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COMPARISON OF MANDELSHTAM'S CONDITIONS WITH CONVERGENCE CONDITIONS FOR ITERATIVE PROCEDURES IN THE ANALYSIS OF COMPLEX DYNAMIC SYSTEMS BY MEANS OF PARTIAL MODELS

T. L. STAŃCZYK (KIELCE)

The convergence conditions for iterative procedures in the analysis of dynamic systems with the use of partial models are compared with Mandelshtam's conditions (uncoupling of vibrations of partial models). Both basic iteration procedures are discussed. The conditions for iterative procedures are found to be much weaker than those formulated by Mandelshtam. A simplified criterion for the selection of the type of procedure (a manner in which a system is to be decomposed) is presented.

1. Introduction

Mandelshtam's theory [1] was formulated to solve a problem of dynamics in the absence of suitable computational tools at that time. Such an approach, consisting in splitting up a complex system into a number of simple partial subsystems was commonly used before the advent of the computer. In many papers the subsystems were distinguished with *a priori* assumptions of very weak links or even complete isolation of the system. The main reason for Mandelshtam's approach has virtually disappeared and very complex systems with many degrees of freedom can be dealt with by means of computers. However, some difficulties arise with computations, interpretations and identifications of increasing numbers of parameters to describe the adopted models. Thus very complicated models still appear to be rather inconvenient to deal with.

Starting from these premises a concept of analysis of complex dynamic systems was put forward [2] consisting in taking advantage of simpler (partial) systems with simultaneous acceptance of weaker couplings to preserve a required accuracy of results. When an analysis of a complete system is made as a series of analysis of partial subsystems, weak couplings can be accounted for as forced perturbations in consecutive steps of the iteration procedure. Following [2], the problem of convergence of iterative procedures was tackled in [3] where an effect of damping was also discussed. This paper is aimed at comparison of Mandelshtam's conditions for uncoupling of vibrations of partial systems with the convergence conditions for iterative procedures given in [3].

2. Weak associations of masses in the complete system (procedure I)

A two-mass model is shown in Fig.1. Corresponding frequencies of free vibrations are

$$\begin{aligned} \omega_{01}^2 &= 0.5\left[\Omega_{01}^2 + \Omega_{02}^2 - (\Omega_{02}^2 - \Omega_{01}^2)\sqrt{1+\sigma^2}\right], \\ \omega_{02}^2 &= 0.5\left[\Omega_{01}^2 + \Omega_{02}^2 + (\Omega_{02}^2 - \Omega_{01}^2)\sqrt{1+\sigma^2}\right], \end{aligned} \tag{2.1}$$

where

$$\Omega_{01}^2 = \frac{k_1 + k_2}{m_1}, \qquad \Omega_{02}^2 = \frac{k_2}{m_2}, \qquad \Omega_{12}^4 = \frac{k_2^2}{m_1 m_2}. \tag{2.2}$$

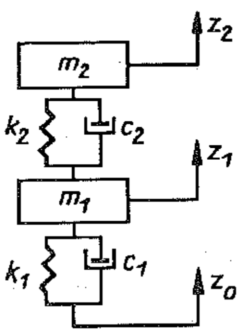

FIG. 1.

According to MANDELSHTAM [1], a coupling can be considered to be weak when the coefficient σ^2 (called here Mandelshtam's coefficient),

$$\sigma^2 = \frac{4\Omega_{12}^4}{(\Omega_{02}^2 - \Omega_{01}^2)^2} \tag{2.3}$$

satisfies the following inequality:

$$\sigma^2 \ll 1\,. \tag{2.4}$$

Then the model shown in Fig.1 can be divided into two partial models to be analysed separately. Such a division corresponds to a case of weak association of masses in the complete model, described in [2]. Convergence conditions in this case are

$$(2.5)_1 \qquad S_1 < \frac{-S_3(S_3-1) - 2S_3\sqrt{S_3}}{S_3^2 - 6S_3 + 1}$$

or

$$(2.5)_2 \qquad S_1 > \frac{-S_3(S_3-1) + 2S_3\sqrt{S_3}}{S_3^2 - 6S_3 + 1}\,,$$

where

$$S_1 = \frac{k_2}{k_1}\,, \qquad S_2 = \frac{m_1}{k_1}\,, \qquad S_3 = \frac{m_2}{m_1}\,. \tag{2.6}$$

To compare the above conditions with those of Mandelshtam, let us express the coefficient σ^2 in terms of S_1, S_2, S_3. On inserting Eqs.(2.2) and (2.6) into Eq.(2.3) and rearranging we obtain

$$\sigma^2 = \frac{4S_1^2 S_3}{(S_1 - S_3 - S_1 S_3)^2}\,. \tag{2.7}$$

Using the first of the convergence conditions $(2.5)_1$ we further get

$$\sigma^2 < \frac{4S_3^2\left(1 - S_3 - 2\sqrt{S_3}\right)^2}{4S_3^2\left(1 - S_3 - 2\sqrt{S_3}\right)^2} = 1\,. \tag{2.8}$$

On inserting the other convergence condition $(2.5)_2$ into Eq.(2.7) it turns out that σ^2 has the same form as in inequality (2.8). Thus the condition for convergence in the procedure I is

$$\sigma^2 < 1\,, \tag{2.9}$$

and appears to be much weaker than Mandelshtam's condition (2.4) and thus easier to satisfy. This situation is according to expectations: Mandelshtam's condition indicates when couplings among partial systems can be considered to be negligibly small and those subsystems to be analysed separately. In the proposed method the weak couplings are accounted for and the introduced criterion indicates at what value of σ^2 the iteration procedures lead to convergent results.

According to Mandelshtam, complete uncoupling of vibrations of partial subsystems can take place under two conditions: the inequality (2.4) and a certain difference in free vibration frequencies (see the denominator in Eq.(2.3)). The latter condition applies also to the convergence of the iteration procedure. Specifically, a ratio of the free vibration frequencies can be determined to ensure convergence. This ratio in the case of weak association takes the form

$$\frac{\omega_{01}}{\omega_{02}} = \sqrt{\frac{(1+S_1)S_3}{S_1}} \,. \tag{2.10}$$

On using the ends of convergence region expressed in terms of S_3 as a function of S_1, the conditions $(2.5)_1$ and $(2.5)_2$ become

$$S_3 > \frac{S_1(3S_1+1)+2S_1\sqrt{S_1(2S_1+1)}}{(1+S_1)^2} \tag*{$(2.11)_1$}$$

or

$$S_1 < \frac{S_1(3S_1+1)-2S_1\sqrt{S_1(2S_1+1)}}{(1+S_1)^2} \,. \tag*{$(2.11)_2$}$$

Accounting for condition $(2.11)_1$ in Eq.(2.10), we arrive at

$$\frac{\omega_{01}}{\omega_{02}} > \sqrt{\frac{3S_1+1+2\sqrt{S_1(2S_1+1)}}{1+S_1}} \,. \tag{2.12}$$

Let us note that

$$\lim_{S_1\to 0}\frac{\omega_{01}}{\omega_{02}} = 1 \,,$$

$S_1 \to 0$ should be here and in what follows understood as corresponding to $k_1 \to \infty$ ($k_2 \to 0$ would have led to a trivial case). The other limiting case is

$$\lim_{S_1\to\infty}\frac{\omega_{01}}{\omega_{02}} = \sqrt{3+2\sqrt{2}} \approx 2.4142 \,.$$

Accounting for condition $(2.11)_2$ in Eq.(2.10) leads to

$$\frac{\omega_{01}}{\omega_{02}} < \sqrt{\frac{3S_1+1-2\sqrt{S_1(2S_1+1)}}{1+S_1}} \,. \tag{2.13}$$

Corresponding limits of the ratio ω_{01}/ω_{02} amount to:

$$\begin{aligned} \lim_{S_1\to 0}\frac{\omega_{01}}{\omega_{02}} &= 1 \,, \\ \lim_{S_1\to\infty}\frac{\omega_{01}}{\omega_{02}} &= \sqrt{3-2\sqrt{2}} \approx 0.4142 \,. \end{aligned}$$

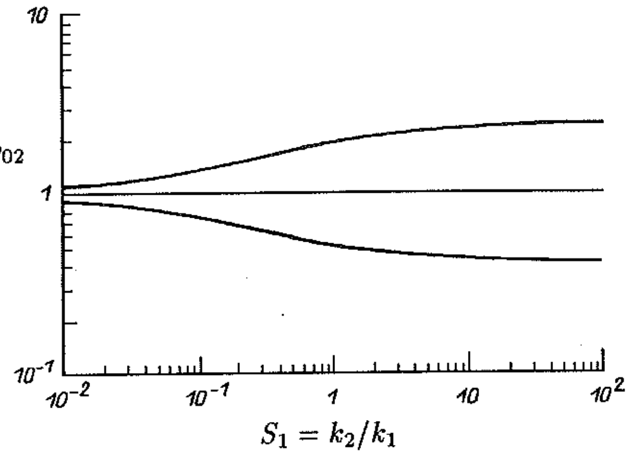

FIG. 2.

The latter number is clearly a reciprocal of 2.4142. This means that for very small values of S_1 (i.e., for $k_2 \ll k_1$) the frequencies ω_{01} and ω_{02} can be close to each other (even equal in the limiting case). For large values of S_1 ($k_2 \gg k_1$), even tending to infinity, the ratio of frequencies cannot exceed 2.4142. This ratio as a function of S_1 is plotted in Fig.2; its values ensure that the iteration procedure of the type I (relations (2.12) and (2.13)) remains convergent.

3. STRONG ASSOCIATIONS OF MASSES IN THE COMPLETE SYSTEM (PROCEDURE II)

The case analysed in the previous section referred to the coupling of a kinematic type. No conditions of Mandelshtam's type have been found in the available literature, suitable for the case of dynamic coupling which exists in the course of decomposition in the iteration procedure II. Let us consider this situation by starting with the set of equations, given in [2], describing vibrations of the model shown in Fig.1:

$$(3.1)\qquad \begin{aligned} (m_1+m_2)\ddot{z}_1 + (\ddot{z}_2-\ddot{z}_1)m_2 + c_1\dot{z}_1 + k_1 z_1 &= c_1\dot{z}_0 + k_1 z_0\,,\\ m_2\ddot{z}_2 + c_2\dot{z}_2 + k_2 z_2 - c_2\dot{z}_1 - k_2 z_1 &= 0\,. \end{aligned}$$

On neglecting the forcing and damping terms and accounting for specific integrals in the form

$$z_1 = z_{01}\sin\omega_0 t\,,\qquad z_2 = z_{02}\sin\omega_0 t\,,$$

we arrive at the following set of algebraic equations:

$$
\begin{aligned}
-m_2\omega_0^2 z_{02} + \left[k_1 + m_2\omega_0^2 - (m_1+m_2)\omega_0^2\right] z_{01} &= 0\,, \\
(k_2 - m_2\omega_0^2) z_{02} + k_2 z_{01} &= 0\,.
\end{aligned}
\tag{3.2}
$$

To yield non-zero solutions, the characteristic determinant of the set must vanish. On necessary rearrangement of this requirement, we get the following bi-quadratic equation:

$$
\omega_0^4 - \omega_0^2\mu(\bar{\Omega}_{01}^2 + \Omega_{02}^2) + \mu^2(\bar{\Omega}_{01}^2\Omega_{02}^2 - \bar{\Omega}_{12}^4) = 0\,,
\tag{3.3}
$$

where

$$
\begin{aligned}
\bar{\Omega}_{01}^2 &= \frac{k_1}{m_1+m_2}\,, & \Omega_{02}^2 &= \frac{k_2}{m_2}\,, \\
\bar{\Omega}_{12}^4 &= \frac{k_1 k_2}{(m_1+m_2)^2}\,, & \mu &= \frac{m_1+m_2}{m_1} = 1 + \frac{m_2}{m_1}\,.
\end{aligned}
\tag{3.4}
$$

The free vibration frequencies are given by:

$$
\bar{\omega}_{01,2}^2 = 0.5\mu\left[\bar{\Omega}_{01}^2 + \Omega_{02}^2 \mp (\Omega_{02}^2 - \bar{\Omega}_{01}^2)\sqrt{1+\bar{\sigma}^2}\right]\,.
\tag{3.5}
$$

The coefficient $\bar{\sigma}^2$ has the form

$$
\bar{\sigma}^2 = \frac{4\bar{\Omega}_{12}^4}{\left(\Omega_{02}^2 - \bar{\Omega}_{01}^2\right)^2}\,,
\tag{3.6}
$$

similar as before, see Eq.(2.3).

The conditions

$$
\bar{\sigma}^2 \ll 1 \qquad \text{and} \qquad \mu \to 1 \Leftrightarrow m_2 \ll m_1
\tag{3.7}
$$

must be satisfied to uncouple the partial subsystem together with the different frequencies of free vibrations for those subsystems. Then the free vibration frequencies of the system are

$$
\begin{aligned}
\bar{\omega}_{01}^2 &= \bar{\Omega}_{01}^2\,, \\
\omega_{02}^2 &= \Omega_{02}^2\,.
\end{aligned}
\tag{3.8}
$$

Let us express the coefficient $\bar{\sigma}^2$ in terms of S_1 and S_3. This leads to

$$
\bar{\sigma}^2 = \frac{4S_1S_3^2}{(S_3 - S_1 - S_1S_3)^2}\,.
\tag{3.9}
$$

Comparing the above formula with the formula (2.7), corresponding to the procedure I, a "symmetry" with respect of S_1 and S_3 is readily seen (S_1 and S_3 are interchangeable).

The conditions for convergence of procedure II are the following:

$$(3.10)\qquad S_3 > \frac{-S_1(S_1-1)+2S_1\sqrt{S_1}}{S_1^2-6S_1+1} \quad \text{or} \quad S_3 < \frac{-S_1(S_1-1)-2S_1\sqrt{S_1}}{S_1^2-6S_1+1}\,.$$

They can be proved to be equivalent to the conditions (2.22) given in the paper [3].

On using those conditions in Eq.(3.9) and rearranging, the condition for convergence of the procedure II, expressed in terms of Mandelshtam's coefficient, is the same as in the procedure I, namely

$$(3.11)\qquad \bar{\sigma}^2 < 1\,.$$

The convergence condition for the iteration procedure appears to be, as before, much weaker than the criterion for uncoupling of partial systems and therefore easier to satisfy.

Consider now the second condition (3.7). In order to uncouple the partial systems, the strong inequality $m_2 \ll m_1$ has to take place, expressible in terms of S_3 as

$$(3.12)\qquad S_3 \ll 1\,.$$

A question arises whether the above condition is also necessary to achieve convergence of the procedure II. An answer can be found in [3]. Suitable analysis showed that the procedure remained convergent in the zones 2, A and 3, indicated in Fig.8 of that paper. It appears that no satisfaction of the condition (3.12) is required to ensure convergence. The procedure II, although stays convergent in the zone 3, is not recommended to be employed [3]. A constraint on the values of S_3 is thus depicted as an interface of the zones A and 3. The condition (3.12) for the convergence of the procedure II can therefore be rewritten to become, approximately,

$$(3.13)\qquad S_3 < 3+2\sqrt{2} \approx 5.8284\,,$$

or, in terms of μ,

$$(3.14)\qquad \mu < 4+2\sqrt{2} \approx 6.8284\,.$$

Similarly as in the case of $\bar{\sigma}^2$, this condition is weaker than that corresponding to the uncoupling of vibrations of partial systems.

Existence of this additional condition for the procedure II can serve as an explanation of the circumstance that this procedure is better only in the zone 2 (see [3]) and is not recommended in the zone 3 in spite of the fact that the condition $\bar{\sigma}^2 < \sigma^2 < 1$ is satisfied for all points lying above the bisectrix $S_3 = S_1$. This condition would have suggested to use the procedure II in this zone but it must be remembered that here the requirement $\mu \to 1$, corresponding to small values of S_3, is drastically violated.

Let us now examine how much the frequencies of free vibrations have to differ to satisfy the convergence conditions for the procedure II. Remembering Eqs.(3.4) and (3.8), the relevant ratio is given by

$$\frac{\bar{\omega}_{01}}{\omega_{02}} = \sqrt{\frac{S_3}{(1+S_3)S_1}} \,. \tag{3.15}$$

Using the convergence conditions in accord with (3.10) – equivalent to those formulated as (2.22) in the paper [3] – we obtain

$$\frac{\bar{\omega}_{01}}{\omega_{02}} < \sqrt{\frac{1+S_3}{3S_3+1+2\sqrt{S_3(2S_3+1)}}} \,, \tag{3.16}$$

or

$$\frac{\bar{\omega}_{01}}{\omega_{02}} > \sqrt{\frac{1+S_3}{3S_3+1-2\sqrt{S_3(2S_3+1)}}} \,. \tag{3.17}$$

Limiting values for (3.16) are

$$\lim_{S_3\to 0} \frac{\bar{\omega}_{01}}{\omega_{02}} = 1 \,, \qquad \lim_{S_3\to\infty} \frac{\bar{\omega}_{01}}{\omega_{02}} = \sqrt{\frac{1}{3+2\sqrt{2}}} \approx 0.4142 \,,$$

and for (3.17) amount to

$$\lim_{S_3\to 0} \frac{\bar{\omega}_{01}}{\omega_{02}} = 1 \,, \qquad \lim_{S_3\to\infty} \frac{\bar{\omega}_{01}}{\omega_{02}} = \sqrt{\frac{1}{3-2\sqrt{2}}} \approx 2.4142 \,.$$

Required differences in the free vibration frequencies are similar as in the case of the procedure I. They now depend on the mass ratio S_3 instead of the stiffness ratio.

The inequalities (3.16) and (3.17) are depicted in Fig.3 to visualize the convergence condition for the procedure II. For the sake of comparison, the ratio ω_{01}/ω_{02} for the procedure I is also shown. Limits of admissible solutions for the procedure I are shown by dashed line.

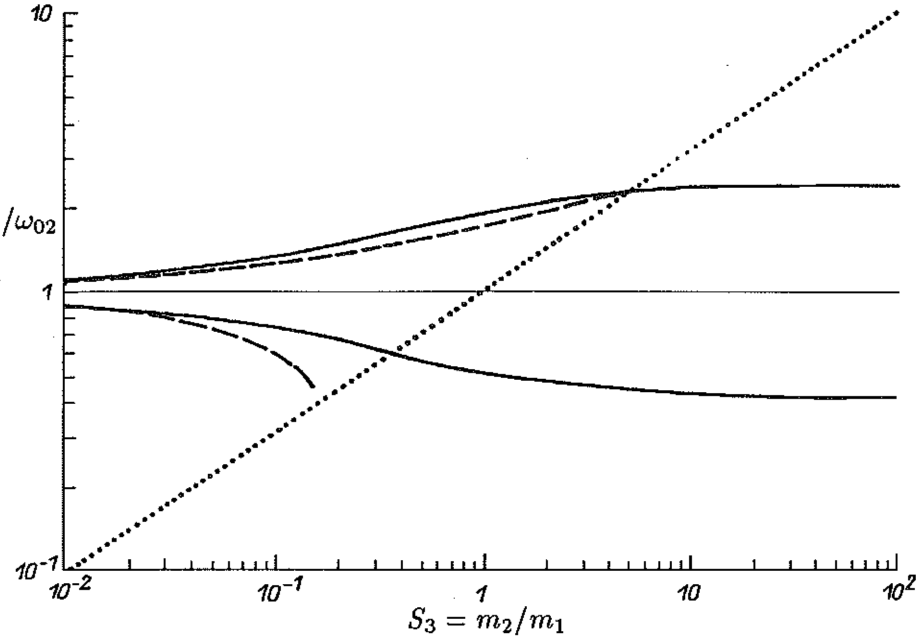

FIG. 3. —— Procedure II, – – – Procedure I, ······ Limits for procedure I.

4. Concluding remarks

The above considerations referred to undamped systems. In the paper [3] damping was found to be advantageous for the enhancement of convergence of iterative procedures. Mandelshtam's coefficients account for no damping terms. In the presence of damping the convergence conditions can be proved to be less stringent than those given by Eq.(2.9) and (3.11), i.e. the procedures remain convergent even when Mandelshtam's coefficients σ^2 and $\bar{\sigma}^2$ are greater than unity. The requirements for differing frequencies are also relaxed in the presence of damping in the system. This is clearly seen in Fig.4 (procedure I) and Fig.5 (procedure II) in which the frequency ratios are indicated to ensure convergence.

Damping in the system was described with the use of dimensionless coefficients $WSP1$ and $WSP2$ having the forms

$$(4.1)\qquad \begin{aligned} WSP1 &= \frac{c_1}{\sqrt{2k_1m_1}}, \\ WSP2 &= \frac{c_2}{\sqrt{2k_2m_2}}. \end{aligned}$$

From the presented diagrams it follows that even for low damping there

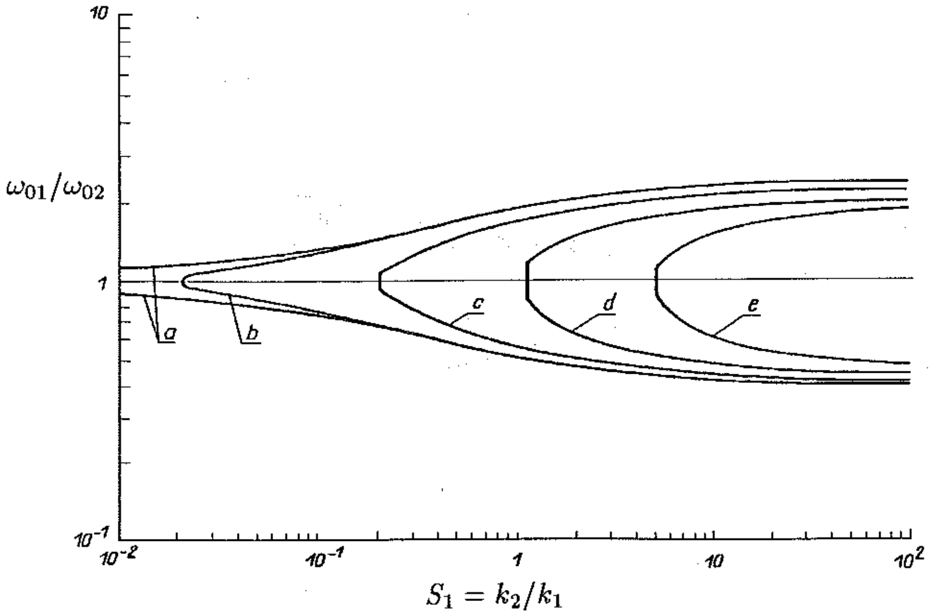

FIG. 4. Procedure I. Divergence regions for: $S_2 = m_1/k_1 = 0.0004(100/250000)$ and a) undamped system, b) $WSP1 = WSP2 = 0.1$, c) $WSP1 = WSP2 = 0.3$, d) $WSP1 = WSP2 = 0.6$, e) $WSP1 = WSP2 = 1.0$.

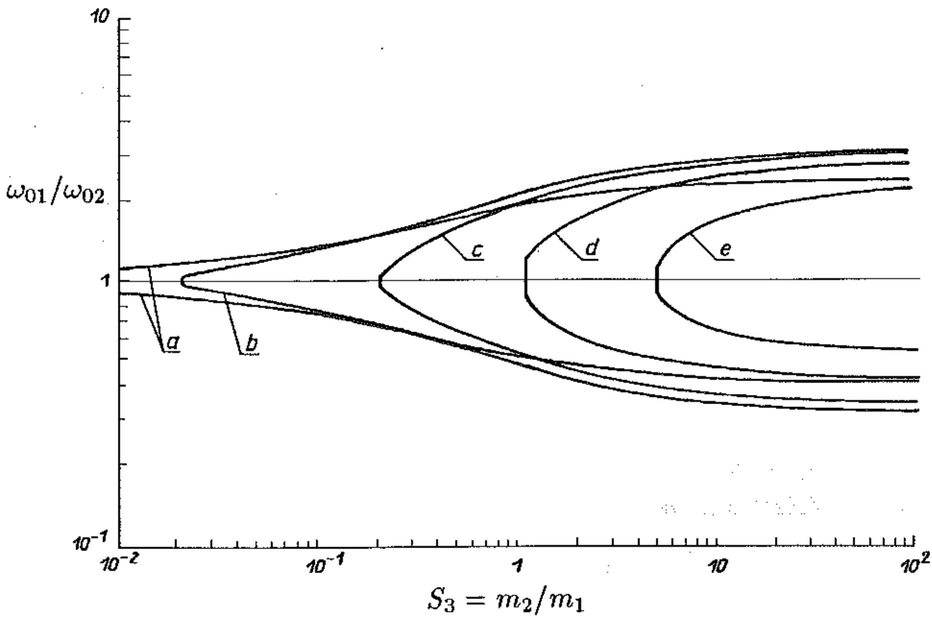

FIG. 5. Procedure II. Divergence regions for: $S_2 = m_1/k_1 = 0.0004(100/250000)$ and a) undamped system, b) $WSP1 = WSP2 = 0.1$, c) $WSP1 = WSP2 = 0.3$, d) $WSP1 = WSP2 = 0.6$, e) $WSP1 = WSP2 = 1.0$.

exist broad ranges of variability of parameters (S_1 for procedure I and S_3 for procedure II) with which the procedures stay convergent despite the free vibration frequencies of partial models are equal.

An exact answer to the question which method of decomposition into partial models to use and which type of the iteration procedure to select can be obtained with the help of analyses presented in [3]. Mandelshtam's coefficients can also be used for the purpose since a definite relationship between them and the convergence has been uncovered.

To put it simply, it can be said that the procedure I should be used when the coefficient σ^2, formula (2.7), is smaller than the coefficient $\bar{\sigma}^2$, formula (3.9). Otherwise the procedure II is to be chosen. On comparing suitable relations

$$\frac{4S_1^2S_3}{(S_1 - S_3 - S_1S_3)^2} < \frac{4S_1S_3^2}{(S_3 - S_1 - S_1S_3)^2},$$

and rearranging we obtain the inequality

$$(S_3 - S_1)\left[S_1^2S_3^2 + (S_1 + S_3)^2 + 2S_1S_3(S_1 + S_3)\right] > 0\,.$$

The bracket expression is always positive, so finally

$$\sigma^2 < \bar{\sigma}^2 \Leftrightarrow S_1 < S_3\,. \tag{4.2}$$

Very simple criterion has just emerged to select provisionally a way of decomposition and a relevant iteration procedure. This criterion is less precise than the conditions given in [3], but is very convenient to use. Another advantage is that it can also be employed in the case of more complicated systems for which any derivation of a more exact criterion is too time-consuming.

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TECHNOLOGICAL UNIVERSITY OF KIELCE, KIELCE.

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