of Mg or Al crystals is not strong and it is strong for Zn or Cu. Note that introduction of such indicators of the intensity of the elastic anisotropy generalizes the

[4)] Note that if $\xi = 0$ and $\lambda_I = \lambda_{II} = ... = \lambda_V$, the material is isotropic.

anisotropy factor introduced for cubic crystals by Zener [32]: $A = (L_{1111} - L_{1122})/(2L_{1212})$. As it could be easily verified, this factor is the ratio of deviatoric Kelvin moduli of cubic crystal, namely $A = \lambda_2^{\text{cub}}/\lambda_1^{\text{cub}}$.

Table 3. Kelvin moduli λ_K [GPa], a stiffness distributor $\xi^3 = \sqrt{2}\eta$ (Eq. (4.52)) and $\Phi = \arctan(3\xi)$ obtained by spectral decomposition of the local elasticity tensor for single crystals of selected metals and alloys [13].

<table>
<tr><th>Material</th><th>λ_{VI}</th><th>λ_V</th><th>λ_{IV}</th><th>λ_{III}</th><th>$\lambda_{II} = \lambda_I$</th><th>ξ</th><th>Φ [°]</th></tr>
<tr><td>Mg</td><td>105.7</td><td>40.8</td><td colspan="2">33.6</td><td>32.8</td><td>−0.0051</td><td>−0.87</td></tr>
<tr><td>Zn</td><td>233.2</td><td>30.4</td><td colspan="2">127.3</td><td>77.6</td><td>−0.0674</td><td>−11.43</td></tr>
<tr><td>Zr</td><td>286.4</td><td>94.5</td><td colspan="2">71.0</td><td>64.2</td><td>0.0117</td><td>2.01</td></tr>
<tr><td>Ti</td><td>322.6</td><td>114.2</td><td colspan="2">72.6</td><td>94.4</td><td>−0.0035</td><td>−0.61</td></tr>
<tr><td>α_2-Ti_3Al</td><td>332.6</td><td>151.1</td><td colspan="2">86.4</td><td>125.2</td><td>0.0161</td><td>2.77</td></tr>
<tr><td>γ-TiAl</td><td>330.0</td><td>105.1</td><td>108.9</td><td>156.8</td><td>210</td><td>−0.0033</td><td>−0.56</td></tr>
<tr><td>Cu</td><td>415.0</td><td colspan="2">49.0</td><td colspan="2">138.2</td><td>0</td><td>0</td></tr>
<tr><td>Al</td><td>228.9</td><td colspan="2">46.5</td><td colspan="2">56.6</td><td>0</td><td>0</td></tr>
</table>

5. Conclusions

In the paper, the spectral theorem for the elasticity tensor has been thoroughly discussed. The main aim of the work was the clarification of the issue of invariance of the spectral decomposition. Therefore, the forms of the decomposition for all elastic symmetry groups have been derived in an original way by imposing the symmetry conditions upon the orthogonal projectors, instead of the stiffness tensor itself. Thanks to that, the uniqueness of the orthogonal projectors for the considered Hooke's tensor in contrast to the non-uniqueness of eigen-states has been demonstrated. For completeness of the review, the number of independent eigenvalues (Kelvin moduli) and the corresponding orthogonal projectors have been explicitly outlined for each elastic symmetry class. Finally, the spectral decomposition of the stiffness tensor has been derived for single crystals of the selected metals and alloys.

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Appendix

The space $\mathcal{S}$ of symmetric second-order tensors possesses all the properties of the six-dimensional Euclidean space with the scalar product defined as follows:

$$\bigwedge_{\mathbf{a},\mathbf{b}\in\mathcal{S}} \qquad \mathbf{a}\cdot\mathbf{b} = \mathrm{tr}(\mathbf{ab}) = a_{ij}b_{ij},$$

where a_{ij}, b_{ij}, $i,j = 1,2,3$ are components of tensors $\mathbf{a}$ and $\mathbf{b}$ in some orthonormal basis $\{\mathbf{e}_i\}$ in the three-dimensional physical space. Therefore, any second-order tensor has all the properties of the vector in the six-dimensional Euclidean space.

Due to this property of $\mathcal{S}$ it is possible to select in $\mathcal{S}$ a subset of six mutually orthogonal and normalized tensors $\{\mathbf{a}_K\}$, $K = I,\ldots,VI$ which constitute the basis. One of the possible bases is the following orthonormal subset of basis diads $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ of the form:

$$\begin{aligned}
\mathbf{a}_I &= \mathbf{e}_1\otimes\mathbf{e}_1 & \mathbf{a}_{IV} &= \frac{1}{\sqrt{2}}(\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2),\\
\mathbf{a}_{II} &= \mathbf{e}_2\otimes\mathbf{e}_2, & \mathbf{a}_{V} &= \frac{1}{\sqrt{2}}(\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_1),\\
\mathbf{a}_{III} &= \mathbf{e}_3\otimes\mathbf{e}_3, & \mathbf{a}_{VI} &= \frac{1}{\sqrt{2}}(\mathbf{e}_2\otimes\mathbf{e}_1+\mathbf{e}_1\otimes\mathbf{e}_2).
\end{aligned}$$

A basis in the six-dimensional space is called a poly-basis. In the above poly-basis, any symmetric tensor of the second order is specified in the following way:

$$\mathbf{a} = a_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = a_K\mathbf{a}_K, \qquad K = I,\ldots,VI, \qquad \text{where} \qquad \mathbf{a}\cdot\mathbf{b} = a_Kb_K$$

and relations between representations a_{ij} and a_K are given by

$$\begin{aligned}
a_I &= a_{11}, & a_{II} &= a_{22}, & a_{III} &= a_{33},\\
a_{IV} &= \sqrt{2}a_{23}, & a_V &= \sqrt{2}a_{13}, & a_{VI} &= \sqrt{2}a_{12}.
\end{aligned} \tag{A.1}$$

Consequently, the linear projection from the space $\mathcal{S}$ into $\mathcal{S}$ treated as the six-dimensional Euclidean space is described by the second-order tensor belonging to tensorial product $\mathcal{S}\otimes\mathcal{S}$. This reasoning brings us to conclusion that the fourth-order tensor $\mathbf{A}$ that represents this projection in the three-dimensional physical space has all the properties of the second-order tensor in the six-dimensional Euclidean space. Therefore, one can write

$$\mathbf{A} = A_{ijkl}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l = A_{KL}\mathbf{a}_K\otimes\mathbf{a}_L.$$

The set of all basis diads $\{\mathbf{a}_I \otimes \mathbf{a}_J\}$ is the basis in the space $\mathcal{S} \otimes \mathcal{S}$. Components A_{KL} depend on components A_{ijkl} of the IV-th order tensor $\mathbf{A}$ in the basis $\{\mathbf{e}_i\}$ in the physical space, in the following way:

$$
\text{(A.2)} \quad [A_{KL}] = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & \sqrt{2}A_{1123} & \sqrt{2}A_{1113} & \sqrt{2}A_{1112} \\ A_{2211} & A_{2222} & A_{2233} & \sqrt{2}A_{2223} & \sqrt{2}A_{2213} & \sqrt{2}A_{2212} \\ A_{3311} & A_{3322} & A_{3333} & \sqrt{2}A_{3323} & \sqrt{2}A_{3313} & \sqrt{2}A_{3312} \\ \sqrt{2}A_{2311} & \sqrt{2}A_{2322} & \sqrt{2}A_{2333} & 2A_{2323} & 2A_{2313} & 2A_{2312} \\ \sqrt{2}A_{1311} & \sqrt{2}A_{1322} & \sqrt{2}A_{1333} & 2A_{1323} & 2A_{1313} & 2A_{1312} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & \sqrt{2}A_{1233} & 2A_{1223} & 2A_{1213} & 2A_{1212} \end{bmatrix}.
$$

The following products can be obtained in two alternative, but fully equivalent ways ($\mathbf{a}, \mathbf{b} \in \mathcal{S}$; $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{S} \otimes \mathcal{S}$):

$$\mathbf{a} \cdot \mathbf{b} = a_{ij} b_{ij} = a_K b_K,$$

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{a} \Leftrightarrow b_{ij} = A_{ijkl} a_{kl} \quad \text{or} \quad b_K = A_{KL} a_L,$$

$$\mathbf{D} = \mathbf{A} \circ \mathbf{B} \Leftrightarrow D_{ijkl} = A_{ijmn} B_{mnkl} \quad \text{or} \quad D_{KL} = A_{KM} B_{ML},$$

where a_{ij}, b_{ij}, A_{ijkl}, B_{ijkl}, D_{ijlk} and a_K, b_K, A_{KL}, B_{KL}, D_{KL} are related by Eqs. (A.1) and (A.2).

It should be stressed that, due to the fact that the tensor $\mathbf{A}$ represents linear projection between spaces of the symmetric second-order tensors, one obtains $A_{ijkl} = A_{jikl} = A_{ijlk}$. Note that in the case of the stiffness tensor $\mathbf{L}$ and the compliance tensor $\mathbf{M}$, additionally one has to do with diagonal symmetry, $A_{KL} = A_{LK}$ $(A_{ijkl} = A_{klij})$.

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