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TETRAHEDRAL SPACE-TIME ELEMENTS IN THE ANALYSIS OF FORCED VIBRATIONS OF PLATES

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The paper presents a generalization of the space-time element method with a triangular net. Space-time is three-dimensional in dynamic analysis of plates, so instead of the triangular elements, tetrahedrons were used. The space-time tetrahedral element of a moderately thick plate was produced and examined. The advantages of a triangular net, especially those leading to equations uncoupled with respect to the nodes, were conserved.

1. INTRODUCTION

There are a lot of possibilities for the generation of space-time element nets. However, the triangular nets described in the ODEN'S paper [1] have special properties. As the result of using these nets we obtain a triangular matrix for the equations of the space-time element method. The solution of the above set is very simple because a triangular matrix does not require any inversion. The applications of this concept to the different problems of structural dynamics are described in the papers of KĄCZKOWSKI [2] and WITKOWSKI [3, 4]. However, all the mentioned works deal with only one-dimensional problems of mechanics, what enables to formulate them in two space-time dimensions. The work [2] indicates the possibility of solving also the plates, shells and solids but without sample problems. In the recently published BAJER'S paper [5], the tetrahedral space-time elements were considered in the analysis of the disks.

In the presented paper the forced vibrations of moderately thick plates are considered. If in one-dimensional problems (for example tension bar, bending beam, axisymmetrical solids with the plane strains) we have the triangular nets, the analysis of the plates requires using of the tetrahedrons. These elements preserve all advantageous properties of the triangular nets what permits to reduce considerably time of calculation.

2. BASIC FORMULATIONS OF THE THEORY OF MODERATELY THICK PLATES

This theory assumes that a displacement field has three components-deflection w and two independent angles ϑ_x and ϑ_y. We can describe it in the form

$$\mathbf{u} = \{w, \vartheta_x, \vartheta_y\}. \tag{2.1}$$

The strain vector is defined as follows:

$$\underline{\varepsilon}_s = \{\beta_x, \beta_y, \kappa_x, \kappa_y, \kappa_{xy}\}. \tag{2.2}$$

The strain-displacement relations look like

$$\underline{\varepsilon}_s = \underline{\partial}_s \mathbf{u}, \tag{2.3}$$

where the operator matrix has a form which is described in the KĄCZKOWSKI'S work [6]

$$\underline{\partial}_s = \begin{bmatrix} \frac{\partial}{\partial x} & -1 & 0 \\ \frac{\partial}{\partial y} & 0 & -1 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}. \tag{2.4}$$

The stress vector contains the shearing forces, bending and twisting moments. It has the form

$$\underline{\sigma}_s = \{Q_x, Q_y, M_x, M_y, M_{xy}\}. \tag{2.5}$$

The stress-strain relation is

$$\underline{\sigma}_s = \mathbf{E}_s \underline{\varepsilon}_s, \tag{2.6}$$

where $\mathbf{E}_s$ is the constitutive matrix which has the following form for the elastic plates:

$$\mathbf{E}_s = \begin{bmatrix} H & 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 & 0 \\ 0 & 0 & D & \nu D & 0 \\ 0 & 0 & \nu D & D & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)D}{2} \end{bmatrix}. \tag{2.7}$$

In the above formulas we admit

$$H = \frac{5Eh}{12(1+\nu)}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{2.8}$$

where E, ν — Young's and Poisson's modulus, respectively; h — thickness of the plate.

In the dynamic problems, the equilibrium equations should be completed by inertia terms. Introducing of inertia terms in the space-time description requires the increase of dimensions of the strain and stress vectors.

They can be written as follows:

$$\underline{\varepsilon} = \{\underline{\varepsilon}_s, \underline{\varepsilon}_t\}, \quad \underline{\sigma} = \{\underline{\sigma}_s, \underline{\sigma}_t\}, \tag{2.9}$$

where

$$\underline{\varepsilon}_t = \{\beta_t, \kappa_{tx}, \kappa_{ty}\}, \quad \underline{\sigma}_t = \{Q_t, M_{tx}, M_{ty}\}. \tag{2.10}$$

The strain-displacement relation is

$$\underline{\varepsilon}_t = \underline{\partial}_t \mathbf{u}, \tag{2.11}$$

where the operator matrix has the diagonal form

$$\underline{\partial}_t = \frac{\partial}{\partial t}[1, 1, 1]. \tag{2.12}$$

In this case we obtain the following stress-strain relation

$$\underline{\sigma}_t = \mathbf{E}_t \varepsilon_t, \tag{2.13}$$

where the constitutive matrix looks like

$$\mathbf{E}_t = -\left[\varrho h, \frac{\varrho h^3}{12}, \frac{\varrho h^3}{12}\right] = -\left[M, B, B\right], \tag{2.14}$$

and ϱ is a density of the mass.

In the space-time description the full displacement–strain-stress relations have forms

$$\underline{\varepsilon} = \underline{\partial}\mathbf{u}, \quad \underline{\sigma} = \mathbf{E}\varepsilon, \quad \underline{\partial} = \begin{bmatrix} \underline{\partial}_s \\ \underline{\partial}_t \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{E}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_t \end{bmatrix}. \tag{2.15}$$

3. The tetrahedral space-time plate element

We take into consideration a tetrahedral space-time element (Fig. 1). Let us assume that the displacement field of this element can be approximated, using the linear shape functions. The position of the point P is defined by three Cartesian coordinates x, y, t. In patricular coordinates of the apices are

$$P_i(x_i, y_i, t_i), \quad i = 1, 2, 3, 4. \tag{3.1}$$

It is more convenient to use the voluminal coordinates in the later analysis. Voluminal coordinates $L_i (i = 1, 2, 3, 4)$ are defined as follows:

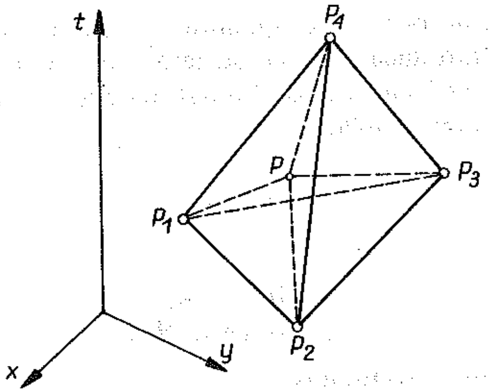

FIG. 1

$$L_i = \frac{V_i}{V}, \tag{3.2}$$

where V is a volume of the whole tetrahedron P_1-P_2-P_3-P_4 and V_i as a volume of the tetrahedron P-P_j-P_k-P_m, $j \neq i$, $k \neq i$, $m \neq i$.

Cartesian coordinates depend linearly upon voluminal coordinates. Let us write it in the form

$$\begin{bmatrix} 1 \\ x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ t_1 & t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}. \tag{3.3}$$

After inversion of above relation we obtain

$$L_i = a_i x + b_i y + c_i t + d_i, \quad i = 1, 2, 3, 4, \tag{3.4}$$

where a_i, b_i, c_i, d_i are constants. The formulas for these terms are presented in the well-known book of ZIENKIEWICZ [7], Chapter 6.

We can describe the operator matrices (2.4) and (2.12), as the functions of voluminal coordinates. The transformation formulas are

$$\begin{aligned} \frac{\partial}{\partial x} &= \sum_{i=1}^{4} \frac{\partial L_i}{\partial x} \frac{\partial}{\partial L_i}, \\ \frac{\partial}{\partial y} &= \sum_{i=1}^{4} \frac{\partial L_i}{\partial y} \frac{\partial}{\partial L_i}, \\ \frac{\partial}{\partial t} &= \sum_{i=1}^{4} \frac{\partial L_i}{\partial t} \frac{\partial}{\partial L_i}. \end{aligned} \tag{3.5}$$

The shape functions can be written as below

$$\mathbf{N} = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4], \tag{3.6}$$

where

$$\mathbf{N}_i = L_i[1, 1, 1].$$

The generalized coordinates vector of the space-time element contains following twelve componets

$$\mathbf{q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}, \quad \mathbf{q}_i = \{q_i^1, q_i^2, q_i^3\}, \tag{3.7}$$

where

$$q_i^1 = w_i, \quad q_i^2 = \vartheta_{xi}, \quad q_i^3 = \vartheta_{yi}.$$

If we still regard the well-known relation

$$\mathbf{K} = \int_V (\partial\mathbf{N})^T \mathbf{E} \partial\mathbf{N} dV \tag{3.8}$$

then substituting Eqs. (2.15) and (3.6) we obtain the formulas for the components of the stiffness matrix.

We take into account that V is a space-time tetrahedron volume, therefore Eq. (3.8) describes the stiffness matrix of space-time element.

It looks like

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} & \mathbf{K}_{24} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} & \mathbf{K}_{34} \\ \mathbf{K}_{41} & \mathbf{K}_{42} & \mathbf{K}_{43} & \mathbf{K}_{44} \end{bmatrix}, \tag{3.9}$$

where the submatrix $\mathbf{K}_{ij}$ can be written

$$\mathbf{K}_{ij} = \begin{bmatrix} K_{ij}^{11} & K_{ij}^{12} & K_{ij}^{13} \\ K_{ij}^{21} & K_{ij}^{22} & K_{ij}^{23} \\ K_{ij}^{31} & K_{ij}^{32} & K_{ij}^{33} \end{bmatrix} V. \tag{3.10}$$

The subindices belong to a set of nodes number, and superindices are conformed to the sequence described in the formula (3.7).

The analytical way for calulation of the integrals (3.8) can be applied and we obtain following formulas for the stiffness components of the space-time element:

$$\begin{aligned} K_{ij}^{11} &= H(a_i a_j + b_i b_j) - M c_i c_j, \\ K_{ij}^{22} &= D\left(a_i a_j + \frac{1-\nu}{2} b_i b_j\right) + \frac{H}{20}(1+\delta_{ij}) - B c_i c_j, \\ K_{ij}^{33} &= D\left(b_i b_j + \frac{1-\nu}{2} a_i a_j\right) + \frac{H}{20}(1+\delta_{ij}) - B c_i c_j, \\ K_{ij}^{12} &= K_{ji}^{21} = -\frac{1}{4} H a_i, \end{aligned} \tag{3.11}$$

$$K_{ij}^{13} = K_{ji}^{31} = -\frac{1}{4}Hb_i,$$

$$K_{ij}^{23} = K_{ji}^{32} = D(a_i b_j + \frac{1-\nu}{2} b_i a_j).$$

In the above formula D, H have forms according to Eq. (2.8), M, B — respectively to Eq. (2.14), a_i, b_i, c_i — to book [7], however δ_{ij} is a Kronecker's symbol.

4. Plate vibrations in the consequence of loading by impact force

In a series of numerical experiments the behaviour of the simply-supported square plate was studied. The geometrical and material properties were assumed as follows:

Span of the plate	$a = 12$ m,
Poisson's modulus	$\nu = 0.3$,
Young's modulus	$E = 2.1 \cdot 10^5$ MPa,
density of the mass	$\varrho = 4000$ kg/m^3,
plate thicknesses	$h = 1$ m, 2 m, 5 m.

All experiments were made for the identical space division of the plate into elements (Fig. 2) and three plate thicknesses (1 m, 2 m, 5 m).

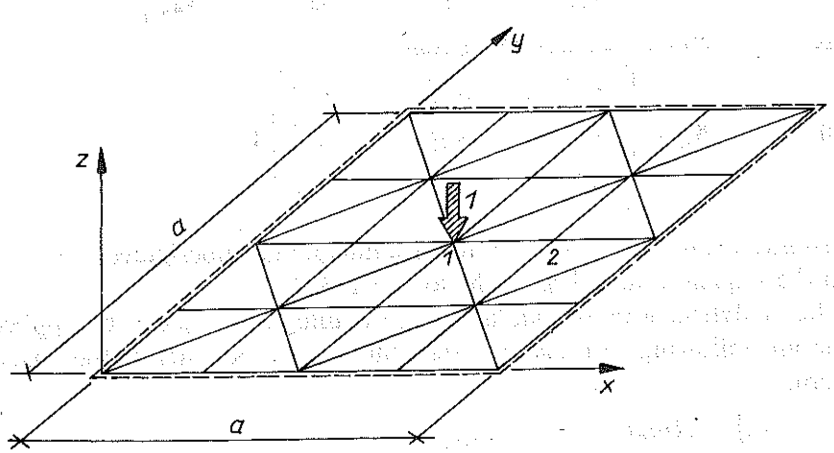

Fig. 2

At first the eigenproblem for the plates was solved. The plates were described using the moderately thick plates theory. The vibration periods of these plates, modelled by the use of the classic finite elements, are presented in the first row of the Table 1.

Table 1

	Vibration period × 10^{-4} [s]		
Thickness of the plate [m]	1.	2.	5.
Moderatorly thick plate. Eigenproblem for finite element method	104	83.4	56.0
Space-time element method. Tetrahedral elements, time step $0.5 \cdot 10^{-4}$ s	105	85	57.5
Thin plate simple supported. Analytical solution	209	104.5	41.8

Next, the free vibrations of the same moderately thick plates were analysed. Initial conditions were assumed as the displacements, corresponding with the first eigenvector, which was calculated earlier. It was computed using the space-time tetrahedral elements, whose net was generated for the base of the space elements, presented in Fig 2. Application of the single-precission gave a stabilization of numerical process only for time step $DT = 2 \cdot 10^{-4}$ s. This process was unstable when time steps were greater or smaller. For the improvement of stable degree it was decided to make calculations with double-precision. In this case, convergence of the result to a certain limit was very good. Analysis of the vibrations enabled to compute the vibration periods, which are presented in the second row of the Table 1.

We can observe slight differences between results obtained by different methods of computing the plates, described using the same theory.

The third row of the Table 1 presents the well-known analytical solutions for the thin plates. Remarkable differences, im comparison with the previous solutions, are caused by the big thickness of considerated plates, what does not allow to use the thin plates theory in this case.

Finally, the simply supported square plate charged by impact force was solved. The force was applied in the middle of the plate. Vibration process for the points *1* and *2* is presented in Fig. 3 using continous and drawing lines, respectively. The same straight lines represented static deflections for these points. We can observe oscillation process around the equilibrium state as well as an effect of wave propagation. UFLAND [8] proved that for an infinite plate the equation of motion has a wave character. However, analytical solutions for finite plates do not permit to regard it.

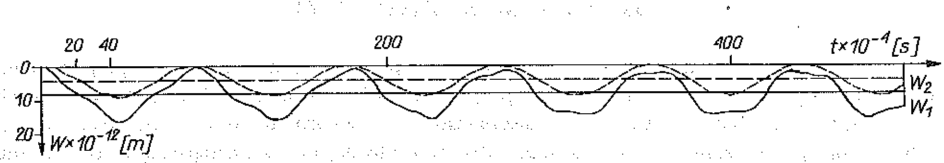

FIG. 3

Triangular nets are very useful not only in the wave propagation problems for one-dimensional systems but also in the dynamic analysis of the plates. Wave effect can be observed particularly in Fig. 4, where variability of deflection of points *1* and *2* is presented.

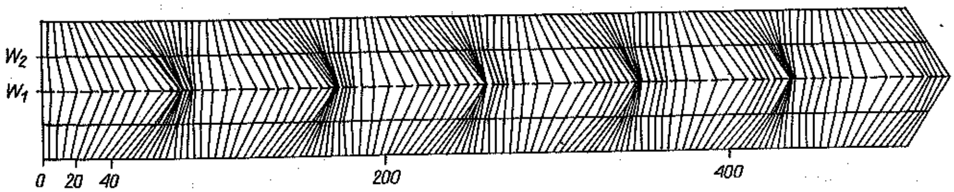

FIG. 4

Using triangular or tetrahedral nets gives not only the decrease of computing costs but also permits the accuracy in description of a phenomen of vibration.

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STRESZCZENIE

CZWOROŚCIENNE ELEMENTY CZASOPRZESTRZENNE W ANALIZIE DRGAŃ WYMUSZONYCH PŁYT

Praca stanowi uogólnienie metody elementów czasoprzestrzenych z siatką trójkątną. W analizie dynamicznej płyt czasoprzestrzeń jest trójwymiarowa, dlatego zamiast elementów trójkątnych zastosowano czworościany. Utworzono czasoprzestrzenny czworościenny element płyty średniej grubości i przeprowadzono badanie testowe. Zalety siatki trójkątnej, wyrażające się w rozdzieleniu równań względem węzłów, zostały zachowane.

Резюме

ЧЕТЫРЕХГРАННЫЕ ВРЕМЕНИПРОСТРАНСТВЕННЫЕ ЭЛЕМЕНТЫ В АНАЛИЗЕ ВЫНУЖДЕННЫХ КОЛЕБАНИЙ ПЛИТ

Работа составляет обобщение метода временипространственных элементов с треугольной сеткой. В динамическом анализе плит времени-пространство является четырехмерным и поэтому вместе трехугольных элементов применены четырехгранные элементы. Образован временипространственный четырехгранный элемент плиты средней толщины и проведены тестовые исследования. Достоинства треугольной сетки, выражающиеся в разделении уравнений по отношению к узлам, сохранены.

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