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CREEP EFFECTS IN THE FINITE ELEMENT ANALYSIS OF THERMO-MECHANICAL PROBLEMS

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A finite element method is employed to analyse creep of a structure under mechanical and thermal load. Theoretical considerations are illustrated by a numerical test in which the effect of temperature on the stress relaxation is analysed.

1. Introduction

In general, application of load leads to a time-dependent stress and strain states in structures. However, such effects can also be encountered in structures under constant load due to creep of material. In metal structures creep becomes pronounced at elevated temperatures (above 300° C, according to [10]) whereas the phenomenon of creep in structures made of lightweight alloys and some man-made materials can be observed at room temperatures as well. For instance, pressurized, overheated to 400-500 ° C, steam pipes are known to have gradually thinner walls and larger diameters. These changes in dimensions are accompanied by a certain relaxation of stresses. The problem becomes increasingly important in flange connections operating at elevated temperatures since the stress relaxation in bolts may lead to the loss of tightness of the whole union. Considerations on creep problems and their analytical solutions can be found in [5, 10, 14].

It is, however, the finite element technique that enables the creep effects to be determined in structures of arbitrary shapes and under complex initial-boundary conditions, account being at the same time taken of the physical nonlinearity of materials and the temperature dependence of material constants such as the yield point. Additivity is here assumed of elastic, plastic, thermal and creep strains [2, 8, 9, 11, 13] and a creep stress-strain relationship must be formulated. It is more often than not that a creep law for the three-dimensional situations is established with the help of a relationship valid for uniaxial test [2, 9, 11, 13, 15].

The paper is aimed at employing FEM to analyse creep effects in a material of which a cantilevered beam is made. The beam free end is subjected to a time-independent load at various temperatures. Calculations were performed with the use af a nonlinear thermomechanical analysis program written for an IBM PC-compatible microcomputer.

2. Constitutive relationships

Analysis of thermo-elastic-plastic problems with creep effects can be obtained from the constitutive relationship in the form [1, 2, 6, 8, 13, 15]

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \varepsilon_{kl}^{(p)} - \varepsilon_{kl}^{(\Theta)} - \varepsilon_{kl}^{(c)}\right), \tag{2.1}$$

where σ_{ij} – stress tensor, $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ – elastic constants tensor, λ, μ – Lamé's constants, δ_{ij} – Kronecker's delta, ε_{kl} – total strain tensor, $\varepsilon_{kl}^{(p)}$ – plastic strain tensor, $\varepsilon_{kl}^{(\Theta)}$ – thermal strain tensor, $\varepsilon_{kl}^{(c)}$ – creep strain tensor.

To determine the plastic strain rate $\dot{\varepsilon}_{ij}^{(p)}$ a flow law is used associated with the yield criterion in the form

$$f(\sigma_{ij}, \sigma_y) = F(\sigma_{ij}) - \sigma_y\left(\bar{\varepsilon}^{(p)}, \Theta\right) = \sqrt{\frac{3}{2}\sigma_{ij}^D\sigma_{ij}^D} - \sigma_y = \bar{\sigma} - \sigma_y = 0, \tag{2.2}$$

where σ_y denotes the uniaxial yield point, $\bar{\sigma}$ stands for a stress intensity, $\sigma_{ij}^D = \sigma_{ij} - \frac{1}{3}\sigma_{nn}\delta_{ij}$ is a stress deviator and $\bar{\varepsilon}^{(p)}$ is an effective plastic strain.

Under the above assumptions the plastic strain rate $\dot{\varepsilon}_{ij}^{(p)}$ can be calculated from the equation

$$\dot{\varepsilon}_{ij}^{(p)} = \frac{1}{h}(n_{kl}\dot{\sigma}_{kl})n_{ij} - \sqrt{\frac{2}{3}}\frac{1}{h}\frac{\partial\sigma_y}{\partial\Theta}\dot{\Theta}n_{ij}, \tag{2.3}$$

where an isothermal strain-hardening parameter h is defined as

$$h = \frac{2}{3}\frac{\partial\sigma_y}{\partial\bar{\varepsilon}^{(p)}}\bigg|_{\Theta=\mathrm{const}}, \tag{2.4}$$

$n_{ij} = \sqrt{\frac{3}{2}}\frac{\sigma_{ij}^D}{\sigma_y}$ is a normal to the yield surface and $\bar{\varepsilon}^{(p)} = \int\limits_{T}\sqrt{\frac{2}{3}\dot{\varepsilon}^{(p)}\dot{\varepsilon}^{(p)}}\,dt$ is an effective plastic strains.

Accounting for Eqs.(2.1), (2.2) and (2.3), the constitutive relationship for the stress rate tensor becomes

$$\dot{\sigma}_{ij} = \left(C_{ijkl} - \frac{C_{ijst}n_{st}C_{prkl}n_{pr}}{h + n_{mn}C_{mntu}n_{tu}}\right)\left(\dot{\varepsilon}_{kl} - \overset{\star}{a}_{kl}(\Theta)\dot{\Theta} - \dot{\varepsilon}^{(c)}_{kl}\right) + \overset{\star}{\sigma}_{ij} + \overset{\star\star}{\sigma}_{ij}, \tag{2.5}$$

where

$$\overset{\star}{\sigma}_{ij} = \dot{C}_{ijkl}\left(\varepsilon_{kl} - \varepsilon^{(p)}_{kl} - \varepsilon^{(\Theta)}_{kl} - \varepsilon^{(c)}_{kl}\right), \tag{2.6}$$

$$\overset{\star\star}{\sigma}_{ij} = C_{ijkl}n_{kl}\left(\frac{-n_{pr}C_{prst}n_{st}\sqrt{\frac{2}{3}}\frac{1}{h}\frac{\partial\sigma_y}{\partial\Theta}\dot{\Theta} - \overset{\star}{\sigma}_{pr}\,n_{pr}}{h + n_{mn}C_{mntu}n_{tu}} + \sqrt{\frac{2}{3}}\frac{1}{h}\frac{\partial\sigma_y}{\partial\Theta}\dot{\Theta}\right). \tag{2.7}$$

3. Creep laws

Creep strain rate that enters Eq.(2.5) is, in the general case of combined stress state, proportional to the stress deviator [1, 2, 9, 13] and can thus be expressed as

$$\dot{\varepsilon}^{(c)}_{ij} = \lambda\sigma^{D}_{ij}, \tag{3.1}$$

where a scalar multiplier λ amounts to

$$\lambda = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}^{(c)}}{\bar{\sigma}}. \tag{3.2}$$

The stress intensity $\bar{\sigma}$ was defined in Eq.(2.2), $\dot{\bar{\varepsilon}}^{(c)}$ is the creep strain intensity rate depending on the temperature Θ and the time t as follows:

$$\dot{\bar{\varepsilon}}^{(c)}(\bar{\sigma}, \bar{\varepsilon}, \Theta, t) = \sqrt{\frac{2}{3}\dot{\varepsilon}^{(c)}_{ij}\dot{\varepsilon}^{(c)}_{ij}}. \tag{3.3}$$

For the $3D$ creep law to be formulated it is necessary to find the creep strain intensity function

$$\bar{\varepsilon}^{(c)} = \bar{\varepsilon}^{(c)}(\bar{\sigma}, t, \Theta) \tag{3.4}$$

from a uniaxial creep test.

A differential increment of the creep strain vector for an instant t can be, in matrix notation, shown as

$$d\varepsilon^{(c)} = k\mathrm{D}\sigma\,dt, \tag{3.5}$$

where **D** is a matrix operator transforming a stress vector σ into its deviator σ^D.

As creep effects can be assumed [14] to depend on stress, time and temperature in a separate manner, the following expression follows

$$\bar{\varepsilon}^{(c)} = f_1(\sigma)\, f_2(t)\, f_3(\Theta)\,. \tag{3.6}$$

In expressions (3.7) to (3.12) are presented (written by the author of paper [14]) some forms of functions $f_1(\sigma)$, $f_2(t)$ and $f_3(\Theta)$ (see Eq.(3.6)) and names of their authors obtained for uniaxial test under the assumption of constant strain.

The function $f_1(\sigma)$ was proposed by Norton and Bailey to have the form

$$\dot{\varepsilon}^{(c)} = K\sigma^m\,, \tag{3.7}$$

whereas Odquist suggested a relationship

$$\dot{\varepsilon} = \frac{d}{dt}\left(\frac{\sigma}{\sigma_{c0}}\right)^{n_0} + \left(\frac{\sigma}{\sigma_c}\right)^n\,, \tag{3.8}$$

where $K, m, \sigma_c, \sigma_{c0}, n, n_0$ are constants.

To describe creep strain as a function of time ($f_2(t)$) the following relationships can be used:

$$\varepsilon^{(c)} = F t^n \tag{3.9}$$

proposed by Bailey, and

$$\varepsilon^{(c)} = G(1 - e^{-qt}) + Ht \tag{3.10}$$

presented by Mc Vetty.

F, n, G, q and H in Eqs.(3.9) and (3.10) represent some constants. When the creep strain depends on temperature, the form of function $f_3(\Theta)$ can be adopted as proposed by Penny and Mariott

$$\varepsilon^{(c)} = [t \exp(-Q/R\Theta)]^n f_1(\sigma)\,, \tag{3.11}$$

where Q denotes an activation energy and R is a gas constant.

Effect of temperature changes is usually taken into account by making the material constants depend on temperature or allowing for structural changes in the material. Increase in temperature is known to accelerate the velocity of creep. To describe creep in metals a law proposed by Nutting, Scott-Blair and Veinglou is often employed in the form

$$\varepsilon^{(c)} = A\sigma^p t^n\,, \tag{3.12}$$

where A, p, n are temperature-dependent parameters. The authors of paper [11] make an additional assumption that the parameter A in Eq.(3.12) depends on the temperature. Specifically,

$$A = A_1 \exp(A_2/\Theta)\,, \tag{3.13}$$

where A_1 and A_2 denote constants obtained from a uniaxial creep test and Θ is measured in Kelvin degrees.

Similar approach is proposed in [12] whereas in [3] all parameters that appear in Eq.(3.12) are assumed to be temperature-dependent. The parameters that enter relationships (3.7)–(3.12) should all be found experimentally. Equation (3.12) can, in a particular case, describe temperature-independent creep (it can be derived from Eqs.(3.7) and (3.9)). Then the parameters K, p and n remain constant.

4. Incremental finite element equilibrium equation

On using an incremental description of the deformation process [1, 2, 4, 6, 13, 15] in its finite element formulation for the small strains, the relevant equilibrium equation assumes the form

$$\left[\int\limits_V \mathbf{B}_L^T \mathbf{C} \mathbf{B}_L \, dV\right] \Delta\mathbf{u} = {}^{t+\Delta t}\mathbf{R} - \int\limits_V \mathbf{B}_L^T \, {}^t\mathbf{\Sigma} \, dV \tag{4.1}$$

or, in a compact manner,

$${}^t\mathbf{K} \Delta\mathbf{u} = {}^{t+\Delta t}\mathbf{R} - {}^t\mathbf{F}\,, \tag{4.2}$$

where

${}^t\mathbf{K} = \left[\int\limits_V \mathbf{B}_L^T \mathbf{C} \mathbf{B}_L \, dV\right]$ secant stiffness matrix,

$\Delta\mathbf{u}$ – nodal displacement increments,

${}^t\mathbf{F} = \int\limits_V \mathbf{B}_L^T \, {}^t\mathbf{\Sigma} \, dV$ nodal forces vector corresponding to the body stress state,

$\mathbf{B}_L$ – geometric, strain-displacement matrix,

$\mathbf{C}$ – elasticity matrix corresponding to a constitutive tensor of material properties,

${}^t\mathbf{\Sigma}$ – current stress vector.

Two methods have found broad applications in the solutions of physically nonlinear incrementally formulated problems – the variable stiffness method and the initial stress method [2, 6, 15]. Each is being used in various modifications to allow for specific features of particular problems. In the initial stress method the elastic-plastic matrix ${}^t\mathbf{K}$ that enters Eq.(4.2) is expressed as a difference of matrices corresponding to elastic ($\mathbf{K}^e$) and plastic ($\mathbf{K}^p$) behaviour of a considered material,

$$ {}^t\mathbf{K} = \mathbf{K}^e - \mathbf{K}^p \,. \tag{4.3} $$

Then the equilibrium equation can be shown to be

$$ \mathbf{K}^e \Delta\mathbf{u} = \Delta\mathbf{R} - \mathbf{J} \,. \tag{4.4} $$

The vector $\mathbf{J}$ is termed an initial load vector. The initial stress method has it that at a given loading step the stiffness matrix is calculated and inverted only once at the cost of many multiplications of this matrix by the initial load vector. Thus the vector $\Delta\mathbf{u}$ has the form

$$ \Delta\mathbf{u} = (\mathbf{K}^e)^{-1}(\Delta\mathbf{R} - \mathbf{J}) \tag{4.5} $$

and it is from this vector that the stress and the plastic strain increments are calculated followed by a check on the convergence of the latter.

5. Numerical example

Consider a cantilever beam loaded by a concentrated force at its free end, Fig.1, and heated to the temperature $T_0 = 300\,^\circ$ C. Material characteristics

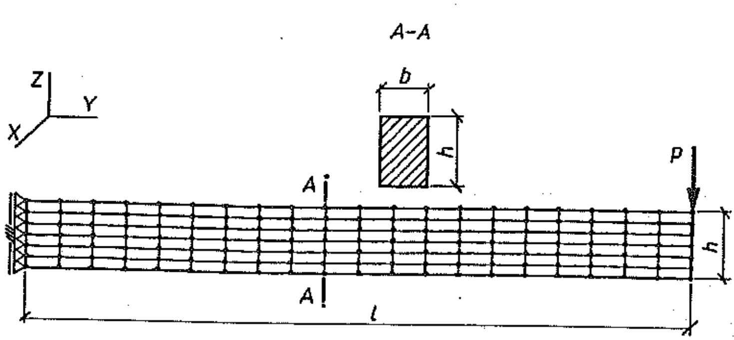

FIG. 1. The model of cantilever beam ($T = 300\,^\circ$ C).

Table 1.

Temperature	Young's modulus E [MPa]	strain-hardening modulus E_T [MPa]	thermal expansion coefficient β	yield point σ_y^Θ [MPa]
100 ° C	182222	5467	$0.1115 \cdot 10^{-4}$	231.3
500 ° C	151111	4533	$0.1141 \cdot 10^{-4}$	143.2

of the material depend on its temperature and vary linearly between 100 and 500 ° C. Their values at 100 and 500 ° C are shown in Table 1.
The yield point shown in the last column of the table varies according to the function given in [7],

$$\sigma_y^\Theta = \sigma_y \left[1 - \left(\frac{\Delta\Theta}{800}\right)^2\right], \tag{5.1}$$

where $\sigma_y = 235$ [MPa] is a yield point in the reference temperature and $\Delta\Theta$ is a temperature increment.

A thermo-elastic-plastic problem with isotropic strain-hardening was dealt with in which the tangent modulus $E_T = 0.03E$.

The considered cantilever was assumed to be in plane stress, its dimensions being denoted as: length l, depth h and thickness b. Finite element mesh consists of 120 two-dimensional four-node elements. Clamped end of the cantilever was modelled with the help of hinged supports free to travel in the direction perpendicular to the beam axis with the exception

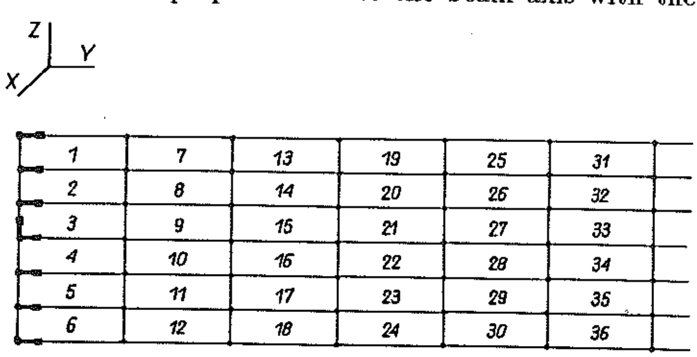

FIG. 2. The model of cantilever beam ($T = 300$ ° C).

of midhight hinge that was fixed. Stresses were calculated at four Gauss integration points. In Fig.2 a fragment of the mesh near the supports is shown on which the numerical results will be presented. The dimensions were assumed: $l = 1\,\mathrm{m}$, $h = 0.1\,\mathrm{m}$, $b = 0.075\,\mathrm{m}$. Poisson's ratio was taken as $\nu = 0.3$.

Propagation of plastic zones was obtained on the basis of equivalent stresses σ_H according to the Huber yield criterion. The load P increased linearly at each step to reach $P_{\max}$.

Plastic zones propagated from the clamped end and their evolution is shown in Fig.3 for the initial temperature $T_0 = 100\,^\circ\mathrm{C}$ and in Fig.4 for $T_0 = 300\,^\circ\mathrm{C}$. Plastic zones can be seen to initiate earlier and to propagate faster for the higher initial temperature.

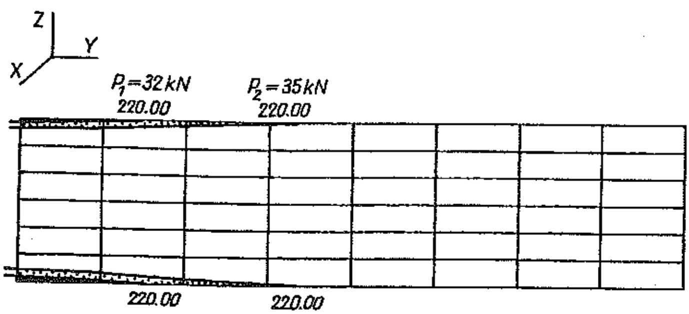

FIG. 3. Cantilever beam – plastic zones ($T = 100\,^\circ\mathrm{C}$).

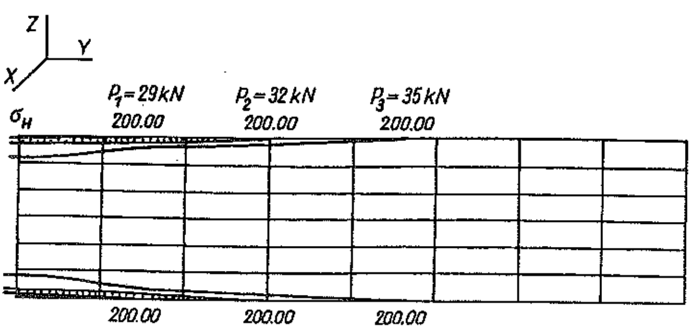

FIG. 4. Cantilever beam – plastic zones ($T = 300\,^\circ\mathrm{C}$).

Creep effects in the cantilever were investigated under a constant load $P = 34\,\mathrm{kN}$. Creep strains for uniaxial test were assumed to vary in an exponential way according to Nutting, Scott-Blair and Veinglou

$$\varepsilon^{(c)} = A\sigma^n t^m . \tag{5.2}$$

Material constants, according to the authors of [11], are to be calculated from the formula

$$A = q \exp\left(\frac{r}{\Delta\Theta + 273}\right), \tag{5.3}$$

where $\Delta\Theta$ is a temperature increment, n, m, q, r are constants. For $\Delta\Theta = 300\,^\circ\mathrm{C}$ the parameters of creep law (5.2) are: $A = 3.77 \cdot 10^{-11}$, $n = 3.142$, $m = 0.615$. Time is measured in hours, stress in $\mathrm{N/mm^2}$.

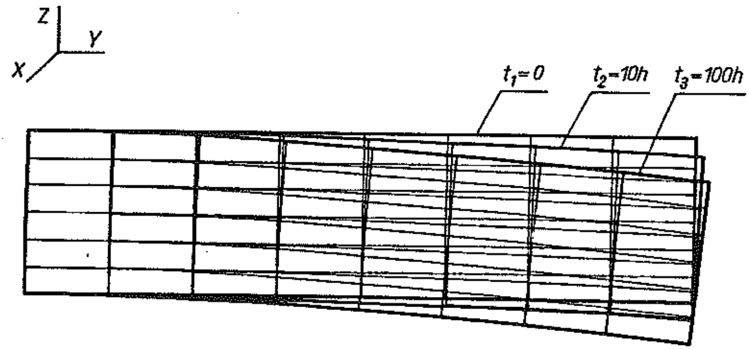

FIG. 5. Cantilever beam – steady creep ($T = 300\,^\circ$C).

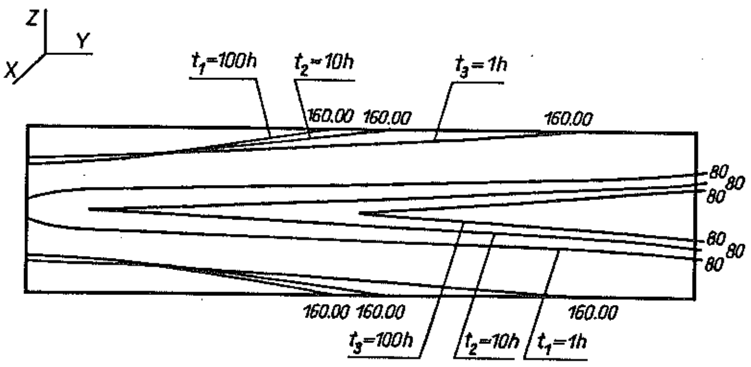

FIG. 6. Cantilever beam – steady creep ($T = 300\,^\circ$C).

Suitable calculations supplied the time-dependent nodal displacements and stresses at the Gauss points in each element. Initial situation and the deformation of the cantilever for $t_2 = 10\,\mathrm{h}$ and $t_3 = 100\,\mathrm{h}$ are shown in Fig.5. Changes in time of the equal stress lines are depicted in Fig.6. Relaxation of the longitudinal normal stress σ_y at a selected point is shown diagrammatically in Fig.7.

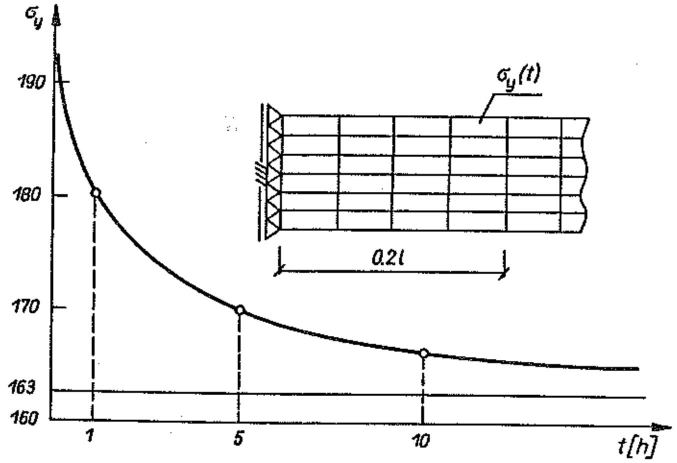

FIG. 7.

The following conclusions can be drawn from the above graphical presentation of results:

- deformations increase faster at the initial stages of the process,
- the equal equivalent stress lines vary in time in such a manner that their gradient in the lateral direction decreases,
- stress relaxation stabilizes as soon as rapid initial changes are over.

Two ranges or the behaviour can be distinguished: I – nonsteady creep characterized by rapid changes in displacements and stresses, II – steady creep in which those magnitudes change much slower.

6. CONCLUSIONS

The presented FEM creep analysis of a structure under mechanical and thermal loading shows its practical applicability. The crucial points is the availability of material characteristics derived from experiments and their dependence on temperature. The knowledge of a temperature-dependent creep law for uniaxial test is also necessary. The finite element technique makes it possible to analyse the strength of structures with complex geometries such as layered beams and sandwich plates as well as the occurrence of local zones with permanent deformations.

When a nonstationary temperature field is to be tackled, there is a possibility to account for the effects of current temperatures on the material characteristics and creep of structures.

However, to analyse the creep of structures over prolonged periods of time a necessity arises to continue calculations for a large number of time steps. This is especially the case when a comparative analysis with various time steps is required. Different lengths of time steps can result from considerable differences in creep strain rates at certain temperature ranges. It is the loss of numerical stability of the calculation process that can result from too long time steps. Special techniques are known to have been used in such situations, for instance a subdivision of time steps into finer intervals [13].

An improvement of stability and accuracy is reached at the cost of much longer computation times.

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