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DISCRETE MODELLING OF WAVE PROPAGATION IN BARS WITH PIECEWISE-LINEAR CHARACTERISTICS

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A method is proposed to model one-dimensional wave propagation problems in a discrete manner as applied to the material with piecewise-linear stress-strain relationship in loading and rigid-elastic behaviour in unloading. The method consists in a simple combination of basic models, i.e. an elastic model and a plastic model with rigid unloading to obtain a combined model. It turns out to be accurate and effective. Numerical algorithm is described. Errors are discussed and an example of unloading stress wave is given.

1. Introduction

Much effort has been devoted in the literature to the problem of propagation of one-dimensional plastic waves. Solutions to specific problems can be found e.g. in [1, 2, 4–12, 18–20] with the use of both analytical and numerical methods that, in turn, were described in [1, 4, 5, 9, 11]. Method of characteristics in conjunction with finite difference technique was pointed out as a powerful tool [1]. Some problems were solved by means of that procedure in [2, 3, 13–16]. However, some difficulties are mentioned in these papers such as general nonlinearity of unloading waves, moving boundaries of loading and unloading regions and a multiphase character of the process. These difficulties can often make it impossible to arrive at an effective solution to some more complex engineering situations. For example, problems of dynamic soil-structure interaction belong to this category.

A discrete modelling method, presented in [21–23], offers an effective tool to solve the above wave propagation problems. Relevant models consist of rigid masses and deformable connections with the following properties: linearly elastic in loading and unloading as well as linearly elastic on loading and rigid on unloading. Respectively, an elastic or plastic model can be obtained with one of above connections. Wave problems that are analysed with the use of plastic model have in general a nonlinear characteristics. The described elastic and plastic models are called BASIC ones. Except for rigid unloading, the wave processes are analysed to within an accuracy of computer truncation errors, due to the applied principle of errorless difference approximation [21].

In this paper some possibilities will be indicated to apply the discrete modelling method for materials with complex $\sigma - \varepsilon$ relationships in the loading and unloading processes. Loading processes will be analysed with the use of the so-called soft $\sigma - \varepsilon$ relationship. In the case of unloading much attention will be focused on a nonlinear characteristics that commences with rigid behaviour. Such stress-strain relations are known to represent mechanical properties of certain soils [1, 2, 4, 5, 17–19, 23–28]. The considered $\sigma - \varepsilon$ relations take no account of viscous effects and cyclic loading. The most general form of the $\sigma - \varepsilon$ function for wave problems can be found in [29] where the constitutive equations are shown for elastic-plastic solids with discrete memory.

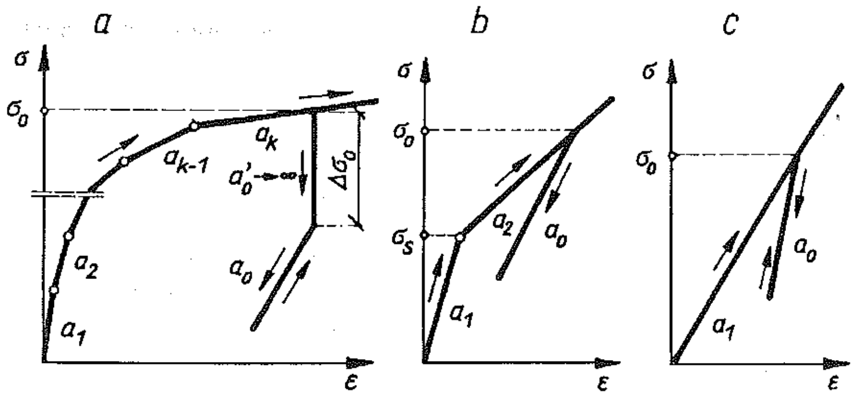

FIG. 1.

The proposed discrete modelling requires the constitutive relationships for both loading and unloading to be linearized. For example, the $\sigma - \varepsilon$ relationship for soils in the range of medium pressures can be approximated by a multisegmental loading branch and rigid-elastic unloading as shown in Fig.1a. No wave propagation processes based on such a deformation model have so far been discussed in the literature. The unloading branch can be simplified by assuming either $\Delta\sigma_0 = 0$ or $\Delta\sigma_0 = \sigma_0$. Particular cases of the $\sigma - \varepsilon$ relationship are shown in Fig.1b and 1c and discussed in detail in [1, 4–10, 18–20, 23, 28].

Construction of a combined model for an elastic layered bar was described in [21]. Such a global model consists of a number of suitably linked basic elastic models. In this paper another combination model will be proposed – namely that correspoding to a piecewise linear $\sigma - \varepsilon$ relationship and consisting of a suitable number of basic elastic and plastic models to form a COMBINED DISCRETE MODEL. Specific type of linearization of the physical relationships will be used, separately for loading and unloading.

2. Discrete model of loading wave propagation for a piecewise linear $\sigma - \varepsilon$ relationship

2.1. Piecewise linear $\sigma - \varepsilon$ relationship for loading

Basic discrete models were introduced for linear $\sigma - \varepsilon$ relationships on loading process. Their structure results from an errorless difference approximation by a relating a spatial step Δz to a temporal step Δt with the use of

$$\Delta z = a\Delta t\,, \tag{2.1}$$

where a is a wave velocity corresponding to an assumed $\sigma - \varepsilon$ relationship. Thus a nonlinear $\sigma - \varepsilon$ function should be transformed to become a piecewise linear expression. Each segment of such a relationship will correspond to a different wave velocity. The linearization cannot, however, be arbitrary since the condition (2.1) is required to be satisfied for each segment of $\sigma - \varepsilon$ plot. Particular relations among characteristic wave velocities a_j must hold good in a multi-segmental $\sigma - \varepsilon$ relationship, Fig.2.

Let us assume that

$$a_j = \frac{k+1-j}{k} a_1\,, \tag{2.2}$$

where $j = 1, 2, \ldots, k$ – number of a $\sigma - \varepsilon$ segment along the loading branch (Fig.2), k – number of segments on the loading branch, a_1 – a suitably assumed first wave velocity (may be treated as an "elastic" one).

Slope of the j-th segment is $E_j = a_j^2\rho$, where ρ denotes density of the material. From the formula (2.2) it follows that consecutive wave velocities satisfy an inequality $a_j > a_{j+1}$. A particular case of $k = 1$ and $j = 1$ leads to the $\sigma - \varepsilon$ relationship in the form of a single straight line with the characteristic velocity a_1.

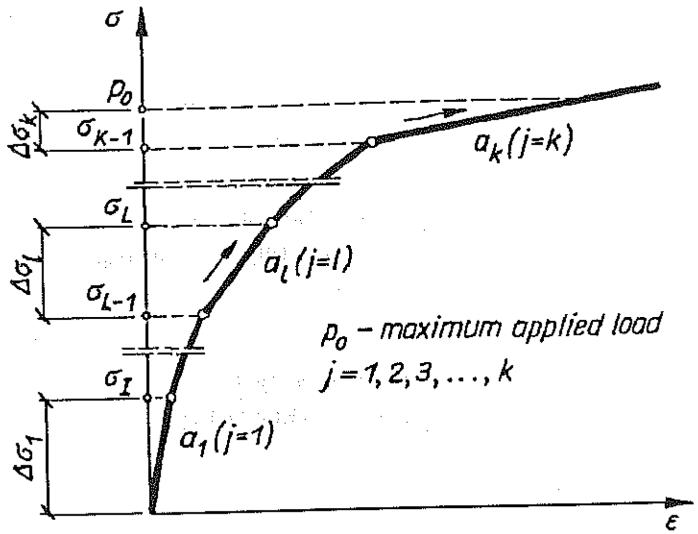

FIG. 2.

The multisegmental $\sigma-\varepsilon$ relationship can be described with the use of characteristic stresses σ_{I}, σ_{II}, ..., σ_{k-1} that depend on a shape of approximated $\sigma-\varepsilon$ curve. The proposed manner of linearization corresponds to the so-called soft characteristic of the $\sigma-\varepsilon$ curve which can be shaped in a desired way by emphasizing a given wave velocity a_1 as referred to a selected plastic wave velocity a_j. Two examples for $k=4$ are depicted in Fig.3.

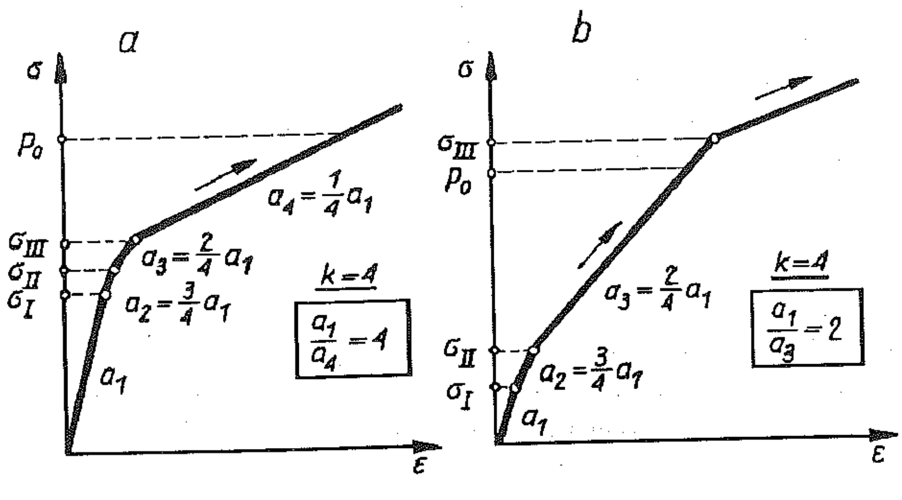

FIG. 3.

2.2. Combined model of loading process

Consider the propagation of loading wave in a semi-infinite bar with a multi-segmental $\sigma-\varepsilon$ relationship as shown in Fig.2 and the unit cross-section surface. The bar is subjected to an arbitrarily increasing end load $p(t)$. The problem will be solved with the help of a suitable discrete model. The linearized $\sigma-\varepsilon$ relationship is chosen in such a way that the combined model can be composed of k independent basic models which were explained in [21]. Each segment of the $\sigma-\varepsilon$ relationship will correspond to a suitable basic model. Each of these, in turn, is capable of propagating stresses within a certain interval described by the values $\Delta\sigma_1$, $\Delta\sigma_2$, ..., $\Delta\sigma_k$, Fig.2, and associated mass velocities Δv_j

$$\Delta v_j = \frac{\Delta\sigma_j}{a_j\rho}, \tag{2.3}$$

where $j = 1, 2, \ldots, k$ – number of a stress interval (of a basic model), a_j – corresponding wave velocity.

Each stress interval is associated with a characteristic maximum deformation that can be conveniently expressed in terms of a mutual displacement Δu_j of two neighbouring masses of the discrete model,

$$\Delta u_j = \frac{\Delta\sigma_j \Delta z}{E_j}, \tag{2.4}$$

where Δz – an assumed spatial step, $E_j = a_j^2\rho$ as before.

The above relations are of a general nature. Further procedure must depend on the assumed concept of modelling. One way to follow is to assume a constant time step Δt which leads to differentiating the spatial division of each basic model according to $\Delta z_j = a_j\Delta t$. This circumstance does not allow for superimposing the actions of basic models. Moreover, some of the modelling principles given in [21, 22] would not be satisfied.

The other concept to follow is to introduce a constant spatial step Δz for each basic model labelled j. This will clearly lead to unequal time steps Δt_j and according to Eq.(2.1) we have

$$\Delta t_j = \frac{\Delta z}{a_j}, \qquad j = 1, \ldots, k. \tag{2.5}$$

However, remembering the relation (2.2), we can synthesize actions of basic models at each step Δt_c

$$\Delta t_c = k\Delta t_1 = (k-1)\Delta t_2 = (k-2)\Delta t_3 = \ldots = \Delta t_k. \tag{2.6}$$

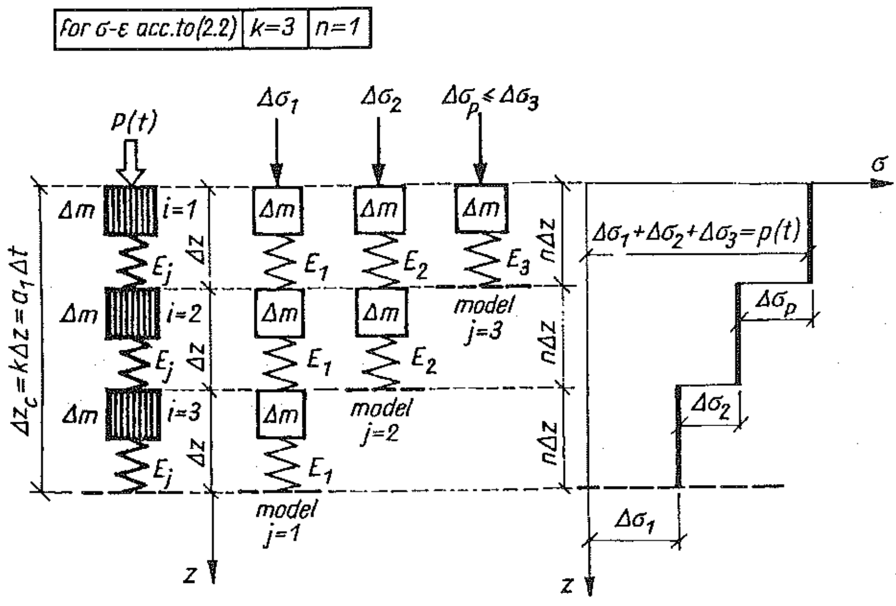

FIG. 4.

This means that the step Δt_c is the longest from all those corresponding to the basic models, that are involved in the wave process. From Eq.(2.5) it follows that the wave fronts in particular basic models are, after n steps Δt_c have possed, spatially displaced by a distance equal to $n\Delta z$. This fact for $n = 1$ is visualized in Fig.4. What is obtained can be called a spatial "smearing out" of the stress distribution that corresponds to a soft $\sigma - \varepsilon$ characteristic. Parameters of the stress wave for the i-th mass and at the consecutive step Δt_c can be calculated from the fromulae:

$$
\begin{aligned}
\Delta u_i^{n,n+1} &= \sum_{j=1}^{j=j_i} \Delta u_{i,j}^{n,n+1}, \\
u_i^{n+1} &= \sum_{j=1}^{j=j_i} u_{i,j}^{n+1}, \\
v_i^{n+1} &= \sum_{j=1}^{j=j_i} \Delta v_{i,j}^{n+1}, \\
\sigma_i^{n+1} &= \sum_{j=1}^{j=j_i} \Delta \sigma_{i,j}^{n+1},
\end{aligned}
\tag{2.7}
$$

where $t^{n+1} = t^n + \Delta t_c$, i – number of the lumped mass in the discrete model, j – number of the discrete model, j_i – the largest number of the basic model comprising the i-th mass $\Delta m = \Delta z \rho$.

Displacement of the i-th mass in the j-th discrete model can be determined according to [21]. Values of $\Delta v_{i,j}^{n+1}$, $\Delta u_{i,j}^{n,n+1}$ are to be found from the formulae (2.3), (2.4), respectively.

Thus, the combined model for loading process has been constructed with the use of a particular type of superposition, namely: A sum of reasons $(\Delta \sigma_j)$ that exhaust particular regions of local linearity lead to a result that is a sum of associated partial results $(\Delta v_j, \Delta u_j)$.

3. Discrete model for propagation of rigid-elastic unloading wave

A unit cross-section surface bar made of the material with the $\sigma - \varepsilon$ relationship as shown in Fig.5 is acted upon by a suddenly applied load $p(t)$ and then monotonically decreasing to zero.

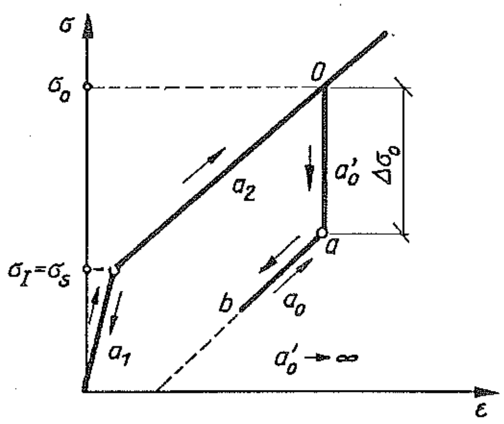

Fig. 5.

Consider an unloading stress wave by following the bisegmental unloading branch. Its first segment $0 - a$ corresponds to rigid unloading whereas the other one, $a - b$, to a linear behaviour as at the initial elastic stage. That is why the branch $0 - a - b$ will be called a rigid-re-elastic unloading, or simply a rigid-elastic one. The loading branch can clearly consist of more than two segments that are considered here for the sake of brevity.

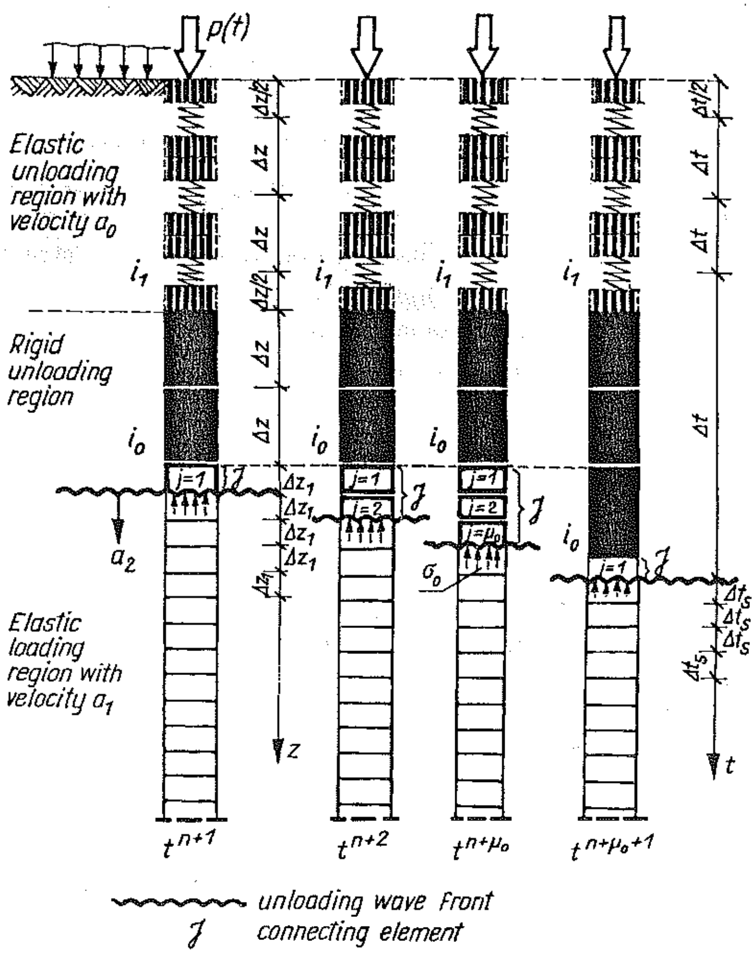

FIG. 6.

The model for solving the stated problem consists of three basic models as described in [21–23] and is shown in Fig.6. Elastic model serves to describe loading process, two models – a rigid (plastic) model and a re-elastic model – to describe unloading process. Front of the rigid model coincides with the front of unloading wave. An important component of the proposed model is constituted by a part of the rigid model that is located between the elastically loaded and the unloaded regions. Although this component remains an integral part of the rigid model, it is here isolated and termed a connecting element J. Its role is to ensure a smooth interaction of the unloading wave front with the remainder of the discrete model. That is why its property must remain rigid, thus ensuring no waves propagated through it. Continuity requires that the front of connecting element must move with the velocity a_2 and interact, at the same time, with the unloaded region in which the velocities a_0' and a_0 are present, both different from a_2. In addition, the front of connecting element acts on the neighbouring elastic region in which the wave velocity is equal to a_1.

Let the unloading region be divided into segments Δz. To allow the unloading wave front to move with the velocity a_2 (under constant step Δt), the connecting element should consist of successive segments each Δz_1 long:

$$\Delta z_1 = \frac{\Delta z}{\mu_0}, \tag{3.1}$$

$$\mu_0 = \frac{a_0}{a_2}. \tag{3.2}$$

Number μ_0 can be conveniently assumed to be an integer. Thus at each step Δt the connecting element get longer by a segment Δz_1. This procedure is shown in detail in Fig.6. When the connecting element reaches its maximum length equal to $\mu_0 \Delta z_1$, it is transferred to the rigidly unloaded region. Such an expansion of the rigid region is realized with a suitable discrete model which is described in [23]. The whole rigidly unloaded region of the model moves with uniform mass velocity v_0^n which is determined at each time step Δt by means of the momentum conservation principle. The following relationship applies:

$$v_0^{n+1} = \frac{v_0^n \overline{\Delta m}^n + \Delta m_1 v_{i_0+j+1}^n + \overline{\Delta\sigma}\Delta t}{\overline{\Delta m}^{n+1}}, \tag{3.3}$$

where $\overline{\Delta m}^n = (l_s + 0.5)\Delta m + j\Delta m_1$, $\overline{\Delta m}^{n+1} = (l_s + 0.5)\Delta m(j+1)\Delta m_1$, $\Delta m = \Delta z \rho$, $\Delta m_1 = \Delta z_1 \rho$, $\overline{\Delta\sigma} = \sigma_{i_1}^n - \sigma_{i_0+j+1}^n$, $j = 0, 1, 2, \ldots, \mu_0 - 1$ – subscript for a current length of the connecting element, $l_s = i_0 - i_1$ – number of rigid segments with the length Δz each.

Knowing the velocity v_0^{n+1}, accelerations $\ddot{u}_r^{n,n+1}$ and displacements u_r^{n+1} of particular masses within the rigid region can be calculated ($r = i_1 + 1$ to $i_0 + j$):

$$\begin{aligned} \ddot{u}_r^{n,n+1} &= \frac{v_0^{n+1} - v_r^n}{\Delta t}, \\ \Delta u_r^{n,n+1} &= \Delta u_r^{n-1,n} + \ddot{u}_r^{n,n+1}\Delta t^2; \\ u_r^{n+1} &= u_r^n + \Delta u_r^{n,n+1}. \end{aligned} \tag{3.4}$$

Stress $\sigma_0(z, t^n)$ at the front of the unloading wave can be found according to the rules given in Sec. 2.2. Stress inside the rigid region is given by the formula

$$\sigma_r^{n+1} = \sigma_{r-1}^{n+1} - \Delta m_r \frac{v_0^{n+1} - v_r^n}{\Delta t}, \tag{3.5}$$

where $i_1 + 1 \leq r < i_0 + j$.

The procedure of successive expansion of the rigid region described above enables the re-elastic unloading to be made with the velocity a_0 accompanied by the constant step Δt. The segment Δz with the number i_1+1 commences re-elastic unloading when the following condition is met

$$\sigma_{i_1+1}^{n+1} = \sigma_{0,i_1+1} - \Delta\sigma_0 = \sigma'_{0,i_1+1}, \tag{3.6}$$

where σ_{0,i_1+1} – stress at the front of unloading wave remembered at an instant of transfer of the connecting element to the unloading region, $\sigma_{i_1+1}^{n+1}$ – stress calculated according to Eq.(3.7).

Stresses inside the re-elastic unloading region are given by the formula

$$\sigma_i^{n+1} = \left(\frac{u_i^{n+1} - u_{i+1}^{n+1}}{\Delta z} - \varepsilon_{0,i} \right) E_0 + \sigma'_{0,i}, \tag{3.7}$$

where $\varepsilon_{0,i}$ – strain at the front of unloading wave at the beginning of rigid unloading of the i-th segment Δz, $E_0 = a_0^2 \rho$.

Displacements u_i^{n+1} are calculated as shown in [21]. Thus, the displacements of interior masses of the re-elastically unloaded region $2 \leq i \leq i_1$ are:

$$\begin{aligned} \ddot{u}_i^{n,n+1} &= \frac{\sigma_{i-1}^n - \sigma_i^n}{\Delta m}, \\ \Delta u_i^{n,n+1} &= \Delta u_i^{n-1,n} + \ddot{u}_i^{n,n+1} \Delta t^2, \\ u_i^{n+1} &= u_i^n + \Delta u_i^{n,n+1}. \end{aligned} \tag{3.8}$$

It is only natural to form an exterior mass ($i = 1$) as $0.5\Delta m$, see [21, 22] for boundary conditions. This results in an accurate realization of the reflection from the free end. Acceleration of the edge mass $\ddot{u}_1^{n,n+1}$ is given by the formula

$$\ddot{u}_1^{n,n+1} = \frac{p_z(t^n) - \sigma_1^n}{0.5\Delta m}. \tag{3.9}$$

Increment of the displacement $\Delta u_1^{n,n+1}$ and the displacement u_1^{n-1} can be calculated from Eqs.(3.8)$_2$ and (3.8)$_3$, respectively. Magnitude of the discrete form of load $p(t)$ is denoted by $p_z(t^n)$. As indicated in [21], the formation of a half-mass is associated with

$$p_z(t^n) = \frac{1}{2} p(t^n). \tag{3.10}$$

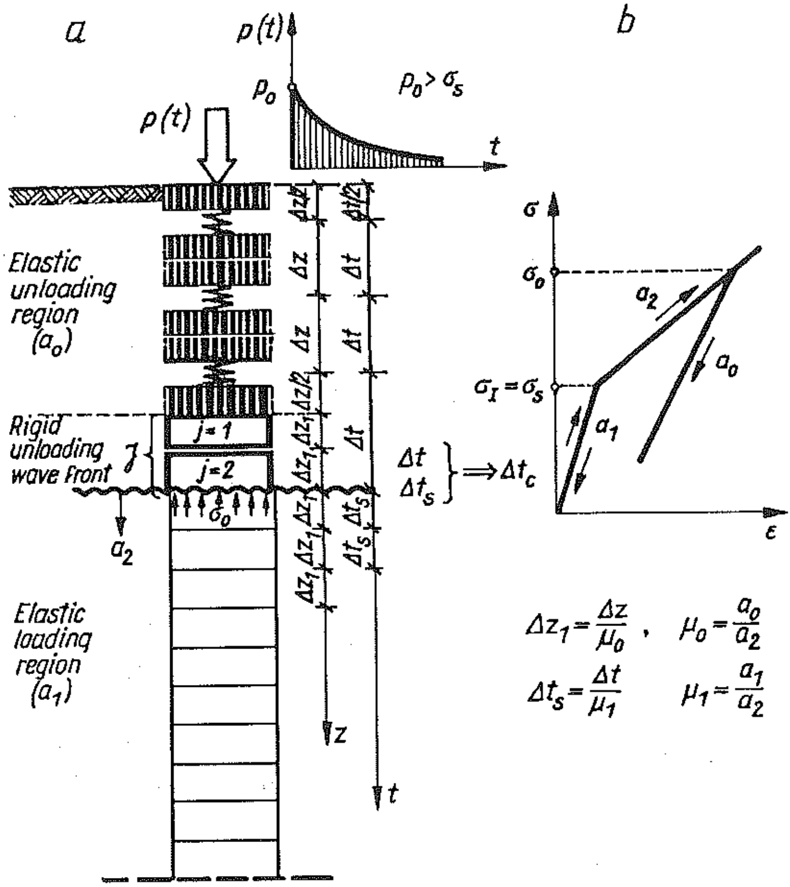

FIG. 7.

It should be noticed that a jump in the stress $\sigma_0(z, t^n) - \sigma_s$ is present at the front of the unloading wave.

Consider a case of the re-elastic unloading as shown in Fig.7b. The corresponding discrete model is given in Fig.7a as a specific case of that shown in Fig.6. The rigid unloading region reduces to the connecting element J which plays here the same role as in the more general case. J separates the loading and the unloading regions in which waves propagate with different velocities. The formula (3.3) should now be simplified by insisting on $l_s = 0$. Moreover, $\Delta\sigma_0 = 0$ should be assumed in Eq.(3.6) which leads to $\sigma'_{0,i} = \sigma_{0,i}$.

The discrete model of Fig.7a propagates the unloading wave with the velocity a_2 so long as

$$\sigma_0(z, t^n) > \sigma_s \,. \tag{3.11}$$

At an instant t^{n^*} the condition

$$\sigma_0(z^*, t^{n^*}) = \sigma_s\,, \tag{3.12}$$

is satisfied and the location of the unloading wave front enables to determine the length of plastically deformed bar z^* together with the length of the connecting element J^*. The latter element will no longer change. In this situation only elastic waves propagate in the bar. Their velocities are: a_0 for $z < z^*$ (re-elastic behaviour) and a_1, for $z \geq z^*$. Interaction of these two regions is ensured by the connecting element with the length J^*, fixed at an instant t^{n^*}.

4. Numerical examples and accuracy of calculations

Two examples will now be given to illustrate the discrete method as applied to the analysis of unloading waves. The first example deals with a straight-line unloading behaviour as shown in Fig.1b, the other with a two-segmental rigid-elastic unloading as depicted in Fig.5. To focus our attention on the unloading process, let the loading behaviour be described by a single straight line. More complicated loading behaviour would not contribute to better understanding of the problem, anyway. Moreover, calculations in the loading regime are made according to the rules of errorless difference approximation.

Let the load $p(t)$ be

$$\begin{aligned} p(t) &= p_0\left(1-\frac{t}{\tau}\right)^{\beta}, && \text{for} \quad t \leq \tau\,, \\ p(t) &= 0, && \text{for} \quad t \geq \tau\,. \end{aligned} \tag{4.1}$$

Example 1

As explained above, it is the $\sigma - \varepsilon$ relationship shown in Fig.1c that is relevant for the purpose of unloading analysis. The obtained solutions will be compared with the exact one given in [10]. Accuracy of the stress at the front of unloading wave will also be discussed. The discrete model from Fig.7 will be adopted, except for the elastic loading region which will be ignored.

The following data are assumed: $\tau = 0.1\,\mathrm{s}$, $\rho = 1800\,\mathrm{kg/m^3}$, $a_0 = 100\,\mathrm{m/s}$ and, for the sake of comparison, $a_0 = 300\,\mathrm{m/s}$. Wave velocity ratio $\mu_0 =$

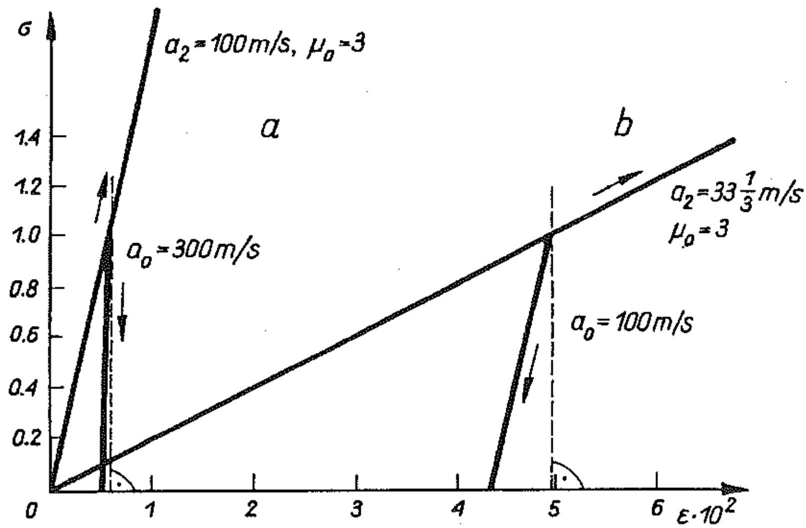

FIG. 8.

a_0/a_2 will change from 1 to 5, the exponent β from 1 to 7. More specific data on the $\sigma - \varepsilon$ curve are shown in Fig.8.

No error arises, within all the assumptions of the method, so far as the location of the unloading wave front is concerned. It is the stress at this front that may attain inaccurace values. Let this stress be expressed in a dimensionless form as $\bar{\sigma}_0(z) = \sigma_0(z)/p_0$. Main source of the discussed error is the presence of a fictitious rigid connection in the discrete model. The obtained results, collected in Table 1, show that already for $\Delta z = 0.1\,\mathrm{m}$

Table 1. Dimensionless stresses $\bar{\sigma}_0(z)$ for $\Delta z = 0.1\,\mathrm{m}$, $a_0 = 100\,\mathrm{m/s}$, $\beta = 1$.

μ_0 \ z [m]		1.	2.	3.	4.	5.
$\mu_0 = 3$	exact solution	0.866666	0.733333	0.599999	0.466666	0.333333
	numerical solution	0.866643	0.734207	0.598679	0.468519	0.335540
$\mu_0 = 4$	exact solution	0.8124999	0.6249999	0.4374998	0.2500	0.0625
	numerical solution	0.8118497	0.6195660	0.4362508	0.252476	0.0626754

(irrespective of the value of μ_0) the numerical results practically coincide with the exact ones [10]. Accuracy of the numerical solution can always be enhanced by decreasing the step Δz. However, in the example worked out it is not worth- while: e.g. for $\beta = 1$, $\mu_0 = 3$, $a_2 = 100\,\mathrm{m/s}$, $z = 2\,\mathrm{m}$ the exact answer is $\bar{\sigma}_0(z) = 0.73333(3)$, whereas the numerical solutions for $\Delta z = 0.1\,\mathrm{m}$ and $\Delta z = 0.005\,\mathrm{m}$ are $\bar{\sigma}_0(z) = 0.734207$ and 0.733742, respectively. Accuracy of the solution increases together with an increasing velocity a_0 of the unloading wave as a result of the type of the $\sigma - \varepsilon$ relationship (compare diagrams *a* and *b* in Fig.8). A certain part of the results is shown diagrammatically in Fig.10 to follow. It enables to compare the present results with those for the rigid-elastic unloading. Assessing the accuracy of results it must be remembered that Prandtl's constitutive model on which the exact solution is based is only an idealization of real behaviour of the material. Thus the proposed numerical model should be verified by means of suitable experiments. Soils have been found to better conform to the discrete model with a rigid connecting element than to Prandtl's model.

Example 2

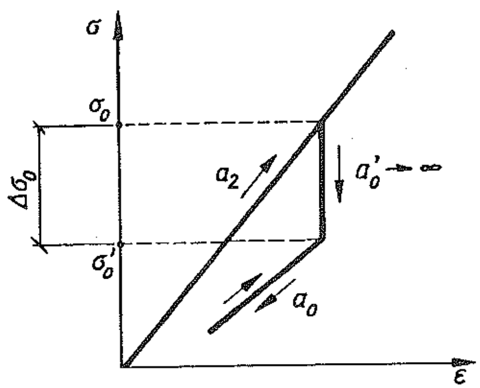

FIG. 9.

An influence of the rigid-elastic unloading behaviour, shown in Fig.9, on the dimensionless stress $\bar{\sigma}_0(z) = \sigma_0(z)/p_0$ and on the time-spatial spread of rigid region will now be examined. Remembering the remarks at the beginning of Sec.4, let us assume the same data as in Example 1. The discrete model will be similar to that given in Fig.6. Effects of the magnitudes $\Delta\bar{\sigma}_0$ on the stress $\bar{\sigma}_0(z)$ at the unloading wave front will be analysed, where $\Delta\bar{\sigma}_0$ is a dimensionless length of the rigid part of unloading branch. Various values of the ratio μ_0 (from 1 to 5) and $\Delta\bar{\sigma}_0(0.1; 0.2; 0.5)$ were used. The obtained results were compared with those corresponding to limiting cases: $\Delta\bar{\sigma}_0 = 0$ (Prandtl's model) and $\Delta\bar{\sigma}_0 = 1$ (perfectly rigid unloading branch). From the results depicted in Fig.10 it follows that for $\mu_0 \geq 3$ the results

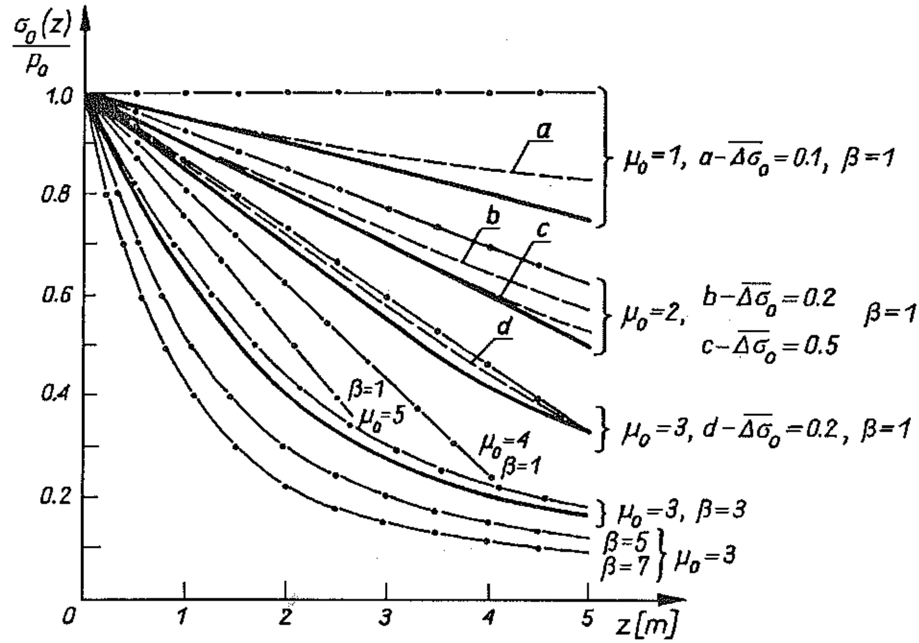

FIG. 10.

for different $\sigma - \varepsilon$ relations practically coincide. Influence of the spatial step length Δz on the accuracy of results is shown in Table 2. The pro-

Table 2. $\mu_0 = 3$, $\overline{\Delta\sigma_0} = 0.2$, $a_0 = 100\,\mathrm{m/s}$, $\beta = 1$.

	z [m]	1.	2.	3.	4.	5.
$\bar{\sigma}_0(z)$	$\Delta z = 0.01\,\mathrm{m}$	0.851573	0.713388	0.578311	0.444116	0.331245
	$\Delta z = 0.1\,\mathrm{m}$	0.850901	0.712886	0.577387	0.442673	0.331559

posed model disposes of the difficulties mentioned in the literature, e.g. in [1], namely: phase type of the studied processes, nonlinearity and arbitrary variations in the load $p(t)$. Time-spatial distribution of the rigidly unloaded region in the worked out example is shown in Fig.11. It follows that, for the assumed data, the rigid unloading affects the wave propagation process in a predominant manner within the soil layers up to 1.5 m or even 2.0 m thick.

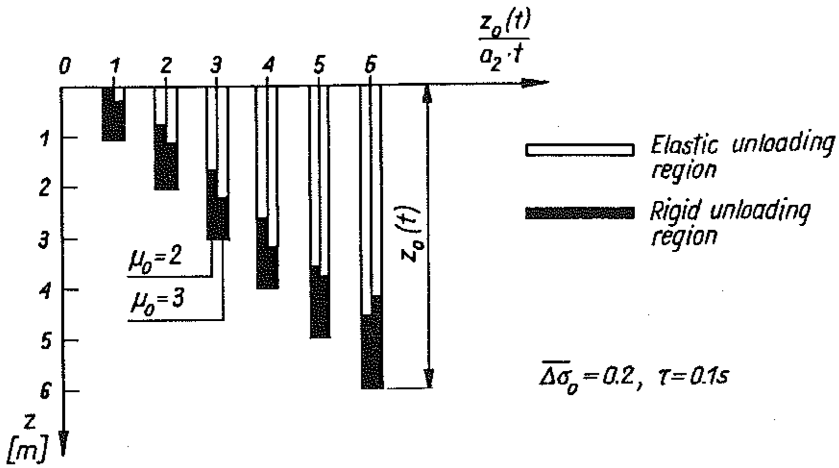

FIG. 11.

Substantial part of these thicknesses are occupied by rigid regions as seen in Fig.10. From the above as well as from [1, 4, 5] it can be observed that – in particular for rather thin soil layers – the unloading branch of the $\sigma - \varepsilon$ curve can be perfectly rigid. Thus even simpler discrete model, described in [23] can be employed.

5. CONCLUSIONS

One-dimensional discrete models for wave propagation problems were put forward in [21–23]. The elastic model is suitable to deal with linearly elastic materials, that are inherently insensitive to loading-unloading. The plastic model differs in that the unloading branch follows a rigid behaviour. These two basic models have the property that, except for rigid unloading, all wave propagation processes are analysed under the condition of errorless difference approximation.

The basic models are applied to solve the problems for materials with piecewise linear $\sigma - \varepsilon$ relationships in loading and a specific, rigid-elastic behaviour in unloading. The method consists in the superposition of basic models in a manner suitable for the particular situation under consideration. Thus a combined model is obtained that preserves the main characteristics of basic models such as a high accuracy and clear physical meaning. The worked out examples indicate adequate efficiency and potential of the proposed method to be used in complex engineering situations.

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