ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSLATIONS • 36, 1, 157–166, 1988
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

A NUMERICAL SOLUTION FOR THE HEAT TRANSFER IN NON-NEWTONIAN FLOW PAST A WEDGE WITH NON-ISOTHERMAL SURFACE

R. VASANTHA (BANGALORE), I. POP (CLUJ) and G. NATH (BANGALORE)

In this paper, we present results obtained by using a numerical method for calculating the development of the thermal boundary layer in a non-Newtonian flow past a wedge having a step discontinuity in surface temperature. In particular, solutions are determined for small values of the Prandtl number and the method is shown to be very accurate in comparison with previous analytical solutions.

1. INTRODUCTION

In addition to the well-known Newtonian fluid, there are real fluids used in the mechanical and chemical industries. Most real fluids exhibit so-called non-Newtonian behaviour, which means that the shear stress is no longer linearly proportional to shear rate. The flow and temperature fields for non-Newtonian fluids have been the subject of many investigations for the past four decades and the number of journal articles dealing with such fluids has been increasing rapidly. Our survey of the literature of non-Newtonian fluids indicated two recent review articles by CHO and HARTNETT [1], and SHENOY and MASHELKAR [2].

During recent years, one of the most interesting problem for a purely viscous fluid is that of determination of the temperature distribution and the rate of heat transfer through the laminar boundary layer in the flow over a body with non-uniform surface temperature; notable papers are those from Refs. [3—8]. The case of a non-Newtonian boundary layer wedge flow with non-uniform surface temperature has been investigated by CHEN and RADULOVIC [9] by means of a series solution method. Although their solution gives good results for large and moderate values of the Prandtl number, Pr, it cannot be relied upon for Pr < 1, where the solution tends to underestimate the value of the wall temperature gradient, the discrepancy increasing as Pr becomes smaller.

The object of the work reported below is to present a numerical solution of Chen and Radulovic's problem by using a very efficient and accurate finite-difference scheme devised by KELLER [10, 11] and also to obtain results for small Prandtl numbers.

2. Analysis

The formulation relates to a wedge of angle $\pi\beta$ immersed in a steady incompressible laminar boundary layer flow of a non-Newtonian fluid which obeys the power-law model of index n. The x-axis is measured along the surface of the wedge and the y-axis is normal to it, respectively (see Fig. 1). Suppose

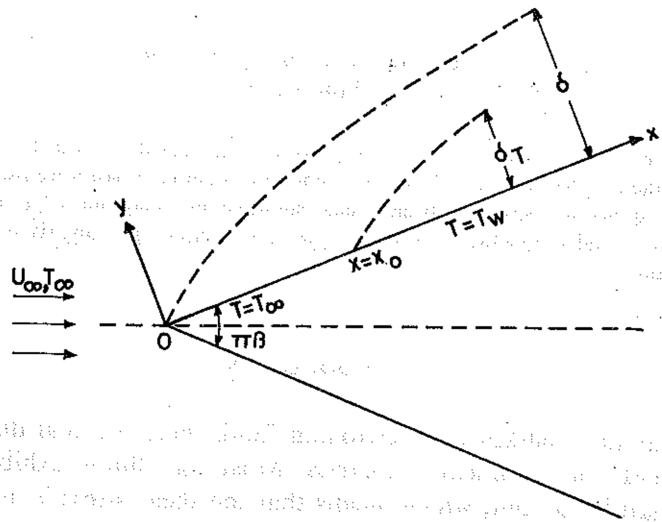

Fig. 1. Physical model and coordinate system.

that the free stream velocity outside the boundary layer in the x-direction is $U = Cx^m$, where C is a constant and $m = \beta/(2-\beta)$. The leading portion of the wedge surface of length x_0 measured from the forward stagnation point is at the same temperature T_∞ of the incoming free stream fluid and the remaining portion of the wedge surface, $x > x_0$, is maintained at a uniform temperature T_w which differs from T_∞. If $f(\eta)$ represents the dimensionless stream function and $\theta(\xi, \eta)$ the dimensionless temperature function in the boundary layer, then these functions satisfy the following equations (see [9])

$$f''' + f(f'')^{2-n} + \alpha_0[1-(f')^2](f'')^{1-n} = 0, \tag{2.1}$$

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1,$$

and

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{\alpha_3}{b}\,\mathrm{Pr}\,(1-\xi^3)^{\alpha_4}\left(\alpha_5\xi f + \frac{1-\xi^3}{2\xi^2}\,\eta\,\frac{df}{d\eta}\right)\frac{\partial\theta}{\partial\eta} = \frac{\alpha_3}{2b}\,\mathrm{Pr}\,\frac{(1-\xi^3)^{\alpha_4+1}}{\xi}\,\frac{df}{d\eta}\,\frac{\partial\theta}{\partial\xi} \tag{2.2}$$

subject to the boundary conditions, for $\xi > 0$

$$\theta(\xi, 0) = 1 \qquad \text{and} \qquad \theta(\xi, \infty) = 0, \tag{2.3}$$

where the coordinate transformation is

$$\xi = [1-(x_0/x)^c]^{1/3}, \qquad \bar{\eta} = b\eta/\xi, \qquad \eta = \alpha_1 y x^{\alpha_2},$$

in which

$$a = f''(0), \qquad b = (a\alpha_3 \,\mathrm{Pr}/6)^{1/3}, \qquad c = 3(m+1)/(2n+2).$$

The primes refer to differentiation with respect to η and the constants α_0, α_3, α_4 and α_5 are given by

$$\alpha_0 = \frac{m(n+1)}{m(2n-1)+1}, \qquad \alpha_3 = \left(\frac{m+1}{n+1}\right)\left[\frac{n(n+1)}{m(2n-1)+1}\right]^{2/(n+1)},$$

$$\alpha_4 = \frac{2(1-n)(3m-1)}{3(m+1)}, \qquad \alpha_5 = \frac{m(2n-1)+1}{m+1}.$$

Here Pr is the generalized Prandtl number.

3. Numerical solution

First, we note that Eq. (2.1) was solved analytically by HSU and COTHERN [12] using MEKSYN's method [13], and numerically by CHEN and RADULOVIC [9]. Our aim here is to solve numerically both Eqs. (2.1) and (2.2). To this end we note that since ξ is bounded between 0 and 1, the coefficient of $\partial\theta/\partial\xi$ is always positive and Eq. (2.2) may be solved by „marching" downstream in the ξ — direction using a finite-difference method suitable for parabolic differential equations. This algorithm is based upon a KELLER [10, 11] box scheme whereby accurate results are obtained using extrapolation on crude grids. In detail, Eq. (2.2) is solved as the set of two simultaneous equations

$$\frac{\partial\theta}{\partial\bar{\eta}} = g, \tag{3.1}$$

$$\frac{\partial g}{\partial\bar{\eta}} + \frac{\alpha_3}{b}\,\mathrm{Pr}\,(1-\xi^3)^{\alpha_4}\left(\alpha_5 \xi f + \frac{1-\xi^3}{2\xi^2}\,\eta\,\frac{df}{d\bar{\eta}}\right) g = \frac{\alpha_3}{2b}\mathrm{Pr}\frac{(1-\xi^3)^{\alpha_4+1}}{\xi}\frac{df}{d\bar{\eta}}\frac{\partial\theta}{\partial\xi}. \tag{3.2}$$

A net is placed on the $(\xi, \bar{\eta})$ plane defined by

$$\xi_0 = 0, \qquad \xi_i = \xi_{i-1} + k_i, \qquad i = 1, 2, \ldots$$

$$\bar{\eta}_0 = 0, \qquad \bar{\eta}_j = \bar{\eta}_{j-1} + h_j, \qquad j = 1, 2, \ldots, N.$$

If w_j^i denotes the value of any variable at $(\xi_i, \bar{\eta}_j)$, then variables and derivatives of Eq. (3.2) at $(\xi_{i-1/2}, \bar{\eta}_{j-1/2})$ are replaced by

$$w_{j-1/2}^{i-1/2} = \frac{1}{4}(w_j^i + w_{j-1}^i + w_j^{i-1} + w_{j-1}^{i-1}),$$

$$\left(\frac{\partial w}{\partial\xi}\right)_{j-1/2}^{i-1/2} = \frac{1}{2k_i}(w_j^i + w_{j-1}^i - w_j^{i-1} - w_{j-1}^{i-1}),$$

$$\left(\frac{\partial w}{\partial\bar{\eta}}\right)_{j-1/2}^{i-1/2} = \frac{1}{2h_j}(w_j^i + w_j^{i-1} - w_{j-1}^i - w_{j-1}^{i-1}),$$

where $\xi_{i-1/2}=\xi_{i-1}+\frac{1}{2}k_i$ and $\bar{\eta}_{j-1/2}=\bar{\eta}_{j-1}+\frac{1}{2}h_j$. Equation (3.1), as it does not involve ξ explicitly, was centered at $(\xi_i, \bar{\eta}_{j-1/2})$ using the relations

$$w^i_{j-1/2}=\frac{1}{2}(w^i_j+w^i_{j-1}), \qquad \left(\frac{\partial w}{\partial \bar{\eta}}\right)^i_{j-1/2}=\frac{1}{h_j}(w^i_j-w^i_{j-1}).$$

The boundary conditions then imply

$$\theta^i_0=1 \quad \text{and} \quad \theta^i_N=0. \tag{3.3}$$

If the problem has been solved up to ξ_{i-1}, then we have $(2N+2)$unknowns (θ^i_j, g^i_j), $j=0, 1, ..., N$. These are nonlinear algebraic equations, which are solved using Newtonian's iteration, the values of the variables at ξ_{i-1} being used as an initial iterate. To start the integration procedure, we require initial profiles of θ and $\partial\theta/\partial\bar{\eta}$. CHEN and RADULOVIĆ [9] showed that a series expansion in powers of ξ (small) can be found to satisfy Eq. (2.2) subjected to (2.3) in the form

$$\theta(\xi, \bar{\eta})=\sum_{i=0}^{\infty}\theta_i(\bar{\eta})\,\xi^i. \tag{3.4}$$

The functions $\theta_i(\bar{\eta})$ satisfy ordinary differential equations of rapidly increasing complexity and $\theta_0(\bar{\eta})$ is given by

$$\theta_0(\bar{\eta})=1-\frac{\gamma(1/3, \bar{\eta}^3)}{\Gamma(1/3)}, \tag{3.5}$$

where γ and Γ are, respectively, incomplete and complete gamma functions. Now $\theta_0(\bar{\eta})$ is taken as the initial profile of $\theta(\xi, \bar{\eta})$ and from Eqs. (3.4) and (3.5) we find that

$$\left(\frac{\partial\theta}{\partial\bar{\eta}}\right)_{\xi=0}=-\frac{3}{\Gamma(1/3)}\exp(-\bar{\eta}^3).$$

To this end, we mention that the quantity of physical significance is the heat transfer rate which is defined through the Nusselt number

$$\mathrm{Nu}\,\mathrm{Re}^{-1/(n+1)}=C_{mn}\,\mathrm{Pr}\,\xi^{-1}[-(\partial\theta/\partial\bar{\eta})_{\bar{\eta}=0}], \tag{3.6}$$

where

$$C_{mn}=\left\{\frac{a}{6}\left(\frac{m+1}{n+1}\right)\left[\frac{m(2n-1)+1}{n(n+1)}\right]^{1/(n+1)}\right\}^{1/3}.$$

4. RESULTS AND DISCUSSION

We have studied the effect of step sizes $\Delta\bar{\eta}$ and $\Delta\xi$, and the edge of the boundary layer $\bar{\eta}_\infty$ on the solution in order to optimize them. The results presented here are independent of $\Delta\bar{\eta}$, $\Delta\xi$ and $\bar{\eta}_\infty$ at least up to the 4th decimal place. Although the results have been obtained for various values of the parameters, only some representative results are presented here.

In order to assess the accuracy of our method, we have compared our skin friction, $f''(0)$, results for non-Newtonian fluids ($n \neq 1$) with those of CHEN and RADULOVIC [9], and they are found to be in excellent agreement (they agree up to the 4th decimal place). Hence, the comparison is not shown here. The heat transfer, Eq. (3.6), results for Newtonian fluids ($n = 1$) for $\beta = 1$ and $\xi = 1$ (or $x_0/x = 0$) have been compared with those of ELZY and SISSON [14], and ECKERT and DRAKE [15], and for non-Newtonian fluids ($n = 0.5$) for $\beta = 0.5$ and 1.0, and $0 < \xi \leqslant 1$ with those of CHEN and RADULOVIC [9]. Our results are found to be again in excellent agreement with those of [14, 15]. However, they differ from those of [9]. The maximum difference is about 12 per cent when x_0/x approaches unity, but the difference decreases as x_0/x decreases. This difference is attributed to series solution method used by CHEN and RADULOVIC [9]. The comparison is shown in Table 1 and Fig. 2.

Table 1. Comparison of heat transfer results (Nu $Re^{-1/2}$) **for** $\beta = 1.0$, $n = 1.0$ **and** $\xi = 1.0$.

Pr	0.7	0.8	1.0
ELZY and SISSON [14]	0.496	0.523	0.570
ECKERT and DRAKE [15]	0.496	0.523	0.570
CHEN and RADULOVIC [9]	0.486	0.514	0.562
Present results	0.496	0.523	0.572

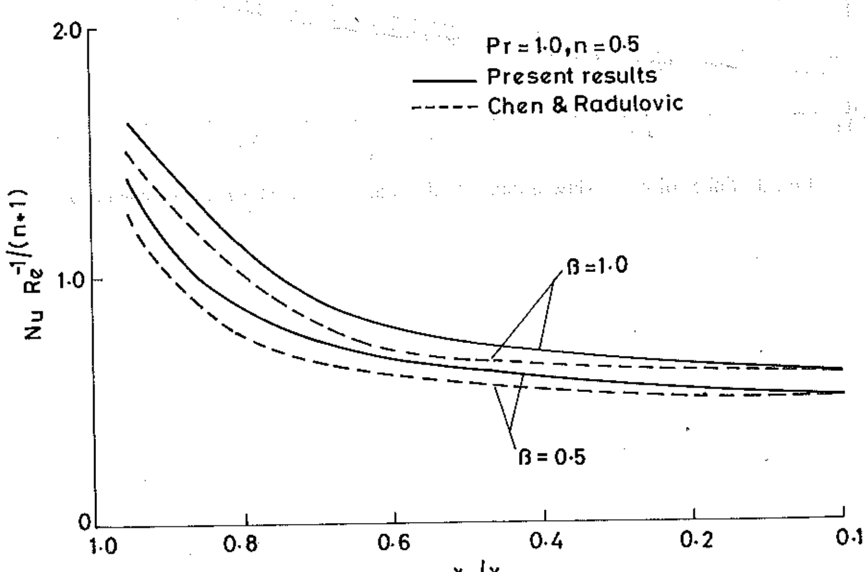

FIG. 2. Comparison of heat transfer results.

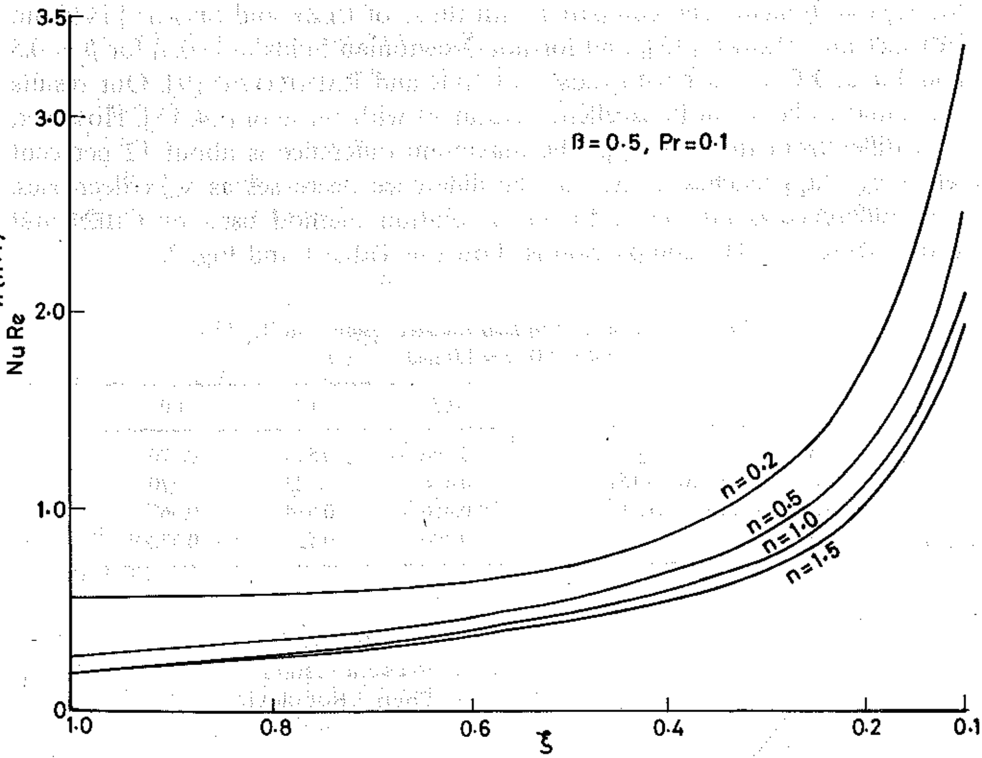

FIG. 3. Effect of power-law index n and distance ξ on heat transfer coefficient.

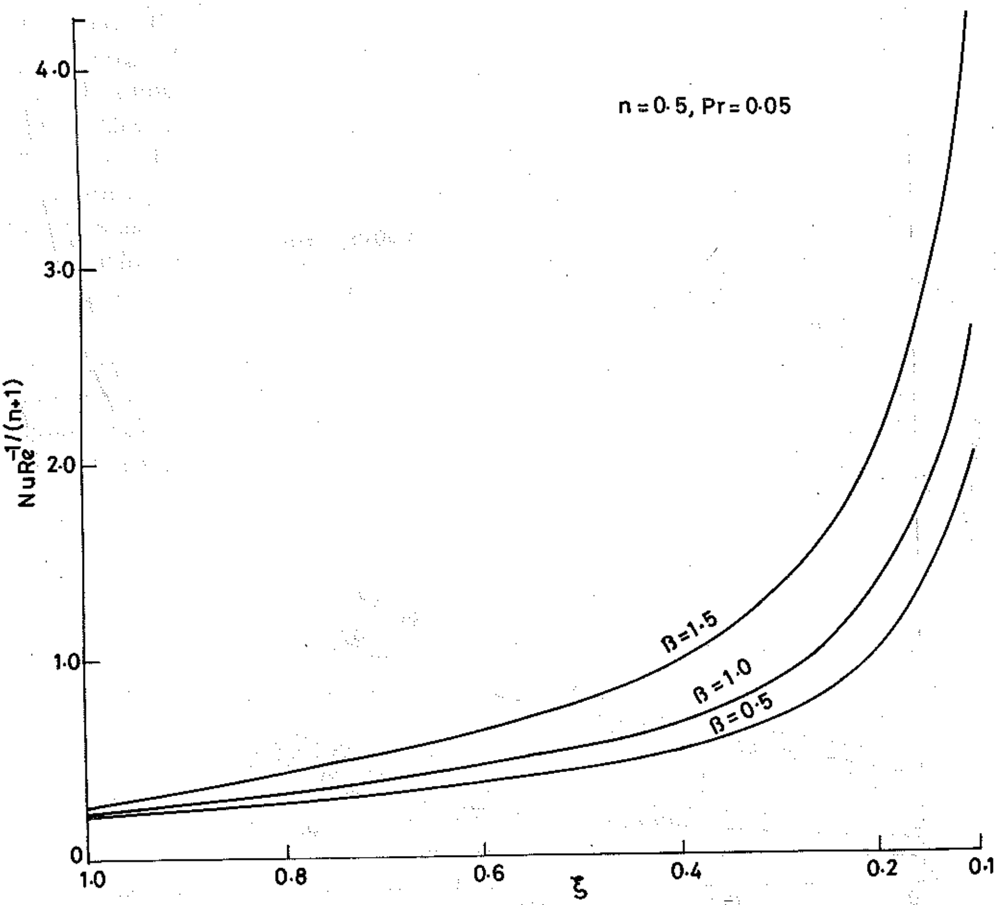

FIG. 4. Effect of pressure gradient parameter β on heat transfer coefficient.

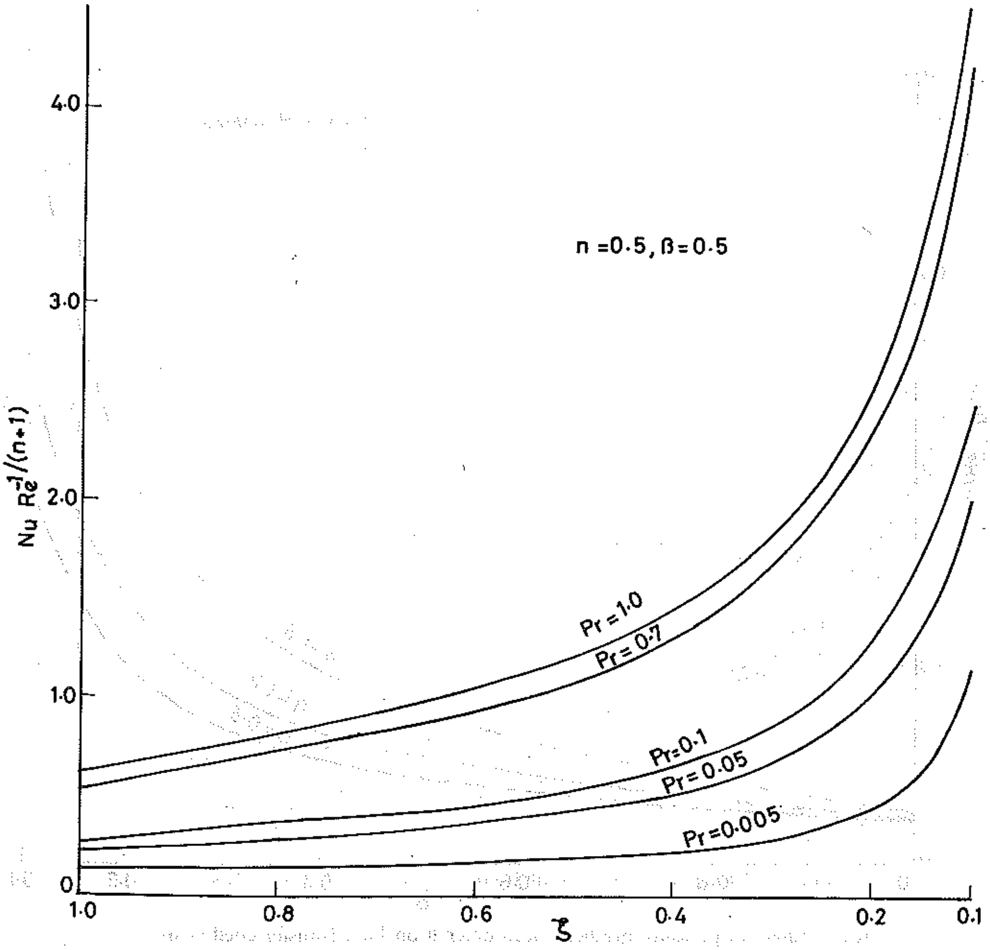

FIG. 5. Effect of the Prandtl number, Pr, on heat transfer coefficient.

The effect of power-law index n and the distance ξ on the heat transfer coefficient given by Eq. (3.6) is shown in Fig. 3. It is observed that for all ξ, the heat transfer is greater for the pseudoplastic fluid ($n < 1$) and smaller for the dilatant fluid ($n > 1$) as compared to the Newtonian fluids ($n = 1$). Also for a given non-Newtonian fluid (i.e., for fixed n), the heat transfer increases as ξ decreases, the effect being more pronounced for small ξ, i.e., away from the leading edge. The increase in the heat transfer is due to the reduction in the thermal boundary layer thickness as n or ξ decreases.

The effect of pressure gradient parameter β on the heat transfer coefficient is presented in Fig. 4. It is seen that the heat transfer increases as β increases. The reason for this behaviour is due to the reduction in the thermal boundary layer thickness as β increases (i.e., increase in the favourable pressure gradient).

The effect of the Prandtl number, Pr, on the heat transfer coefficient is shown in Fig. 5. We note from this figure that for a given ξ, the Nusselt number decreases as Pr decreases. This is due to the fact that a lower Prandtl number fluid has a relatively high thermal conductivity which promotes conduction and thereby reduces the flow convection. This results in increase in the thermal boundary layer thickness and reduction in the convective heat transfer at the wall. The effect of high Prandtl number is just reverse. Also the effect of ξ is pronounced only for small ξ, i.e., for large x_0/x.

5. Conclusions

The results found in this paper indicate that the heat transfer is greater for pseudoplastic fluids and smaller for the dilatant ones as compared to Newtonian fluids. The heat transfer increases as the pressure gradient parameter or the Prandtl number or the distance along the wedge surface increase. The rate of increase is higher at large distance. Finally, we have shown here that the series solution method does not give accurate results for large ξ and it always underestimates the value of the heat transfer coefficient.

References

1. Y. I. Cho and J. P. Hartnett, *Non-Newtonian fluids in circular pipe flow*, Advances in Heat Transfer, **15**, 59—141, 1982.
2. A. V. Shenoy and R. A. Mashelkar, *Thermal conduction in non-Newtonian fluids*, Advances in Heat Transfer, **15**, 143—225, 1982.
3. B. T. Chao and L. S. Cheema, *Forced convection in wedge flow with non-isothermal surfaces*, Int. J. Heat Mass Transfer, **14**, 1363—1375, 1971.
4. D. G. Drake and D. S. Riley, *Heat transfer from an isothermal flat plate with an unheated starting length*, ZAMP, **25**, 799—815, 1974.
5. D. G. Drake and D. S. Riley, *On the heat transfer from a wedge with non-isothermal surfaces*, ZAMP, **26**, 199—202, 1975.
6. D. R. Jeng, M. H. Lee and K. J. DeWitt, *Convective heat transfer through boundary layers with arbitrary pressure gradient and non-isothermal surfaces*, Int. J. Heat Mass Transfer, **21**, 499—509, 1978.

7. D. R. Jeng, K. J. DeWitt and M. H. Lee, *Forced convection over rotating bodies with non-uniform surface temperature*, Int. J. Heat Mass Transfer, **22**, 89—98, 1979.
8. K. Unsworth and T. C. Chiam, *A numerical solution for the heat transfer from a wedge with non-isothermal surfaces*, ZAMP, **31**, 493—501, 1980.
9. J. L. S. Chen and P. T. Radulovic, *Heat transfer in non-Newtonian flow past a wedge with non-isothermal surfaces*, J. Heat Transfer, **95**, 498—504, 1973.
10. H. B. Keller, *A new difference scheme for parabolic problems*, In: Numerical Solutions of Partial Differential Equations. II; B. Hubbard [ed.], Academic Press, New York 1971.
11. H. B. Keller, *Numerical methods in boundary layer theory*, Ann. Rev. Fluid Mech., **10**, 417—433, 1978.
12. C. C. Hsu and J. H. Cothern, *Falkner-Skan flows of power-law fluids*, ASME Paper No. 71-FE-35, 1971.
13. D. Meksyn, *New methods in laminar boundary layer theory*, Pergamon Press, New York 1961.
14. E. Elzy and R. M. Sisson, *Tables of similar solutions in the equations of momentum, heat and mass transfer in laminar boundary layer flow*, Enging. Experiment Station Bull., No. 40, Oregon State University, Corvallis 1967.
15. E. R. G. Eckert and R. M. Drake Jr., *Heat and mass transfer*, 2nd Ed., McGraw-Hill, New York 178—184, 1959.

Streszczenie

ROZWIĄZANIE NUMERYCZNE PROBLEMU PRZEPŁYWU CIEPŁA PRZY NIENEWTONOWSKIM OPŁYWIE KLINA O NIEIZOTERMICZNYCH POWIERZCHNIACH

W pracy przedstawiono wyniki obliczeń numerycznych dotyczących rozwoju termicznej warstwy przyściennej podczas nienewtonowskiego opływu klina ze skokową nieciągłością temperatury powierzchni. W szczególności uzyskano rozwiązania dla małych wartości liczby Prandtla stwierdzając zarazem, że proponowana metoda jest dokładniejsza od dawniej stosowanych metod analitycznych.

Резюме

ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ, ПРИ НЕНЬЮТОНОВСКОМ ОБТЕКАНИИ КЛИНА С НЕИЗОТЕРМИЧЕСКИМИ ПОВЕРХНОСТЯМИ

В работе представлены результаты численных расчетов, касающихся развития термического пограничного слоя во время неньютоновского обтекания клина со скачкообразным разрывом температуры поверхности. В частности получены решения для малых значений числа Прандтля, одновременно констатируя, что предлагаемый метод более точный, чем раньше применяемые аналитические методы.

DEPARTMENT OF APPLIED MATHEMATICS
INDIAN INSTITUTE OF SCIENCE, BANGALORE, INDIA
and
FACULTY OF MATHEMATICS
UNIVERSITY OF CLUJ, CLUJ, ROMANIA

Received April 22, 1987.