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NON-UNIQUE NUMERICAL SOLUTIONS IN VISCO-PLASTICITY

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We present the numerical analysis of a series of cutting problems for a visco-plastic material. The model is implemented in a finite element program written in the flow formulation framework. Almost ideal plasticity is achieved as the limit case of a visco-plastic material. Solution dependence on the chosen initial configuration (the final one being not known) suggests non-uniqueness inherent to the visco-plastic material with free surfaces, since a residual rate dependence is conserved in the model.

1. Introduction

The behaviour of a perfectly plastic material under continuous deformation has attracted the attention of researchers for several decades. Although in most of the contributions severe assumptions have been stated — namely, no strain hardening, no stress rate dependence, two dimensional flow, isothermal conditions, etc. — astonishingly accurate results can be obtained as compared with experimental data.

The mathematical model for rigid-perfectly plastic material yields a hyperbolic system of differential equations. Among the traditional methods for solving this system of equations, the slip line field theory [1] is for sure the most popular, and the one which has been used in most different problems. Special attention was devoted to the presence of non-unique solutions, all of which are complete in the sense that they satisfy all the static and kinematic requirements. This non-uniqueness is partly due to the idealized nature of the material model, but in addition, it has been proved [2–4] that free surfaces introduce the possibility of non-unique solutions, since the final configuration is not known. Different positions of the free surface can be shown to satisfy the governing equations, by application of the slip line theory.

On the other hand, computer numerical methods have constituted an alternative and complementary approach to these problems. It has been possible to reproduce slip line theory results and, in addition, to find out solutions where that method is not applicable. A historical account of the main features which constitute the so-called flow approach, dealing with rigid-visco-plastic materials in metal-forming processes by the finite element method, can be found in [5]. The perfectly plastic material is placed as a strain-rate-independent limit of a visco-plastic material. However, this limit involves a singular problem, from the mathematical and, therefore, numerical point of view. ANTÚNEZ and IDELSOHN [7] have proposed an effective way, based on matrix scaling, to make the problem well posed and, as byproduct, to extend essentially the range of strain rates for which the model can be applied without being distorted.

In this paper, we solve the problem of cutting a semi-infinite domain of a plastic material with a residual viscous effect by a rigid sharp wedge moving parallel to the metal surface. The model considers the material as a non-Newtonian fluid, and therefore the crack formation and propagation is not taken into account. A free surface is found in most of the boundary which can have large variations in shape and orientation from the initial configuration. This behaviour is taken into account by an integration along a streamline using an intrinsic coordinate system with an additional condition for determining the updated configuration. In order to avoid excessive mesh distortion we also introduce an algorithm for mesh rearrangement.

It is found that the final configuration depends on the initial guess. Sensitivity of this results to the specific configuration, i.e. to the wedge angle, is observed.

2. Physical and computational model

In order to model the behaviour of the visco-plastic material during forming processes, it is assumed that the overall strains are large enough so that elastic ones can be neglected. With additional assumptions of associated plastic flow and a power-type law for the plastic potential, the constitutive equation can be written in analogy with the one of a non-Newtonian fluid,

$$s_{ij} = \frac{1}{2\mu}\dot{\varepsilon}_{ij}\,, \tag{2.1}$$

where the viscosity is given by

$$\mu = \frac{\sigma_Y + \left(\frac{\dot{\overline{\varepsilon}}}{\gamma\sqrt{3}}\right)^{\frac{1}{n}}}{\sqrt{3}\,\dot{\overline{\varepsilon}}}, \tag{2.2}$$

and

$\dot{\varepsilon}_{ij}$	strain rate tensor,
s_{ij}	stress deviator tensor,
$\sigma_Y = \sigma_Y(T, \overline{\varepsilon}, \sigma_m)$	yield stress,
$\dot{\overline{\varepsilon}} = \sqrt{2\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}$	second invariant of the strain rate tensor,
γ	fluidity parameter,
$n \geq 1$	power of the visco-plastic law,
T	temperature,
σ_m	mean stress,
$\overline{\varepsilon}$	equivalent total deformation.

In general, the yield stress can be dependent on the total strain, the temperature, and the pressure. For the rigid-perfectly-plastic material we take it as constant.

Bearing this in mind and following standard finite element patterns, it can be shown that the equilibrium equations for a non-Newtonian fluid are satisfied by solving the system of equations which results from weighting them throughout the domain with suitable functions W_i

$$\int_{\Omega} W_i(\nabla \cdot \boldsymbol{\sigma} + \mathbf{f})d\Omega = 0\,, \tag{2.3}$$

where $\boldsymbol{\sigma}$ denotes stresses and $\mathbf{f}$ applied forces. For elliptic problems the weighting functions W_i are best chosen (with respect to convergence of the discretized to the exact solution) as the same shape functions used for approximating the solution (Galerkin approximation).

Equation (2.3) is completed by specifying appropriate boundary conditions: velocities

$$\mathbf{u} = \bar{\mathbf{u}}, \qquad \text{on} \;\; \Gamma_{\mathbf{u}}\,, \tag{2.4}$$

and tractions

$$\mathbf{t} = \bar{\mathbf{t}}, \qquad \text{on} \;\; \Gamma_{\mathbf{t}}\,, \tag{2.5}$$

where the boundary is $\Gamma = \Gamma_{\mathbf{u}} \cup \Gamma_{\mathbf{t}}$.

To the given equations, the incompressibility condition is added by requiring the compressive work rate to vanish

$$\int_{\Omega} |p\,\dot{\varepsilon}_{ii}| d\Omega = 0 . \tag{2.6}$$

The perfectly plastic limit is reached by imposing $n \to \infty$, in Eq.(2.2).

3. Free surfaces

A particularly important feature of the present problem is to handle free surfaces appropriately, in order to reach the final configuration as a part of the solution. Since for steady state the free surfaces are streamlines, the updated coordinates can be calculated by integration from a fixed point. The most straightforward way of performing this integration is, as suggested by ZIENKIEWICZ *et al.* [6],

$$y = y_0 + \int_{x_0}^{x} \frac{dy}{d\overline{x}} d\overline{x} \cong y_0 + \int_{x_0}^{x} \frac{u_y}{u_x} d\overline{x}, \tag{3.1}$$

u_x and u_y being the velocity components in global coordinates. In Eq.(3.1), the nodal coordinates in one direction are kept fixed, and the free surface is followed by changing the coordinates in the other direction. This scheme is convenient, for its simplicity, where free surfaces are roughly parallel to the coordinate axes, and if it is known in advance that the variations from an initial, estimated configuration will not be very large. However, in cases where the free surfaces are expected to have considerable variations, or even to change the orientation with respect to the global coordinate system, a more general algorithm is required. For such cases, we state the free surface condition as

$$\int_{S} |u_n| dS = 0 , \tag{3.2}$$

where S is the unknown free surface, and u_n is the velocity component normal to the surface.

When calculating this integral in the discretized model, we have the velocities and the configuration for the iteration n, and we find the configuration for the iteration $n + 1$. Equation (3.2) will require a rotation of each

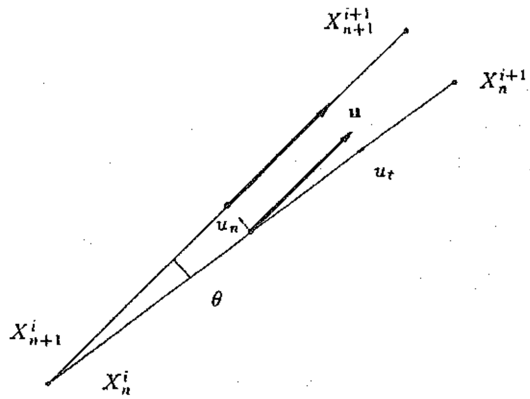

FIG. 1. Free surface updating by curvilinear integration.

segment $i, i+1$, in order to have the velocity tangent to it, Fig.1. Here, the condition of a constant length of each segment must be included, because Eq.(3.2) leaves this magnitude undetermined.

In addition, we perform a node rearrangement within the whole domain in order to have the least distorsion in the mesh. To this end, we consider separately the displacements in two orthogonal directions from two successive configurations in the free surface as the imposed boundary values in the Laplace equation. Calling w the nodal displacements in one direction, we solve

$$\nabla^2 w = 0, \tag{3.3}$$

where the boundary conditions are given by the imposed or null displacements on the boundary, according to either free or fixed boundaries, respectively. This gives a proportional displacement field for all the nodes which keeps a low distortion in the mesh.

4. Plastic material with a residual viscous effect

For the numerical applications we have taken $n = 40$ in Eq.(2.2) to obtain the visco-plastic limit. If, in addition, $\sigma_Y = 200$, $\gamma = 1$, and $\sigma_{vp} = \sigma_Y + (\dot{\bar{\varepsilon}}/\gamma\sqrt{3})^{\frac{1}{n}}$, we have

$\dot{\bar{\varepsilon}}$	σ_{vp}
10^{-6}	200.70
1	200.99
10^{6}	201.39

This means a very small, but non-zero, viscous effect. A perfectly plastic material would yield a singular stiffness matrix, making impossible the numerical solution by this method. Besides, we are interested in solving this problem with this residual viscous effect, since it has been an open question whether such behaviour would eliminate any non-uniqueness. It should be kept in mind that Hill's uniqueness theorem [8] does not apply to problems involving unspecified boundaries.

5. Numerical results

For the numerical calculations a steady state module is used of a finite element code written by the author [9]. The program implements the model presented in Secs. 2 to 4, following standard finite element techniques: mixed, velocity/pressure formulation with isoparametric nine and four node elements for velocities and pressure, respectively. The nonlinear system of equations is iteratively solved by the frontal method. A simple back-substitution scheme is adopted, because the Newton-Raphson method is not applicable – it does not converge – when the material behaviour is close to perfectly plastic.

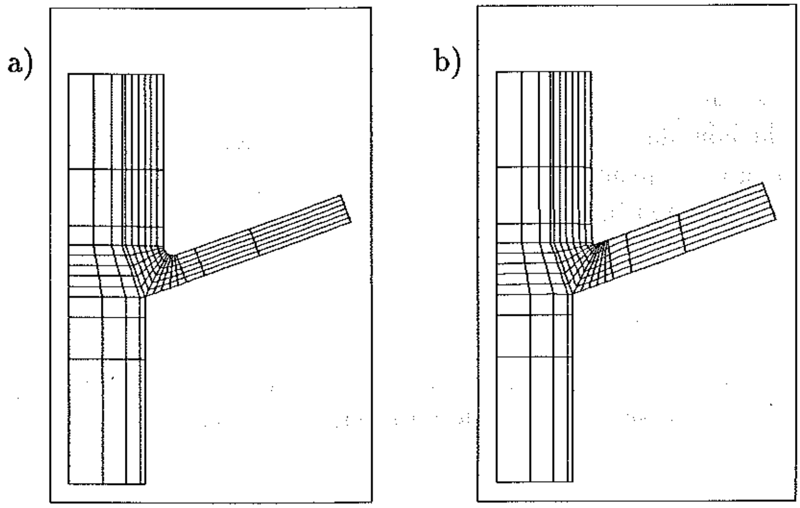

FIG. 2. Wedge angle = 110°, mesh 1; a. Initial mesh; b. Final mesh.

We model a cutting problem as a semi-infinite domain, a narrow strip of which, h, is removed by the motion of a rigid sharp wedge parallel to the metal surface. A uniform, velocity directed downwards is imposed on the upper boundary. The horizontal component is enforced to be zero on the left boundary, as well as the component normal to the solid boundary. The metal chip and the undeformed matrix constitute two regions with a uniform velocity field, separated by a narrow transition zone where all the plastic deformation is concentrated. In correspondence to these two regions, the free surface has two straight segments which form a sharp angle. The line from this corner to the tool edge separates the two rigid body displacement regions.

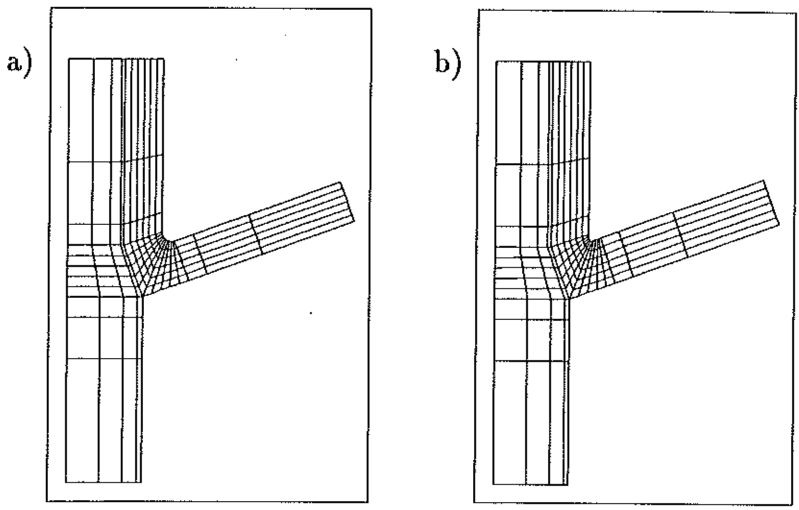

FIG. 3. Wedge angle = 110°, mesh 2; a. Initial mesh; b. Final mesh.

Figures 2 to 4 show three initial and final meshes for a wedge angle of 110° with respect to the imposed velocity. Initial configurations are generated parametrically, where the assumed chip thickness, e_0, is the main variable. It is pointed out that the different final chip thicknesses correspond to the steady-state solutions and result from assuming different initial configurations, as shown in the figures. We state that the solution is not mesh dependent, since we have refined the mesh both in the direction parallel and normal to the free surface, and obtained essentially the same results. Minor variations are produced by changing the initial radius connecting both parts of the domain, which can also be seen as producing different initial configurations. The range of obtained solutions is shown in Fig.5, where, in terms of the initial chip thickness/cutting depth ratio (e_0/h), both the final chip

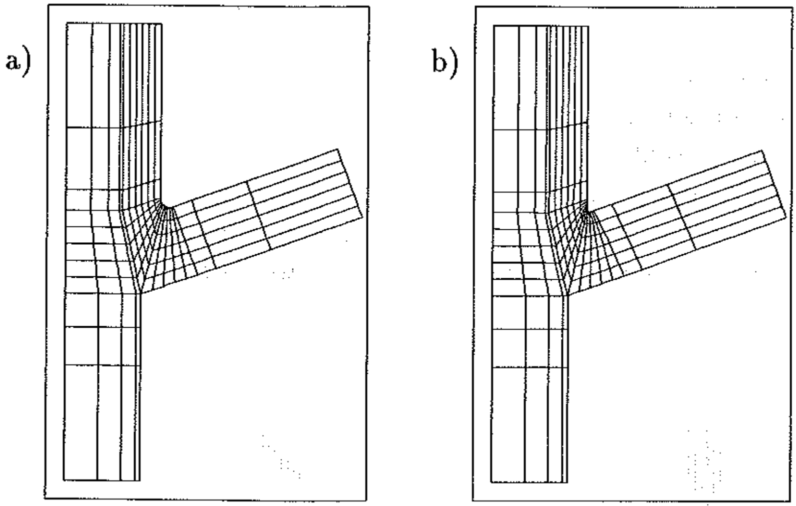

FIG. 4. Wedge angle = 110°, mesh 3; a. Initial mesh; b. Final mesh.

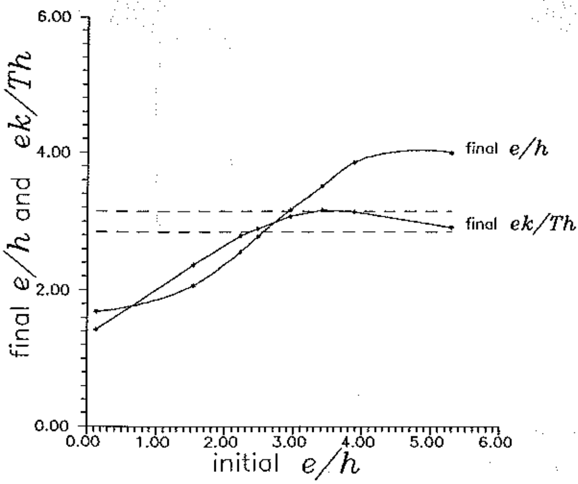

FIG. 5. Wedge angle = 110°, chip thickness/cutting depth ratio.

thickness/cutting depth ratio (e/h) and a thickness/cutting depth ratio normalized in terms of the applied force and the stress at pure shear (ek/Th) are visualized. The latter can be compared with PETRYK'S non-unique solution [3], whose extreme values are shown by the dashed lines in the same figure. In Fig.6 the strain rate contours are plotted, showing how the velocity gradients are concentrated in a narrow band.

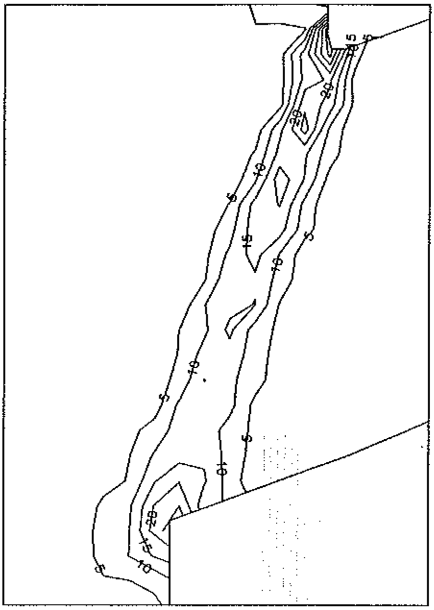

FIG. 6. Strain rate contours.

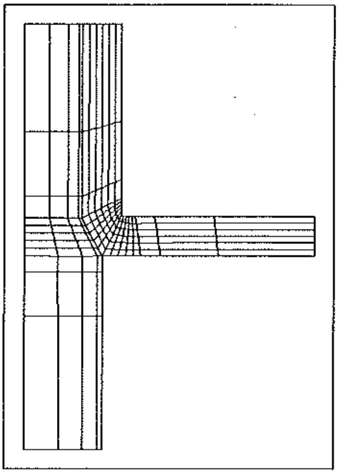

FIG. 7. Wedge angle = 90°, final mesh.

Similar calculations are carried out for tool angles of 90° and 70° (Figs. 7 to 10) showing final meshes and thickness/cutting depth ratios. It can be pointed out that with increasing angles, sensitivity to initial guesses decreases.

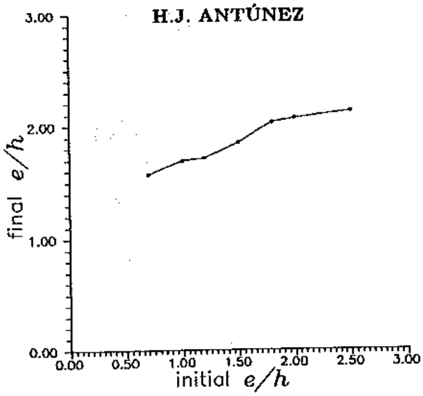

FIG. 8. Wedge angle = 90°, chip thickness/cutting depth ratio.

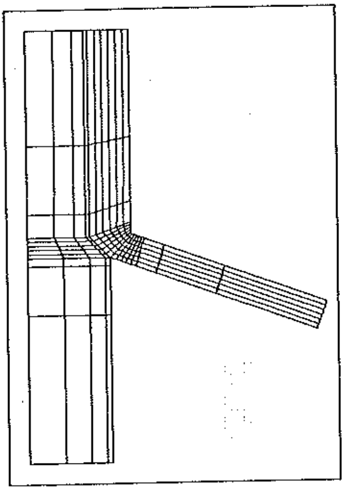

FIG. 9. Wedge angle = 70°, final mesh.

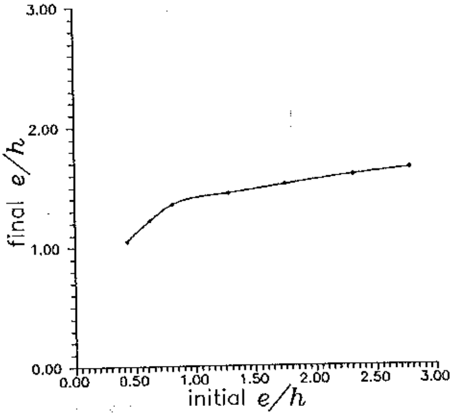

FIG. 10. Wedge angle = 70°, chip thickness/cutting depth ratio.

6. Conclusions

We have shown the numerical solution of some cutting problems where non-uniqueness is found by considering different initial guesses. Because the rigid-perfectly plastic material is reached as a limit of the rigid-visco-plastic one, a residual rate sensitivity is present, and still the non-uniqueness is present. This means that a small amount of rate dependence does not eliminate the non-uniqueness in the cutting problem.

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