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TIME AND CRACK LENGTH DEPENDENT STOCHASTIC MODELS OF FATIGUE CRACK GROWTH STATE-OF-THE-ART REVIEW

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Random character of fatigue crack growth observed in experiments inspires many researchers to apply the methods of probability theory and theory of stochastic processes in modeling of the phenomenon. Among many proposals the continuous stochastic models seem to be very promising both in theoretical consideration and engineering applications. In the paper two concepts of this kind of modeling are presented: the time and crack length dependent models. Some advantages and objections concerning both approaches are extensively discussed. It allows us to draw some more general conclusions on the necessary improvements which should be accounted for in the stochastic modeling of the fatigue crack growth.

1. Introduction

Material nonhomogeneity has random character but its significance depends on the kind of phenomenon which we observe or intend to describe. It always produces a random scatter of the quantities which characterize the results. The more local character have the factors affecting the process, the more significant become the random fluctuations of the material properties. The fatigue crack growth is an example of a locally conditioned process which is affected by material properties from a small neighbourhood of the crack tip. The random nature of crack growth was always recognized to produce a great scatter of experimental results. Various load conditions, different materials, crack and specimen geometries, different measuring systems used in different laboratories and limited number of data could not supply a reliable basis to apply some general and advanced statistical approaches. Mean value and eventually variance of time to reach a given crack length and a regression analysis to fit a crack growth law were usually the most ambitious achievements in statistical analysis of the phenomena. The prediction capability of such an analysis cannot be satisfactory. The analysis describes rather the results of an experiment than provides some general information about stochastic nature of the investigated material under fatigue due to the crack growth.

Among many experimental results which one can find in the literature on the fatigue crack growth under constant amplitude loading there are only a few which allow us to consider them not only as a set of data requiring statistical analysis but give us a possibility to employ evolutionary methods providing an insight into stochastic nature of the fatigue crack propagation process. The most frequently investigated results are given by VIRKLER *et al.* [22]. As usually in fatigue crack growth experiments the data set contains the couples: crack length, a, and the corresponding number of cycles to reach it, N. In Virkler's experiment 68 specimens of 2024-T3 aluminum alloy shaped as panels of $558.8 \times 152.4 \times 2.54$, all in millimeters (length $\times$ width $\times$ thickness), with a central slit of 2.54 mm length and 0.18 mm width were tested. The stress amplitude was kept constant as $\Delta S = 48.28$ MPa value with the stress ratio $R = 0.2$. The relevant measurements started at the crack length $a = 9$ mm as an initial crack length and proceed every 0.2 mm increment within (9.0 mm, 36.2 mm), then every 0.4 mm within (36.2 mm, 44.2 mm) and finally every 0.8 mm within (44.2 mm, 49.8 mm) providing 164 couples (a_i, N_i) for every test. All experiments were conducted by the same operator and on the same machine. The results are shown in Fig.1a.

Another set of very valuable fatigue crack test data was provided by GHONEM and DORE [7]. They used 7075-T6 aluminum alloy rectangular specimens of $320 \times 101 \times 3.175$, all in millimeters (length $\times$ width $\times$ thickness), with a central crack initiating notch of 14.3 mm. After some initial loading application the measurement started at the crack length $a = 9$ mm and continued to $a = 23$ mm as the final length. These tests were conducted at three different load (stress) amplitude conditions, namely: 1) $\Delta S = 28.41$ MPa, $R = 0.6$; 2) $\Delta S = 34.68$ MPa, $R = 0.5$; 3) $\Delta P = 28.41$ MPa, $R = 0.4$ where $R = S_{\min}/S_{\max}$. 60 identical specimens were tested at each load condition providing the sequences of couples (a_i, N_i) for every specimen. The results are shown in Figs.1b–d.

The first glance at the figures allows us to notice that even in well-controlled experiments under constant amplitude loading the scatter of the results is not negligible, e.g. the number of cycles which are necessary to advance the crack from its initial length to the final one can be different for different specimens of the same material. It is obvious that the scatter

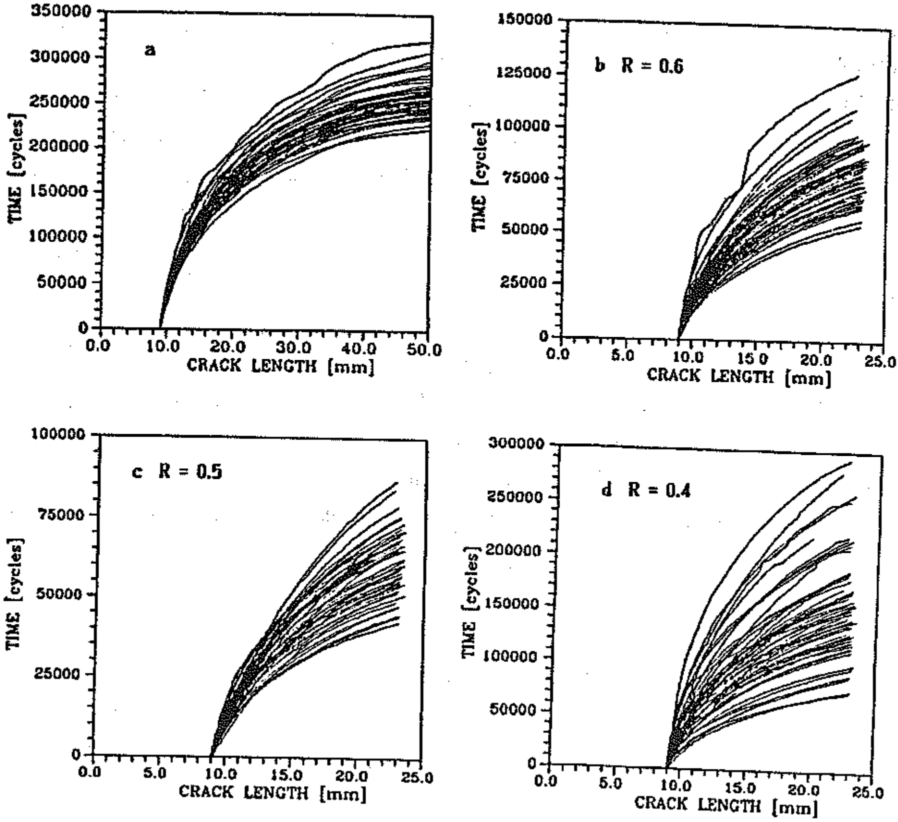

FIG. 1.

originates from the randomness of the material properties around the crack tip. It would seem that the assumption of parameters in a crack growth law as some random functions allows us to account for all these randomness in the most proper way. Unfortunately, substitution of all material parameters with random functions, identification of their statistical properties with keeping the model applicable is hardly realistic.

More careful analysis of the experimental results shows a tendency of keeping slow or fast rate for the specimens for which the crack started out to propagate slower or faster, respectively. It suggests a statistical scatter of material properties between the specimens, i.e. some statistical inhomogeneity in a large scale. Moreover, some $a - N$ curves cross each other suggesting a local change of the material properties at the crack tip front which leads to temporal acceleration or retardation of crack growth accidentally for various specimens. It means that the specimens differ from each other by the mean values of the material properties over a specimen and the microstructural nonhomogeneity of material within a specimen leads additionally to the observable disturbances of the fatigue crack growth. The former is usually accounted for by simple randomization of the crack growth law, while the material constants or some of them only are replaced with random variables. The statistical parameters of them are derived from data and such a randomized crack growth law describes the effect of the statistical nonhomogeneity. It neglects the microstructural effects. The latter should be modeled by random processes which reflect the local variations of the material parameters during fatigue crack growth. It is believed that this much more advanced modeling will be able to describe appropriately not only the experimental results but also to provide the methods of both material parameter identification and fatigue crack behaviour prediction. The prediction capability is the most important feature which verifies the usefulness of any model.

In any crack growth model there are some parameters involved. They are usually some model parameters which depend on test conditions. They cannot be considered as material parameters. In the classical fatigue crack growth equation proposed by PARIS and ERDOGAN [15]

$$\frac{\Delta a}{\Delta n} = C \cdot \Delta K^m, \tag{1.1}$$

where

$$\Delta K = \Delta S \cdot Y(a) \cdot \sqrt{\pi a} = (S_{\max} - S_{\min}) \cdot Y(a) \cdot \sqrt{\pi a} \tag{1.2}$$

means the range of the stress intensity factor, a is the crack length, S denotes the far stress and $Y(a)$ is a function which depends on the crack and specimen geometry, the parameters C and m have got no mechanical interpretation and do not suit the dimensional analysis. In the most simple nondeterministic models all or, more frequently, some of the parameters are assumed as random variables. Statistical characteristics of the variables, namely: means, variances, correlations, types of probability distributions have to be determined from experimental data. In general, the equation of the fatigue crack growth can be written in the following form:

$$\frac{\Delta a}{\Delta n} = g(a; \Theta), \tag{1.3}$$

where Θ denotes the random vector whose components are the random model parameters. The function $g(\cdot; \Theta)$ depends on these random variables. It depends, moreover, implicitly on the loading parameters but does not depend on the cycle number. Thus, the number of load cycles which is necessary to advance the crack of the length a_0 to the length a is called the lifetime of a structural element, $N_0(a;\Theta)$, and is given by the integral

$$N_0(a;\Theta) = \int_{a_0}^{a} \frac{da}{g(a;\Theta)} . \tag{1.4}$$

For the constant amplitude loading and non-varying frequency of cycles, ω, the lifetime can be expressed alternatively in time units as $T_0(a;\Theta) = N_0(a;\Theta)/\omega$.

The lifetime, $T_0(a;\Theta)$, as well as the fatigue crack length increment, $A_0(t;\Theta) = A(t;\Theta) - a_0$, within a given time interval, $\Delta t = t - t_0$, are some functions of the random vector $\mathbf{\Theta}$. Thus, they are some random variables themselves. For deterministic initial conditions, $A(t_0) = a_0$, $T(a_0) = t_0$, the monotone property of the fatigue crack growth process, $A(t;\mathbf{\Theta})$, assures the following relation between the probability distribution, $F_{A_0}(a;t)$, of the crack length increment within a given time interval, $\Delta t = t - t_0$, and the probability distribution, $F_{T_0}(t;a)$, of the lifetime for a given crack length increment, $A_0(t;\mathbf{\Theta}) = a - a_0$, to be valid

$$\mathbb{P}[A_0(t;\mathbf{\Theta}) \le a] = F_{A_0}(a;t) = 1 - F_{T_0}(t;a) = \mathbb{P}[T(a;\Theta) \ge t] , \tag{1.5}$$

where $\mathbb{P}[A]$ denotes the probability of a random event A.This relation originates from the renewal theory, e.g. FELLER [6] and means that the probability distribution of the lifetime defines the probability distribution of the fatigue crack length and vice versa.

A critical review of the probabilistic approach is given by KOZIN and BOGDANOFF [10]. The modeling of the fatigue crack growth process solely by means of the random variables excludes the possibility to describe the effect of local, stochastic fluctuations of material properties. It may be sufficient if we consider some great crack length increments. However, an information about the probability distribution of small crack length increments becomes more and more important, especially in design of the inspection and repair strategy. Therefore, a great effort is being done to find an adequate model which allows us to account for the stochastic nonhomogeneity of material within a specimen. Two main directions of the research on this field will be presented in the following sections.

2. Models with time dependent stochastic function

Considering the fatigue crack growth for moderate values of the stress intensity factor range, ΔK, the Paris–Ergodan equation is usually admitted. In the simple models the parameter C, Madsen [12], or both C and m, Doliński [4], are assumed to be some random sequences, e.g.

$$C_i = C_0 \cdot C_{1i} \qquad \text{and} \qquad m = M_0 , \tag{2.1}$$

where C_0 and M_0 are the random variables, and C_{1i} are the elements of a random sequence $\{C_{1i}\}$ so that for every crack length increment, ΔA_i, we have

$$\Delta A_i = C_0 \cdot \Delta K(a_i)^{M_0} \cdot C_{1i} . \tag{2.2}$$

The model without random fluctuations, i.e. $\{C_{1i}\} = \{1\}$, was often compared with experimental data, e.g. Virkler *et al.* [23], Kozin and Bogdanoff [10]. Statistical analysis has shown that the random variables, C_0 and M_0, are very strongly correlated, $\rho_{CM} \approx 0.90$, and have, respectively, log-normal and normal probability distributions. Assuming the exponent to be deterministic, $M_0 = m$, and the sequence elements, C_{1i}, to be mutually independent and independent of C_0, Madsen [12] simulated numerically the curves $a-n$ which appeared to agree very well, at least qualitatively, with the ones from the experiments. For a great number of the load cycles, $n \to \infty$, such a model gives a possibility to estimate easily the conditional probability distribution of the random variable $\Psi_0(n; A|c_0, m_0)$, which is related to the random crack length, $A(n)$, as follows

$$\Psi_0(n; A|c_0, m_0) = \int\limits_{a_0}^{A(n)} \frac{da}{c_0 \cdot \Delta K(a)^{m_0}} = \sum_{i=1}^{n} C_{1i} , \tag{2.3}$$

given $C_0 = c_0$ and $M_0 = m_0$. Assuming independence and common probability distribution of all sequence elements, C_{1i}, with the means and variances, respectively, $\mathbb{E}[C_{1i}] = \mathbb{E}[C_1] = \bar{C}_1$ and $\mathrm{Var}[C_{1i}] = \mathrm{Var}[C_1] = \sigma_{C_1}^2$, we obtain according to the central limit theorem of the probability theory

$$F_{\Psi_0|C_0M_0}(\psi; n|c_0, m_0) = \Phi\left(\frac{\psi - n \cdot \bar{C}_1}{\sigma_{C_1} \cdot \sqrt{n}}\right), \tag{2.4}$$

where $\Phi(\cdot)$ denotes the standard normal probability distribution function

$$\Phi(x) = \frac{1}{\sqrt{2 \cdot \pi}} \cdot \int\limits_{-\infty}^{x} \exp\left(-\frac{1}{2} \cdot z^2\right) dz = \int\limits_{-\infty}^{x} \varphi(z)\, dz . \tag{2.5}$$

The probability distribution, $F_{A_0|C_0,M_0}(a;n|c_0,m_0)$ of the crack length, $A(n)$, after n load cycles given $A(0)=a_0$, $C_0=c_0$ and $M_0=m_0$ can be directly derived from the probability distribution (2.4) as

$$F_{A_0|C_0,M_0}(a;n|c_0,m_0)=F_{\Psi_0|C_0,M_0}(\psi(a);n|c_0,m_0)\,. \tag{2.6}$$

If we know the two-dimensional probability density function, $f_{C_0M_0}(c_0,m_0)$, of the random variables C_0 and M_0, then the unconditional probability distribution, $F_{A_0}(a;n)$, of the crack length results from the integration

$$F_{A_0}(a;n)=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f_{C_0M_0}(c;m)\cdot F_{A_0|C_0M_0}(a;n|c,m)\,dc\,dm\,. \tag{2.7}$$

The product form of the parameter C, Eq.(2.1), requires the elements of the sequence $\{C_{1i}\}$ to be nonnegative random variables. Thus, the probability distribution, $F_{N_0}(n;a)$, of the random cycle number, $N_0(a)$, which is necessary to advance the crack from its initial length a_0 to a can be derived according to Eq.(1.5). If moreover the random fluctuations, C_{1i}, around the mean, $\bar{C}_1$, are sufficiently small so that the probability $\Phi(-\bar{C}_1/\sigma_{C_1})$ may be neglected, then the relation (1.5) can be used for the conditional limit distribution (2.4). The conditional probability distribution, $F_{N_0|C_0,M_0}(n;\psi(a)|c_0,m_0)$ of the number of cycles, $N_0[\psi(a)|c_0,m_0]$, takes the following form:

$$F_{N_0|C_0,M_0}[n;\psi(a)|c_0,m_0)=\Phi\left(\frac{n-\psi(a)/C_1}{\sigma_{C_1}/C_1\cdot\sqrt{n}}\right)\,. \tag{2.8}$$

Similar form of the probability distribution was obtained by BIRNBAUM and SAUNDERS [1] who investigated a damage process as a sum of independent, nonnegative random variables. The mean and variance for the distribution (2.8) are, respectively, as follows:

$$\mathbb{E}\,[N_0[\psi(a)|c_0,m_0]=\frac{\psi(a)}{C_1}\cdot\left(1+\frac{1}{2}\cdot\frac{\sigma^2_{C_1}}{C_1\cdot\psi(a)}\right)\,, \tag{2.9}$$

$$\mathrm{Var}\,[N_0[\psi(a)|c_0,m_0]=\frac{\sigma^2_{C_1}\cdot\psi(a)}{C_1^3}\cdot\left(1+\frac{5}{4}\cdot\frac{\sigma^2_{C_1}}{C_1\cdot\psi(a)}\right)\,. \tag{2.10}$$

From this model in which random fluctuations are represented by a stationary sequence of random variables we obtain the conditional coefficient of variation, $\nu_{\Psi_0|C_0,M_0}=\sigma_{\Psi_0|C_0,M_0}/\bar{\Psi}_{0|C_0,M_0}$, of the random increment of $\Psi_0(n;A|c_0,m_0)$, Eq.(2.3), given $C_0=c_0$ and $M_0=m_0$ as a decreasing function of the number of cycles, $\nu_{\Psi_0|C_0,M_0}=\sigma_{C_1}/(C_1\cdot\sqrt{n})$, cf. Eq.(2.4). It means that for long lifetimes the effect of stochastic nonhomogeneity diminishes and the random variables C_0 and M_0 suffice then to account for the random properties of the fatigue crack growth process.

Many researchers take an advantage of the rate form of fatigue crack growth equation by considering it as a differential equation of time. The stochastic fluctuations of $a-n$ curves are then modeled by means of a stochastic function of time alone, $X(t)$. It should be pointed out that in this kind of modeling the general tendency is to keep the function $g(a)$ to be deterministic with the model parameters, Θ, as some constants. Thus, the random function $X(t)$ accounts for the whole randomness of the process in the equation

$$\frac{dA(t)}{dt} = g(A)\cdot X(t)\,, \tag{2.11}$$

where $A(t)$ denotes the random crack length at the time instant t. For the constant amplitude loading the function $g(A)$ does not depend on the time. The separation of the variables allows us to introduce a function

$$\Gamma_0(t) = \Gamma_0[A(t)] = \int\limits_{a_0}^{A(t)} \frac{da}{g(a)}\,, \tag{2.12}$$

which defines the random increment of a non-normalized damage parameter, $\Gamma_0(t)$, over a random interval $[a_0, A(t)]$ (within a time interval $[t_0, t]$). Equation (2.11) takes now the very simple form

$$\frac{d\Gamma(t)}{dt} = X(t)\,. \tag{2.13}$$

From the mathematical point of view Eq.(2.11) or, equivalently, Eq.(2.13) are very convenient. In the most simple models it is assumed that

$$X(t) = \mu_X + \xi(t)\,, \tag{2.14}$$

where $\xi(t)$ is the zero mean Gaussian white noise with intensity α, i.e. with the covariance function

$$\mathbb{E}[\xi(t')\cdot\xi(t'')] = 2\alpha\cdot\delta(t'-t'') \tag{2.15}$$

with $\delta(\cdot)$ as the Dirac function. It leads to the Itô stochastic differential equation for the damage parameter $\Gamma_0(t)$ in the following form:

$$d\Gamma_0(t) = \mu_X dt + \sqrt{2\alpha}\cdot dW(t)\,, \tag{2.16}$$

where $W(t)$ is the Wiener process with the initial condition $W(t_0) = 0$ almost surely and the variance, $\mathrm{Var}[W(t)] = t - t_0$. The mean and variance of $\Gamma_0(t)$ are easily determined as

$$\begin{aligned} \mathbb{E}[\Gamma_0(t)] &= \mu_X \cdot (t - t_0), \\ \mathrm{Var}[\Gamma_0(t)] &= 2\alpha \cdot (t - t_0). \end{aligned} \tag{2.17}$$

Since the process $X(t)$ is Gaussian and Eq.(2.13) is linear, the probability distribution, $F_{\Gamma_0}(\gamma; t)$ of the damage parameter, $\Gamma_0(t)$ at any time instant t is Gaussian with the mean and variance like in Eq.(2.17) i.e.

$$F_{\Gamma_0}(\gamma; t) = \mathbb{P}[\Gamma_0(t) \leq \gamma] = \Phi\left[\frac{\gamma - \mu_X \cdot (t - t_0)}{\sqrt{2\alpha \cdot (t - t_0)}}\right]. \tag{2.18}$$

The probability distribution, $F_A(a; t)$, of the crack length, $A(t)$, at time t given $A(t_0) = a_0$ can be directly calculated from the probability distribution of the damage parameter as

$$F_A(a; t) = F_{\Gamma_0}(\gamma; t). \tag{2.19}$$

The probability distribution, $F_T(t; a)$, of the time, $T(a)$, when the crack reaches the given length $a(t_0) = a_0$ is determined as the probability of the first crossing of the level a by the process $A(t)$. Keeping the damage parameter notation valid for its simplicity we can alternatively consider the crossing of a level $\gamma = \gamma(a)$ by the damage parameter process, $\Gamma_0(t)$, which both are respectively related to a and $A(t)$ by Eq.(2.12). Hence, the first crossing time, $T(\gamma)$, is defined as

$$T(\gamma) = \sup\{t : \sqrt{2\alpha} \cdot W(t) \leq \gamma - \mu_X \cdot t\}. \tag{2.20}$$

The similar crossing problem was investigated by E. Schrödinger in his research on the Brownian motion [19]. A quite simple method of determination of the probability distribution function of the random variable given by equation of the type (2.20) was proposed by Ditlevsen [2]. It is the inverse Gaussian probability distribution with the density and cumulative probability functions in the following forms:

$$\begin{aligned} f_T(t; \gamma) &= \frac{\gamma}{\sqrt{2\alpha \cdot t} \cdot t} \cdot \varphi\left(\frac{\gamma - \mu_X \cdot t}{\sqrt{2\alpha \cdot t}}\right), \\ F_T(t; \gamma) &= \Phi\left(\frac{\mu_X \cdot t - \gamma}{\sqrt{2\alpha \cdot t}}\right) + \exp\left(\frac{\mu_X \cdot \gamma}{\alpha}\right) \cdot \Phi\left(-\frac{\mu_X \cdot t + \gamma}{\sqrt{2\alpha \cdot t}}\right), \end{aligned} \tag{2.21}$$

with the mean and variance, respectively,

$$\mathbb{E}[T(\gamma)] = \frac{\gamma}{\mu_X}, \qquad \operatorname{Var}[T(\gamma)] = \frac{2\alpha \cdot \gamma}{\mu_X^3}. \tag{2.22}$$

A comprehensive discussion of consequences of the time-dependent white noise modeling in the fatigue crack analysis was given by SOBCZYK [17]. He pointed out that there exists a non-zero probability of the instable crack growth, $a(t) \to \infty$, for some finite time, $t < \infty$, if the function $g(a)$ in Eq.(2.11) grows faster than the linear one. For the damage parameter notation it corresponds to a limit value, $\gamma_0(a) \to \gamma^*$ for $a \to \infty$. The problem of the so-called explosion time in the fatigue crack growth analysis was also discussed in TSURUI and ISHIKAWA [21].

The additive form of the function $X(t)$ with the Gaussian white noise term, Eq.(2.14), results in a non-zero probability of physically inadmissible negative crack length increments. Therefore, the usual relation (1.5) based on the monotone increase of the crack length may not be applied directly. If, however, the intensity of the white noise is sufficiently small in comparison with μ_X so that the probability of negative values of $X(t)$ can be practically neglected, then the probability distribution $F_{T_0}(t;a)$ of the lifetime, $T_0(a)$, which is necessary for the damage parameter, $\gamma(t)$, to reach a given value γ takes the following form

$$F_{T_0}(t;a) = \mathbb{P}[T_0(\gamma) \le t] = 1 - F_{\Gamma_0}(t;a) = \Phi\left[\frac{\mu_X \cdot (t-t_0) - \gamma}{\sqrt{2\alpha \cdot (t-t_0)}}\right]. \tag{2.23}$$

Assuming $t_0 = 0$ and comparing Eq.(2.23) with Eq.$(2.21)_2$ it is seen that for small fluctuations, i.e. for $2\alpha \ll \mu_X^2$, and great increments of the damage parameter, $\gamma \gg \mu_X$, the first term in Eq.$(2.21)_1$ dominates and the inverse Gaussian probability distribution tends to this one given in Eq.(2.23). Once again, c.f. Eq.(2.8), there appears the probability distribution proposed by BIRNBAUM and SAUNDERS. The mean and variance now are, respectively, given by

$$\begin{aligned} \mathbb{E}[T(\gamma)] &= \frac{\gamma}{\mu_X} \cdot \left(1 + \frac{1}{2} \cdot \frac{2\alpha}{\mu_X \cdot \gamma}\right), \\ \operatorname{Var}[T(\gamma)] &= \frac{2\alpha \cdot \gamma}{\mu_X^3} \cdot \left(1 + \frac{5}{4} \cdot \frac{2\alpha}{\mu_X \cdot \gamma}\right). \end{aligned} \tag{2.24}$$

It is worth to notice that the lifetime probability distribution (2.23) was discussed by BIRNBAUM and SAUNDERS [1] and derived from the investigation of a sum of independent random variables. Assuming many terms, the probability distribution of the sum tends to the Gaussian one with the mean and variance as in Eq.(2.24) provided that $(t - t_0) = N \cdot \Delta t$, where N is the number of terms and Δt means the load cycle duration, say. Now, the coincidence is not surprising anymore because the assumption that all terms in the sum are positive had to be made to validate the relation (2.23) which takes its origin in general renewal theory.

The time-dependent white noise model of the fatigue crack growth were extensively studied by Virkler *et al.* [24] and Lin and Yang [11]. Comparison of the theoretical results and experimental data shows that the white noise model leads to smaller variations of lifetimes than it comes out from the analysis of test data. The white noise is the stochastic process with correlation radius equal to zero. On the opposite pole is the random variable with the infinite correlation radius. Since the one random variable model where $X(t)$ is assumed to be a random variable, $X(t) = X$, leads to too great variations, cf. Kozin and Bogdanoff [10], it was very natural to assume some correlation which should improve the theoretical results. Such an approach was presented in Sobczyk [16, 17], Lin and Yang [11]. Lin and Yang investigated the crack growth model with a zero mean, stationary, stochastic process $Y(t)$ instead of the white noise in Eq.(2.14). It was assumed that $Y(t)$ is linearly correlated over a time interval Δ, i.e.

$$K_Y(\tau) = \mathbb{E}[Y(t) \cdot Y(t+\tau)] = 2\beta \cdot \left(1 - \frac{|\tau|}{\Delta}\right) \qquad \text{for} \quad |\tau| < \Delta\,, \tag{2.25}$$

and the correlation is equal to zero otherwise. Taking the advantage of the stochastic averaging method proposed by Stratanovitch and proved by Khasminskii [9], they appropriately transformed by differential equation (2.11) with a stationary process $Y(t)$ into the Itô stochastic differential equation (2.16) with the averaged drift and diffusion parameters $\alpha(a)$ and $\sigma(a)$ in the form

$$\begin{aligned} \alpha(a) &= g(a) \cdot \left[\mu_X + \beta \cdot \Delta \cdot \frac{dg(a)}{da}\right], \\ \sigma^2(a) &= 2g(a) \cdot \beta \cdot \Delta\,. \end{aligned} \tag{2.26}$$

Hence, the theory of diffusion Markov process might be applied to determine, in particular, the moments of the lifetime $T(a)$. The closed-form solutions were found under the assumption of a reflecting boundary at $a = 0$. The Paris type crack growth equation with $g(a) = Q \cdot a^p$ was assumed. In fact the stochastic process $X(t)$ was assumed to have the log-normal probability distribution. The model parameters and the mean and variance of the Gaussian process $\ln X(t)$ were estimated from the test data for the logarithmic form of the crack rate equation and then recalculated to give the mean and variance of the process $X(t)$. These moments and the correlation length are in fact only necessary to get the solutions in terms of lifetime moments. The correlation length Δ was changed to reach the best agreement with the data. The assumption that the crack growth process $A(t)$ is a diffusion Markov process in an approximate sense and appropriate determination of the drift and diffusion parameters replace in fact the stochastic process $X(t)$ by the sum of the mean and a white noise and define the respective Itô equation for an equivalent diffusion Markov process. Thus, it could be expected that at least for the long lifetimes, $T(a) \gg \Delta$, the probability distributions (2.21) or (2.23) remain approximately valid. The authors, however, assumed rather arbitrarily the Weibull probability distribution of the lifetime. Though they are satisfied comparing the experimental results, the agreement is far from being excellent. Nevertheless, it is explicitly shown that the correlation length significantly affects the variance of the lifetime and it is possible to find such a correlation interval Δ which assures a very good agreement with the mean and variance considered.

Very consequent application of the theory of Markovian processes is presented by SPENCER and TANG [18], TANG and SPENCER [20]. They assumed the process $X(t)$ to be generated by an exponentially correlated stationary standard Gaussian process $Z(t)$ so that

$$X(t) = F_X^{-1}[\Phi[Z(t)]], \tag{2.27}$$

where $F_X^{-1}(\cdot)$ denotes the inverse of the one-dimensional probability distribution, $F_X(x)$, of the process $X(t)$. Assuming the process $Z(t)$ to satisfy the stochastic differential equation

$$\frac{dZ(t)}{dt} = -\alpha \cdot Z + \xi(t), \tag{2.28}$$

where $\xi(t)$ is the white noise, cf. Eq.(2.15), the crack length, $A(t)$, is a component of the two-dimensional Markov process $[A(t), Z(t)]$ which is defined by the system of two differential equations: Eq.(2.28) and

$$\frac{dA(t)}{dt} = g(A) \cdot F_X^{-1}[\Phi(Z)]. \tag{2.29}$$

The stationary form of the correlation function of $Z(t)$ has the exponential form

$$K_Z(\tau) = \mathbb{E}[Z(t) \cdot Z(t+\tau)] = \exp(-\alpha \cdot |\tau|). \tag{2.30}$$

Using the correlation radius definition, $\tau_{\rm cor}$, as

$$\tau_{\rm cor} = \frac{\int\limits_0^\infty \tau \cdot K_Z(\tau)\, d\tau}{\int\limits_0^\infty K_Z(\tau)\, d\tau}, \tag{2.31}$$

we get $\tau_{\rm cor} = 1/\alpha$ for the random function $Z(t)$ and, approximately, for the random fluctuations, $X(t)$, as well.

The proposed formulation of the problem allows the Authors to apply the methods of the general theory of stochastic diffusion Markovian processes to derive a recursive set of boundary value problems for statistical moments and probability distribution of the lifetime. The finite difference method is then proposed to find the solutions. In TANG and SPENCER [20] the results of the model prediction are compared with the Virkler, Ghonem and Dore data. The least square method was applied to establish the mean and variance of the stochastic process $X(t)$ as well as the parameters, ϑ_i, of the assumed crack growth equation, $g(a;\vartheta)$, which is considered further to be deterministic. The model prediction of the mean and variance of lifetimes given various initial and critical crack lengths approaches the experimental data very well, being almost insensitive to the choice of the probability distribution type of the process $X(t)$. There were considered three types of the probability distributions of $X(t)$: normal, log-normal and Weibull. All of them led to the results which are very close to each other.

In SPENCER and TANG [18] the effect of the correlation length, $\tau_{\rm cor} = 1/\alpha$, was also studied. The similar conclusions to those in LIN and YANG [11] could be drawn about its significant influence on the lifetime variance. And again it appears that an appropriate choice of the correlation length can assure very good agreement between the lifetime variances obtained from the model and experiment.

Summarizing the presentation of the models in which the time-dependent stochastic process is assumed to describe the random fluctuations of the fatigue crack growth process, the following remarks can be formulated:

1. The random function $X(t)$ as a factor in a crack growth equation accounts for all uncertainties which affect the crack growth process. The model parameters, ϑ, in the crack growth law function, $g(A;\vartheta)$, are considered as some deterministic quantities. Their values as well as the statistical parameters of $X(t)$ are determined from the least square regression analysis. Sometimes $X(t)$ is considered to account simultaneously for both loading

and material uncertainties. All of these are not the inherent limitations of the model but the usual procedure to handle it.

2. Only the Gaussian white noise model can be analyzed analytically leading to the inverse Gaussian probability distribution of lifetimes. It gives, however, too small variances of them. It seems that there are two sources of this discrepancy. One of them has been discussed above and concerns the zero correlation length of the white noise. Another one has been just noted in the first remark. Since the crack growth paths show simultaneously some statistical and stochastic uncertainties, it is not realistic to describe both of them by a quantity that has purely stochastic nature. Mixed, random variable and white noise model should give better agreement with experiments but unfortunately such an analysis is not known by the author.

3. A significant improvement of the calculated lifetime variations by introducing a finite correlation length of the process $X(t)$ points out the importance of the correlation in the stochastic model. The exact analytical solutions are not available but the numerical results indicate a negligible dependence of the lifetime probability distributions on the probability distribution of $X(t)$, Tang and Spencer [20]. Since the correlation length is expected to be short, the stochastic averaging is justified for longer lifetimes. It, however, leads to the equivalent stochastic differential equation which is driven by an equivalent white noise. Thus, the conclusions about the inverse Gaussian or the Birnbaum-Saunders probability distribution of the lifetime and normal distribution for the damage parameter, $\Gamma_0(t)$, hold to be valid provided that a long lifetime and great crack length increment are considered. The solutions, however, are very sensitive to the length of the correlation radius.

4. The stochastic modeling of the fatigue crack growth by means of the time-dependent stochastic process, $X(t)$, was very extensively discussed in the literature. It was natural then to apply the theory of diffusion Markov processes which has very strong mathematical foundations and offers very effective and well-developed mathematical tools based on the Fokker-Planck-Kolmogorov equations to solve the practical problems. It seems however that the modeling of the fatigue crack growth under constant amplitude loading, when the random properties of the material are solely responsible for the random behaviour of the crack growth process, by a stochastic quantity which depends on time alone does not describe properly the randomness of the phenomena. It is obvious that the randomness of the crack growth depends on the random properties of material around the crack tip. It means

it depends on a random quantity (random field, say) which depends primarily on the crack tip coordinates. Because the relation between the time and crack length is random, any mapping of the space coordinates onto the time one would require at least an averaging procedure. Any transformation of this kind depends on the load conditions so that the parameters of the function $X(t)$ would not be objective but load-dependent. Some proposals of the crack length-dependent random models are presented in the next section.

3. Models with crack length-dependent random function

The modeling of stochastic fatigue crack growth by means of a space-dependent stochastic function has not found so much interest like the models previously discussed. The contributions to this subject are restricted to a few papers only. They provide, however, some very interesting results which combine the methods of material parameter identification and usual reliability problems connected with derivation of the lifetime or crack length probability distributions.

An original proposal of the random fatigue crack growth model is given by Ditlevsen [2]. A single cycle crack length increment Δa_i, is considered to be equal

$$\Delta a_i = g(a_i, \Theta) \cdot X_i \,, \tag{3.1}$$

where X_i is the random variable reflecting the effect of material nonhomogeneity over the crack increment, $\Delta a_i = a_{i+1} - a_i$. The parameter vector, Θ, is considered to be random. The analysis is done separately for every specimen so that all results are conditioned given the realizations, ϑ_j, of the random vector Θ. It will not be explicitly written in the further notation. From the logarithmic form of the Eq.(3.1)

$$\ln \Delta a_i = \ln g(a_i) + R(\Delta a_i; a_i) \,;$$

the residuals, $R(\Delta a_i; a_i) = \ln X_i$, are defined in the following integral form, (the subscript i is omitted)

$$R(\Delta a; a) = \frac{1}{\sqrt{\Delta a}} \cdot \int_0^{\Delta a} \xi(a+s)\, ds \,, \tag{3.2}$$

where $\xi(a)$ denotes the Gaussian white noise depending on the crack tip location a and of the intensity 2α. The crucial assumption in the approach is the averaging the integral in Eq.(3.2) with respect to the square root of the integration interval $\sqrt{\Delta a}$, and not to the interval length itself, Δa. It assures the variance of the residuals to be independent of Δa. Moreover, the covariance function, $K_R(\Delta a_1, \Delta a_2)$, of the residuals, $R(\Delta a; \cdot)$ depends only on the ratio $\Delta a_1/\Delta a_2$

(3.3) $$K_R(\Delta a_1, \Delta a_2) = 2\alpha \cdot \sqrt{\frac{\Delta a_1}{\Delta a_2}} \qquad \text{for} \quad \Delta a_1 \le \Delta a_2 .$$

Hence, because the process $R(\Delta a; \cdot)$ is Gaussian, it assures the equivalence of two Gaussian processes

(3.4) $$R(\Delta a; a) = R\left[g(a) \cdot \frac{\Delta a}{g(a)}; a\right] = R\left[\frac{\Delta a}{g(a)}; a\right] = R(x; a) .$$

Using the notation $x = \Delta\gamma = \Delta a/g(a)$ the crack length increment, Δa, can be found in terms of x as the solution of the first passage problem defined by

(3.5) $$\sqrt{x} \cdot \ln x = \sqrt{2\alpha} \cdot W(x) ,$$

where $W(x)$ is the Wiener process starting at the origin, cf. Eq.(2.16). The first order Taylor expansion of the left-hand side of Eq.(3.5) in $x = 1$ leads to the first passage problem of a linear boundary by the Wiener process

(3.6) $$X = \sup\{x : W(x) \le x - 1\}$$

which has been already discussed in the previous section but in a quite different context, cf. Eq.(2.20). Equation (2.20) defined the random time of the first crossing of a damage parameter level γ by the random damage parameter process $\Gamma_0(t)$. Now, Eq.(3.6) defines a random increment of the damage parameter corresponding to the crack length increment Δa_i. Hence, the resulting inverse Gaussian probability distribution of $X_i = \Delta\Gamma_i$ has the density function as follows:

(3.7) $$f_X(x) = \frac{1}{x \cdot \sqrt{2\alpha \cdot x}} \cdot \varphi\left(\frac{x-1}{\sqrt{2\alpha \cdot x}}\right)$$

with the mean and variance, respectively, $\mathbb{E}[X] = 1$ and $\mathrm{Var}[X] = 2\alpha$. Returning to the damage parameter notation, cf. Eq.(2.12), the following equation

(3.8) $$\Gamma_0(t) = \int\limits_{a_0}^{A} \frac{da}{g(a)} = \sum_{i=1}^{n} X_i$$

defines the random damage parameter increment which corresponds to the crack length increment $\Delta A = A - a_0$ after n load cycles. The Eq.(3.8) is similar to Eq.(2.3) but the stochastic properties of the random variables X_i result from the first crossing problem for the damage parameter. They are mutually independent and all with inverse Gaussian probability distributions, cf. Eq.(3.7). Thus, their sum (3.8) has the inverse Gaussian distribution as well

$$F_{\Gamma_0}(\gamma; n) = \Phi\left(\frac{\gamma - n}{\sqrt{2\alpha \cdot \gamma}}\right) + \exp\left(\frac{n}{\alpha}\right) \cdot \Phi\left(-\frac{\gamma + n}{\sqrt{2\alpha \cdot \gamma}}\right) \tag{3.9}$$

with the mean and variance, respectively, $\mathbb{E}[\Gamma_0(n)] = n$ and $\mathrm{Var}[\Gamma_0(n)] = 2\alpha \cdot n$. The probability distribution, $F_A(a; n)$, of the crack length, $A(n)$, can be determined from Eq.(2.19) with $F_{\Gamma_0}(\gamma; n)$ as in Eq.(3.9). Random variables with inverse Gaussian probability distribution are nonnegative. Thus, the relation (1.5) can be applied to determine the lifetime distribution. Assuming the constant frequency, ω, of the load cycles, the probability distribution function of the lifetime, $T(\gamma) = N(\gamma)/\omega$ satisfies the equation

$$F_T(t; \gamma) = 1 - F_{\Gamma_0}(\gamma; n) \,. \tag{3.10}$$

The second term in the probability distribution function (3.9) tends to zero with increasing n. Neglecting this term the probability distribution of the damage parameter now approaches the Birnbaum-Saunders probability distribution, cf. Eq.(2.23). The probability distribution of the lifetime corresponding to a damage parameter value γ now becomes Gaussian with mean and variance, respectively, $E[N] = \gamma$ and $\mathrm{Var}[N] = 2\alpha \cdot \gamma$.

Furthermore, the sum of independent, identically distributed random variables X_i suggests the application of the central limit theorem of probability theory. Thus, for a large number of cycles, $n \to \infty$, or, equivalently, long time, $t \to \infty$, the probability distribution, $F_{\Gamma_0}(\gamma; t)$, of the damage parameter, $\Gamma_0(t)$, is approximately Gaussian with the mean and variance, respectively, $\mathbb{E}[\Gamma_0(t)] = t$ and $\mathrm{Var}[\Gamma_0(t)] = 2\alpha \cdot t$. But the same type of the probability distribution can be obtained by substitution $X_i = 1 + \xi_i$, where ξ_i, $i = 1, 2, \ldots$, are the zero-mean, independent, normal random variables with variances $\mathrm{Var}[\xi_i] = 2\alpha \cdot t$. Substituting now the sum in Eq.(3.8) with an integral over the interval $[0, t = n/\omega]$, an equivalent continuous model for the damage parameter

$$\Gamma_0(t) = t + \sqrt{2 \cdot \alpha} \cdot W(t) \tag{3.11}$$

can be written down with the standard Wiener process $W(t)$. Thus, the lifetime, $T_0(\gamma)$, is the random variable as it has been already defined in Eq.(2.20) but now with $\mu_X = 1$. The probability distribution of the lifetime is again the inverse Gaussian one, cf. Eq.(2.21) with $\mu_X = 1$. Table 1 summarizes the sequence of the assumptions and their consequences.

Table 1. Probability distribution types of the damage parameter and lifetime resulting from the assumptions in the Ditlevsen model.

Assumption	$F_{\Gamma_0}(\gamma; t)$	$F_T(t; \gamma)$
$\ln \dfrac{\Delta a}{g(a)} = \dfrac{1}{\sqrt{\Delta a}} \cdot \int\limits_0^{\Delta a} \xi(a+s)\, ds$	inverse Gaussian	$1 - F_{\Gamma_0}(t; \gamma)$
$n \gg \gamma$	Birnbaum-Saunders	Gaussian
$n \to \infty$	Gaussian	Birnbaum-Saunders
$\Delta \Gamma_i = 1 + \xi_i$	Gaussian	inverse Gaussian

The logical sequence of the assumptions leads from the model in which the stochastic material nonhomogeneity was described by the crack length dependent white noise to the model in which the damage parameter is defined as a sum of independent Gaussian random variables. Thus, the probability distribution of the damage parameter changes from the inverse Gaussian one to Gaussian, and the lifetime probability distribution finally appears to be the inverse Gaussian one. For the long times (a great number of cycles) there is an equivalence to the model discussed in the previous section, where the time dependent white noise reflected the effect of the stochastic nonhomogeneity of material.

Ditlevsen and Olesen [3] presented a very extensive study of the Virkler crack growth data. They admit the inverse Gaussian probability distribution of the lifetime. The Paris-Erdogan equation, cf. Eq.(1.1), is assumed with the parameters C and m as random variables. In order to improve the results the Authors introduce, rather arbitrarily, a Tshebyshev polynomial up to the seventh order. The material parameters, intensity of the white noise and polynomial coefficients, were estimated from the maximum likelihood method for every specimen. Very good agreement was shown between the experimental results and the combined method where both the random variables and the random process are simultaneously considered to describe the random properties of the fatigue crack growth process.

The approach just presented is virtually set up on the assumption that the variance of the damage parameter increments does not depend on this increment. Although the crack length dependent white noise is assumed at the beginning to reflect the material stochastic nonhomogeneity, the eventual consequence of subsequent assumptions is that increments of the damage parameter may be modeled by independent, identically distributed, Gaussian random variables. It leads back to the stochastic time dependent models and results in the inverse Gaussian probability distribution of lifetimes. In the previous section this model was discussed to yield inappropriate variations of the lifetimes. In DITLEVSEN and OLESEN [3] the agreement is very good. However, contrary to those proposals the Authors consider here the statistical, Θ, and the stochastic, X_i, uncertainties separately. The next step would be to take the stochastic crack length dependent process more consequently to describe the variations of material properties with account for their correlations along the crack growth path.

Such an approach is proposed by ORTIZ and KIREMIDJIAN [13, 14]. The crack growth rate equation is there assumed to be

$$\frac{da}{dt} = g(a,\Theta) \cdot X(a)\,, \tag{3.12}$$

where $X(a)$ is a stationary, log-normally distributed stochastic process with unit median. The logarithmic form of this equation

$$\ln \frac{da}{dt} = \ln g(a;\Theta) + Z(a) \tag{3.13}$$

involves the zero mean, Gaussian, stationary process, $Z(a) = \ln X(a)$, with the variance, σ_Z^2, and the covariance function

$$K_Z(\alpha) = \mathbb{E}[Z(a) \cdot Z(a+\alpha)] = \sigma_Z^2 \cdot \rho_Z(\alpha)\,, \tag{3.14}$$

with $\rho_Z(\alpha)$ as the correlation function. Every crack growth path sample, j, is analyzed by the least square regression procedure to obtain the respective sample, $\boldsymbol{\vartheta}_j$, of the parameter vector, Θ, and the averaged residuals

$$Z_{\Delta a_j,j} = \ln \frac{\Delta a_i}{\Delta N_i} - \ln g(a_i;\boldsymbol{\vartheta}_j)\,, \tag{3.15}$$

where $\Delta a_i/\Delta N_i$ is the averaged crack rate given from the experimental data over the interval $\Delta a_i = a_{i+1} - a_i$. In order to calculate the covariance function of the locally defined process $Z(a)$, the discrete averaged spectral density, $S_{Z_{\Delta a}}(\lambda)$, is estimated by the Fourier transform of the sequence $Z_{\Delta a_i,j}$ and then the spectral density, $S_Z(\lambda)$, of the process $Z(a)$ is obtained by the relation from the filter theory, i.e.

$$S_{Z_{\Delta a}}(\lambda) = |H_{\Delta a}(\lambda)| \cdot S_Z(\lambda)\,, \tag{3.16}$$

where

$$H_{\Delta a}(\lambda) = \left[\frac{\sin\left(\lambda \cdot \dfrac{\Delta a}{2}\right)}{\lambda \cdot \dfrac{\Delta a}{2}}\right]^2 \tag{3.17}$$

is the filter function of an averaging filter with the window Δa. It is required that all measurements are done at the same crack length interval, $\Delta a_i = \Delta a = \text{const}$. The inverse Fourier transform

$$K_Z(\alpha) = \int_{-\infty}^{\infty} S_Z(\lambda) \cdot \exp(i\lambda\alpha) d\lambda \tag{3.18}$$

gives the covariance function of the local stochastic process $Z(a)$ while the variance of $Z(a)$ is given by $\sigma_Z^2 = K_Z(0)$. Repeating this procedure for every crack growth path sample the set of parameter vectors, $\boldsymbol{\vartheta}_j$, variances, $\sigma^2_{Z,j}$, and spectral densities, $S_{Z,j}(\lambda)$ are obtained. Hence, the statistical ensemble parameters are easily calculated. In particular, the mean of the covariance function, $\bar{K}_Z(\alpha)$ is estimated from the mean spectral density

$$\bar{S}_Z(\lambda) = \frac{1}{J} \cdot \sum_{j=1}^{J} S_{Z,j}(\lambda) \tag{3.19}$$

applying the inverse transformation (3.18).

The authors demonstrate this procedure using the Virkler crack growth data and assuming the Paris crack growth equation. The analysis shows that the exponent correlation function may be admitted, i.e.

$$\rho_Z(\alpha) = \exp(-|\alpha|/\alpha_0)\,, \tag{3.20}$$

where α_0 denotes the correlation length estimated to be $\alpha_0 = 0.12\,\text{mm}$.

In order to estimate the probability distribution of the lifetime, the effect of the two types of randomness, the statistical and stochastic ones, is separated in the integral solution of Eq.(3.12), i.e.

$$(3.21)\qquad T_0(a;\Theta) = \int_{a_0}^{a} \frac{da}{g(a;\Theta)\cdot X(a)} = \int_{a_0}^{a} \frac{da}{g(a;\Theta)} \cdot \frac{\displaystyle\int_{a_0}^{a} \frac{da}{g(a;\bar{\Theta})\cdot X(a)}}{\displaystyle\int_{a_0}^{a} \frac{da}{g(a;\bar{\Theta})}}$$

$$= I_0(a;\Theta)\cdot I_X(a,\bar{\Theta}),$$

where $\bar{\Theta} = \mathbb{E}[\Theta]$. Moreover, it is assumed that both random functions, $Z(a)$ and $X(a)$, have the same correlation function (3.20) and the second factor in Eq.(3.21), $I_X(a;\Theta)$ has the log-normal probability distribution. Its effect on the lifetime, especially on the lifetime variance, can be easily investigated. Comparison with the Virkler data shows very good agreement but the results are also very sensitive to the correlation radius. ORTIZ and KIREMIDJIAN [14] examined three possibilities for the process $X(a)$:

1. $X(a)$ is equal to one – it gives the random variable model with C and m as random variables;
2. $X(a)$ is the white noise with log-normal probability distribution;
3. $X(a)$ is the log-normal stationary stochastic process with the exponential correlation function (3.20).

The analysis allows the Authors to draw the following conclusions:

1) the random variable model predicts properly the variance of the lifetime for great crack length increments but underestimates it for the shorter ones;

2) the white noise model predicts properly the variance of the lifetime for median and large crack length increments but overestimates it for the short ones;

3) the model with finite correlation length is able to predict very well the lifetime variance over the entire crack length increment range provided that the correlation length is properly chosen. The prediction is very sensitive to the change of the correlation length as it could also be concluded comparing the random variable and white noise models.

4. Concluding remarks

The fatigue crack rate depends, in general, on the stress/strain state in the neighbourhood of the crack tip. Thus, the material properties in a material volume surrounding the crack tip affect, in fact, the crack growth process. In the most stochastic models of the fatigue crack growth this feature is not considered. The time or crack length dependent stochastic functions which are assumed to account for the effect of stochastic material nonhomogeneity are not related explicitly to the crack tip affected zone. Since the size of this zone depends on the crack length, the properties of the random functions should change while the crack is growing up. Thus, the assumption that the functions are stationary does not reflect the real circumstances of the material nonhomogeneity effect. The stationarity of the crack length dependent random function, $X(a)$, means that the crack rate is affected by the material properties averaged over a constant size volume attached to and moving with the crack tip, say. The correlation function and, particularly, the correlation radius correspond, implicitly, to the shape and dimensions of the volume. In the limit white noise case the volume reduces to the point or a surface given its gradient is not orthogonal to the crack growth direction almost everywhere. Because of the nonlinear relation between the time and crack length the stationary time-dependent random function, $X(t)$, does introduce a non-stationary effect of the material nonhomogeneity but its specification is hardly possible. An attempt to derive the statistical properties of the random function $X(a)$ from a mechanical model of the fatigue crack growth is presented in the forthcoming paper, DOLIŃSKI [5].

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