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EQUATIONS OF THE FIBRE COMPOSITE PLATES

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The paper is concerned with elastic fibre-composite plates. For the fibre composite the concept of two-phase medium presented in the works of Hołnicki-Szulc was used. For the plate the hypothesis of Kirchhoff – Love is applied. The displacement equations of the plates loaded in-plane and transversely are developed, in which the coupling between the "disk state" and the "plate state" can be observed. The bending of the fibre composite plate is presented in more detail.

1. Introduction

Mechanical properties of many structural materials can efficiently be improved by fibre reinforcement. The obtained conglomerate in which fibres are disposed in the base material in a regular way is called fibre composite. Our considerations are adressed mainly to modern plastics and alloys, reinforced with glass, carbon and steel fibres. A further improvement of the properties of the fibre composite may consist in introducing to it some desirable states of distortion. An example of this may be a pre-tensioned prestressed concrete slab. Another kind of distortions, not necessarily desirable, are thermal distortions. This issue will be taken into account in the present work.

Fibre composites are materials in which mechanical properties can be formed in a very wide range and in a very simple manner by an appropriate choice of the component materials. In virtue of this one can think that fibre composites will play a more and more significant role in material engineering.

Fibre reinforced elastic media were the subject of many works. In the majority of these works, fibre composite is treated as a homogeneous and anisotropic medium [1, 2, 3, 5, 6, 7, 13, 20]. The problem of determination of the anisotropic constant was left open [5], or resulted from some additional considerations, in general the simplified ones [6, 7, 13] or eventually, was solved via experimental tests. For instance, in the work [1] reference is made to the experimental data, while the monograph [2] is, among other things, devoted to laboratory tests on properties of various fibre composites. Recently, experimental investigations, covering the inelastic range, are intensively developed also in Poland [4, 12, 16, 21].

A particular example of application of the engineering methods to determine the constants of orthotropy of two-way reinforced slabs calculated due to the phase Ia are different formulae given in [11, 15].

Another view on the mechanics of fibre composite can be found in the papers of HOLNICKI-SZULC [8, 9, 10], in which fibre composite is a particular case of two-phase media. A similar approach is presented by MARKS in [14]. The Holnicki-Szulc's concept of fibre composite has been developed further by OLEJNICZAK [17, 18, 19], who worked out theoretical foundations and computer programs for viscoelastic disks reinforced with viscoelastic fibres, taking into account prestressing.

The aforenamed works are concerned with the plane state of stress. The present work is devoted to elastic plates (including "shallow shells") made of fibre composite. We will here consider the coupling between the "disk state" and the "plate state". More details will be given on fibre composite plates in bending.

2. ASSUMPTIONS

According to the concept of two-phase medium (in general, of multi-phase medium) presented by HOLNICKI-SZULC [10], the model of one continuum embedded into the other is assumed. As an example of this we can indicate a continuous medium (phase I) in which a "dense" lattice structure (phase II) was embedded. By applying a continuous description of the two phases in such a composite, we get an idealized continuous two-phase medium which has the property that two material points belonging to the phases I and II, respectively, correspond to the same, common, geometric point of the domain occupied by the body. This yields a model of Voigt-type, i.e. a parallel connection of both phases, hence

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{I}} + \boldsymbol{\sigma}^{\mathrm{II}}, \qquad \boldsymbol{\varepsilon}^{\mathrm{I}} = \boldsymbol{\varepsilon}^{\mathrm{II}} = \boldsymbol{\varepsilon}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and the strain tensors in the composite, respectively, $\boldsymbol{\sigma}^{\mathrm{I}}$ and $\boldsymbol{\varepsilon}^{\mathrm{I}}$ – stress and strain tensors in the phase I, $\boldsymbol{\sigma}^{\mathrm{II}}$ and $\boldsymbol{\varepsilon}^{\mathrm{II}}$ – stress and strain tensors in the phase II.

For each phase the following constitutive equation may be assumed

$$\boldsymbol{\sigma}^{\mathrm{I}} = \mathbf{A}^{\mathrm{I}}(\boldsymbol{\varepsilon}^{\mathrm{I}} - \mathring{\boldsymbol{\varepsilon}}^{\mathrm{I}}), \qquad \boldsymbol{\sigma}^{\mathrm{II}} = \mathbf{A}^{\mathrm{II}}(\boldsymbol{\varepsilon}^{\mathrm{II}} - \mathring{\boldsymbol{\varepsilon}}^{\mathrm{II}}). \tag{2.2}$$

Here $\mathbf{A}^{\mathrm{I}}$ and $\mathbf{A}^{\mathrm{II}}$ are the constitutive tensors (in general, constitutive operators) of the phases I and II, respectively, whilst $\mathring{\boldsymbol{\varepsilon}}^{\mathrm{I}}$ and $\mathring{\boldsymbol{\varepsilon}}^{\mathrm{II}}$ denote distortions. The distortions $\mathring{\boldsymbol{\varepsilon}}$ do not generally satisfy the continuity equations of strains. The relations inverse to (2.2) can be written as

$$\boldsymbol{\varepsilon}^{\mathrm{I}} = \mathbf{B}^{\mathrm{I}}(\boldsymbol{\sigma}^{\mathrm{I}} + \mathring{\boldsymbol{\sigma}}^{\mathrm{I}}), \qquad \boldsymbol{\varepsilon}^{\mathrm{II}} = \mathbf{B}^{\mathrm{II}}(\boldsymbol{\sigma}^{\mathrm{II}} + \mathring{\boldsymbol{\sigma}}^{\mathrm{II}}), \tag{2.3}$$

with $\mathbf{B} = \mathbf{A}^{-1}$, $\mathring{\boldsymbol{\sigma}} = \mathbf{A}\,\mathring{\boldsymbol{\varepsilon}}$, $\mathring{\boldsymbol{\varepsilon}} = \mathbf{B}\,\mathring{\boldsymbol{\sigma}}$.

The described above model of two-phase medium can naturally be employed to the fibre composite, which has been outlined in the work [10]. The phase I of the fibre composite is a filling base material (matrix), the fibres constitute the phase II. The loss in material of the phase I due to a portion of the volume filled by the phase II is neglected.

Considering a fibre composite plate we take the following assumptions:

1. The plate is made of a matrix and of any number of fibre families (wires, reinforcement bars). Each fibre family is placed in a plane parallel to the middle plane of the plate. The fibres of a given family have a common constant direction and are uniformly and "densely" distributed.

2. The materials of the phases are linearly elastic.

3. The influence of the stress σ_{33} on strains is neglected, which implies the following constitutive equations for the matrix

$$\begin{aligned} \sigma_{ij} &= \frac{E}{1-\nu^2}\left[(1-\nu)(\varepsilon_{ij} - \mathring{\varepsilon}_{ij}) + \nu(e - \mathring{e})\delta_{ij}\right], \\ \varepsilon_{ij} &= \frac{1}{E}\left[(1+\nu)(\sigma_{ij} + \mathring{\sigma}_{ij}) - \nu(s + \mathring{s})\delta_{ij}\right], \quad i,j = 1,2, \end{aligned} \tag{2.4}$$

$e = \varepsilon_{kk}$, $s = \sigma_{kk}$, E, ν – elastic constants of the matrix.

4. The hypothesis of Kirchhoff-Love is used. Taking into account the fact that in the case of the fibre composite, especially of the prestressed one, the middle plane of a plate is not its neutral layer and, actually, the latter does not exist (the position of the layer on which the normal stress in a given cross-section is equal to zero depends upon orientation of the cross-section), the conclusion from the Kirchhoff-Love's hypothesis should be written in the form

$$u_i = w_{,i}x_3 + u_i^*, \qquad i = 1,2. \tag{2.5}$$

3. Displacement equations of the plate

A part of the type of plates we consider in the sequel is shown in Fig.1. The plate may be loaded by forces in its plane or in the direction orthogonal to it. In the case of homogeneous plates, such loadings produce the state of stress which can be treated as the superposition of the "disk state" and the "plate state". For the fibre composite plate, splitting of these two states is possible, as we shall see later, only in some very special cases.

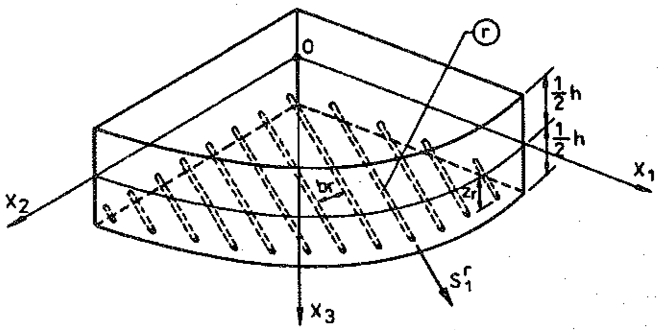

Fig. 1.

The material of the plate can be reinforced with a number of fibre families. The r-th fibre family being at the distance z_r from the middle plane is displayed in Fig.1. The distance between the fibres is $b_r \cdot S_1^r$ denotes a force in the single fibre. If the fibres are sufficiently densely distributed, then one can replace the concentrated forces S_1^r by the continuously distributed line force S^r (see Fig.2), and

$$S^r = \frac{E_r A_r}{b_r}(\varepsilon^r - \overset{\circ}{\varepsilon}{}^r). \tag{3.1}$$

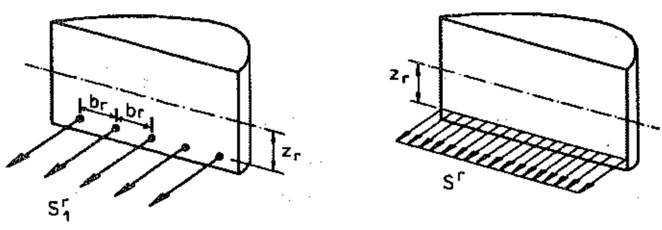

Fig. 2.

In the relation (3.1) E_r is the elasticity modulus of a typical fibre in the r-th family, A_r is the cross-section of area of the fibre and ε^r is its relative elongation, while $\overset{\circ}{\varepsilon}_r$ indicates distortions.

After such a process which yields the continuous distribution of the phase II, we can define the stress tensor in this phase as

(3.2) $$S^r_{ij} = \frac{E_r A_r}{b_r}(\varepsilon^r - \overset{\circ}{\varepsilon}{}^r) s^r_i s^r_j \,,$$

where s^r_i is the i-th component of the unit vector $\mathbf{s}^r$ (cf. Fig.3).

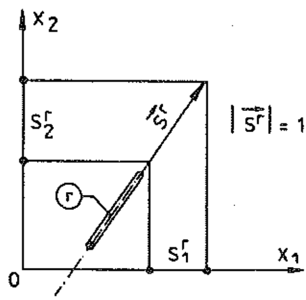

FIG. 3.

Relation (2.5) implies

(3.3) $$\varepsilon_{ij} = -w_{,ij}x_3 + \varepsilon^*_{ij} \,,$$

with $\varepsilon^*_{ij} = \frac{1}{2}(u^*_{i,j} + u^*_{j,i})$.

By making use of (3.3) and (2.4) we get for the matrix

(3.4) $$\sigma_{ij} = -\frac{Ex_3}{1-\nu^2}\left[(1-\nu)w_{,ij} + \nu w_{,kk}\delta_{ij}\right] + \frac{E}{1-\nu^2}\left[(1-\nu)\varepsilon^*_{ij} + \nu e^*\delta_{ij}\right] - \overset{\circ}{\sigma}_{ij} \,,$$

where $e^* = u^*_{k,k}$, and

(3.5) $$\overset{\circ}{\sigma}_{ij} = \frac{E}{1-\nu^2}\left[(1-\nu)\,\overset{\circ}{\varepsilon}_{ij} + \nu\,\overset{\circ}{e}\,\delta_{ij}\right] .$$

In view of (3.3) and (3.2) we obtain for the fibre phase (family r)

(3.6) $$\varepsilon^r = -w_{,kl}z_r s^r_k s^r_l + \varepsilon^*_{kl}s^r_k s^r_l \,,$$

and

$$S^r_{ij} = \frac{E_r A_r}{b_r}(\varepsilon^*_{kl} - w_{,kl} z_r) s^r_i s^r_j s^r_k s^r_l - \overset{\circ}{S}{}^r_{ij}\,, \tag{3.7}$$

with

$$\overset{\circ}{S}{}^r_{ij} = \frac{E_r A_r}{b_r}\, \overset{\circ}{\varepsilon}{}^r s^r_i s^r_j\,. \tag{3.8}$$

The tensor of internal forces within the plate plane may be defined by the formula

$$N_{ij} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}\, dx_3 + \sum_r S^r_{ij}\,. \tag{3.9}$$

By utilizing (3.4) and (3.7) we get

$$\begin{aligned} N_{ij} = {} & \frac{Eh}{1-\nu^2}\left[(1-\nu)\varepsilon^*_{ij} + \nu e^* \delta_{ij}\right] \\ & + \varepsilon^*_{kl} \sum_r \frac{E_r A_r}{b_r} s^r_i s^r_j s^r_k s^r_l - w_{,kl} \sum_r \frac{E_r A_r}{b_r} z^r s^r_i s^r_j s^r_k s^r_l - \overset{\circ}{N}_{ij}\,, \end{aligned} \tag{3.10}$$

with

$$\overset{\circ}{N}_{ij} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \overset{\circ}{\sigma}_{ij}\; dx_3 + \sum_r \overset{\circ}{S}{}^r_{ij}\,. \tag{3.11}$$

The tensor of moments in the plate is given by

$$M_{ij} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} x_3\, dx_3 + \sum_r S^r_{ij} z_r\,, \tag{3.12}$$

which, in virtue of (3.4) and (3.7), may be written in the form

$$\begin{aligned} M_{ij} = {} & -D[(1-\nu) w_{,ij} + \nu w_{,kk}\delta_{ij}] \\ & - w_{,kl} \sum_r \frac{E_r A_r}{b_r} z_r^2 s^r_i s^r_j s^r_k s^r_l + \varepsilon^*_{kl} \sum_r \frac{E_r A_r}{b_r} z_r s^r_i s^r_j s^r_k s^r_l - \overset{\circ}{M}_{ij}\,, \end{aligned} \tag{3.13}$$

with

$$\overset{\circ}{M}_{ij} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \overset{\circ}{\sigma}_{ij}\; x_3\, dx_3 + \sum_r \overset{\circ}{S}{}^r_{ij} z_r\,. \tag{3.14}$$

The equations of equilibrium are written for a plate element (this does not mean that they will be satisfied in each volume element), taking into account the large displacements (however, we consistently employ the tensor of small strains). Thus the equilibrium equations are defined in a deformed configuration of the plate, and the inertial forces are included:

$$
\begin{aligned}
N_{ij,j} + p_i - \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \ddot{u}_i \, dx_3 &= 0, \quad i,j = 1,2, \\
T_{i,i} + N_{ij} w_{,ij} + p_3 - \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \ddot{w}_i \, dx_3 &= 0\,, \qquad (3.15) \\
M_{ij,j} - T_i + \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \ddot{u}_i x_3 \, dx_3 &= 0\,.
\end{aligned}
$$

Using the last equation of (3.15) we can evaluate the transverse forces T_i. The other three equations lead to the following displacement equation for the fibre composite plate:

$$
\begin{aligned}
(3.16) \qquad \nabla^2 u_i^* + \frac{1+\nu}{1-\nu} e^*_{,i} + \left[2(1+\nu) \sum_r \lambda_r s_i^r s_j^r s_k^r s_l^r \right] u^*_{j,kl} \\
- \left[h(1+\nu) \sum_r \lambda_r \zeta_r s_i^r s_j^r s_k^r s_l^r \right] w_{,jkl} - \frac{2(1+\nu)}{E} \rho \ddot{u}_i^* \\
+ \frac{2(1+\nu)}{Eh} (p_i - \overset{\circ}{N}_{ij,j}) = 0\,, \qquad i = 1,2\,,
\end{aligned}
$$

$$
\begin{aligned}
(3.17) \qquad \nabla^4 w + \left[3(1-\nu^2) \sum_r \lambda_r \zeta_r^2 s_i^r s_j^r s_k^r s_l^r \right] w_{,ijkl} \\
- \left[(1-\nu^2) \frac{h}{6} \sum_r \lambda_r \zeta_r s_i^r s_j^r s_k^r s_l^r \right] u^*_{i,jkl} \\
-(1-\nu) \frac{12}{h^2} u^*_{i,j} w_{,ij} - \frac{12\nu}{h^2} e^* \nabla^2 w \\
- \left[(1-\nu^2) \frac{12}{h^2} \sum_r \lambda_r s_i^r s_j^r s_k^r s_l^r \right] u^*_{i,j} w_{,kl} \\
+ \left[(1-\nu^2) \frac{6}{h} \sum_r \lambda_r \zeta_r s_i^r s_j^r s_k^r s_l^r \right] w_{,ij} w_{,kl} + \frac{1}{D} \overset{\circ}{N}_{ij} \, w_{,ij} \\
+ \frac{1-\nu^2}{E} \rho \nabla^2 \ddot{w} + \frac{\rho h}{D} \ddot{w} = \frac{1}{D} (p_3 - \overset{\circ}{M}_{ij,ij})\,.
\end{aligned}
$$

In the above equations D denotes the flexural rigidy of the plate

$$D = \frac{Eh^3}{12(1-\nu)^2},$$

and

$$\lambda = n_r\mu_r\,, \quad n_r = E_r/E\,, \quad \mu_r = A_r/(b_r h)\,, \quad \zeta_r = 2z_r/h\,, \quad \zeta_r \in (-1,1)\,.$$

For reinforced concrete, μ_r is known as the "percentage of reinforcement".

Let us remark that the "disk problem" is coupled with the "plate problem". The coupling disappears only then, when $\zeta_r = 0$ or $\lambda_r = 0$. If $\zeta_r = 0$ and $\lambda_r = 0$, then Eq.(3.16) describes in displacements a plane elasticity problem for an isotropic disk, whereas Eq.(3.17) becomes the equation of an isotropic plate in bending. The third term in Eq.(3.16) and the second one in Eq.(3.17) result from an anisotropy of the fibre composite, which is enforced by the presence of the fibre phase (reinforcement). The next terms of these equations are responsible for the coupling between the plane problem and the problem of plate in bending. In the general case, these two problems can not be separated. The nonlinear terms of Eq.(3.17) describe the buckling problem of the fibre composite plate. In Eqs.(3.16) and (3.17) one can distinguish the equations of longitudinal and of transverse vibrations (the term before last on the left-hand side of Eq.(3.17) represents the rotational inertia of the plate). Finally, Eqs.(3.16) and (3.17) can describe any types of distortions in the plates.

The boundary conditions seem to need no separate consideration. However, it is worth to note that if the boundary conditions are expressed in terms of forces (static boundary conditions), then the relations (3.10) and (3.13) yield a coupling of the disk state and the plate state via the boundary conditions too.

4. Bending of the fibre composite plate

If the plate is subject only to bending, then

$$N_{ij} = 0\,. \tag{4.1}$$

In this way one can obtain three equations in which the components ε^*_{11}, ε^*_{12} and ε^*_{22} of the tensor ε^*_{ij} may be chosen as unknows. The resulting system of equations may conveniently be written in the matrix form,

$$\mathbf{K}\boldsymbol{\varepsilon}^* = w_{,kl}\sum_r \frac{E_r A_r}{b_r} z_r s^r_k s^r_l \mathbf{s}^r + \overset{\circ}{\mathbf{N}}\,, \tag{4.2}$$

where

$$\begin{aligned} \boldsymbol{\varepsilon}^* &= \mathrm{col}(\varepsilon_{11}^*,\ \varepsilon_{12}^*,\ \varepsilon_{22}^*)\,, \\ \mathbf{s}^r &= \mathrm{col}(s_1^r s_1^r,\ 2s_1^r s_2^r,\ s_2^r s_2^r)\,, \\ \overset{\circ}{\mathbf{N}} &= \mathrm{col}(\overset{\circ}{N}_{11},\ 2\,\overset{\circ}{N}_{12},\ \overset{\circ}{N}_{22}) \end{aligned}$$

and $\mathbf{K} = \mathbf{K}^T = [k_{ij}]_{(3\times3)}$ is a matrix, the elements of which are defined as follows:

$$\begin{aligned} k_{11} &= \frac{Eh}{1-\nu^2} + \sum_r \frac{E_r A_r}{b_r}(s_1^r)^4\,, \\ k_{12} &= 2\sum_r \frac{E_r A_r}{b_r}(s_1^r)^3 s_2^r\,, \\ k_{13} &= \frac{Eh\nu}{1-\nu^2} + \sum_r \frac{E_r A_r}{b_r}(s_1^r)^2(s_2^r)^2\,, \\ k_{22} &= 2\left[\frac{Eh}{1+\nu} + 2\sum_r \frac{E_r A_r}{b_r}(s_1^r)^2(s_2^r)^2\right], \\ k_{23} &= 2\sum_r \frac{E_r A_r}{b_r} s_1^r (s_2^r)^3\,, \\ k_{33} &= \frac{Eh}{1-\nu^2} + \sum_r \frac{E_r A_r}{b_r}(s_2^r)^4\,. \end{aligned}$$

A solution of the equation (4.2) may be presented in the form

$$\boldsymbol{\varepsilon} = w_{,kl}\sum_r \mathbf{c}^r z_r s_r^r s_l^r + \overset{\circ}{\boldsymbol{\varepsilon}}{}^*\,. \tag{4.3}$$

Here

$$\mathbf{c}^r = \mathrm{col}(c_{11}^r,\ c_{12}^r,\ c_{22}^r) = \frac{E_r A_r}{b_r}\mathbf{K}^{-1}\mathbf{s}^r\,, \tag{4.4}$$

$$\overset{\circ}{\boldsymbol{\varepsilon}}{}^* = \mathrm{col}(\overset{\circ}{\varepsilon}{}_{11}^*,\ \overset{\circ}{\varepsilon}{}_{12}^*,\ \overset{\circ}{\varepsilon}{}_{22}^*) = \mathbf{K}^{-1}\,\overset{\circ}{\mathbf{N}}\,. \tag{4.5}$$

Having used the result (4.3) we arrive at the following formula for the stress tensor

$$\begin{aligned} \sigma_{ij} = \frac{Ex_3}{1-\nu^2}&[(1-\nu)w_{,ij} + \nu w_{,kk}\delta_{ij}] \\ &+ \frac{E}{1-\nu^2}w_{,kl}\sum_r z_r s_k^r s_l^r[(1-\nu)c_{ij}^r + \nu c^r\delta_{ij}] \\ &+ \frac{E}{1-\nu^2}[(1-\nu)\,\overset{\circ}{\varepsilon}{}_{ij}^* + \nu\,\overset{\circ}{e}{}^*\delta_{ij}] - \overset{\circ}{\delta}_{ij}\,, \end{aligned} \tag{4.6}$$

$$c^r = c_{kk}^r = c_{11}^r + c_{22}^r\,, \qquad \overset{\circ}{e}{}^* = \overset{\circ}{e}{}_{kk}^*\,.$$

Then, the tensor of plate moments is

$$
\begin{aligned}
(4.7)\qquad M_{ij} = -D[(1-\nu)w_{,ij} + \nu w_{,kk}\delta_{ij}] - \Bigg[\sum_r \frac{E_r A_r}{b_r} z_r^2 s_i^r s_j^r s_k^r s_l^r \\
- \sum_r \frac{E_r A_r}{b_r} z_r s_i^r s_j^r s_p^r s_q^r \sum_s c_{pq}^s z_s s_k^s s_l^s \Bigg] w_{,kl} \\
+ \sum_r \frac{E_r A_r}{b_r} z_r \, \overset{\circ}{\varepsilon}{}^*_{kl} s_i^r s_j^r s_k^r s_l^r - \overset{\circ}{M}_{ij} .
\end{aligned}
$$

The equilibrium equations of the plate element read

$$
(4.8)\qquad
\begin{aligned}
T_{i,i} + p &= 0 , \\
M_{ij,j} - T_i &= 0 , \qquad p = p_3 .
\end{aligned}
$$

First, using Eq. $(4.8)_2$ we easily obtain for shearing forces the following formula

$$
\begin{aligned}
(4.9)\qquad T_i = -D\nabla^2 w_{,i} - \Bigg[\sum_r \frac{E_r A_r}{b_r} z_r^2 s_i^r s_j^r s_k^r s_l^r \\
- \sum_r \frac{E_r A_r}{b_r} z_r s_i^r s_j^r s_p^r s_q^r \sum_s c_{pq}^s z_s s_k^s s_l^s \Bigg] w_{,jkl} \\
+ \sum_r \frac{E_r A_r}{b_r} z_r \, \overset{\circ}{\varepsilon}{}^*_{kl,j} s_i^r s_j^r s_k^r s_l^r - \overset{\circ}{M}_{ij,j} .
\end{aligned}
$$

Then, substitution of $(4.8)_2$ and (4.7) into Eq.$(4.8)_1$ eventually leads to the following equation for the fibre composite plate:

$$
\begin{aligned}
(4.10)\qquad D\nabla^4 w + \sum_r \frac{E_r A_r}{b_r} z_r^2 s_i^r s_j^r s_k^r s_l^r \\
- \sum_r \frac{E_r A_r}{b_r} z_r s_i^r s_j^r s_p^r s_q^r \sum_s c_{pq}^s z_s s_k^s s_l^s \, w_{,ijkl} \\
= p + \sum_r \frac{E_r A_r}{b_r} z_r \, \overset{\circ}{e}{}^*_{kl,ij} s_i^r s_j^r s_k^r s_l^r - \overset{\circ}{M}_{ij,ij} .
\end{aligned}
$$

We have obtained the anisotropic plate's equation in which all constants of anisotropy are given explicitly. The second term on the right-hand side of Eq.(4.10) presents disk-type distortions, whereas the third one – bending distortions.

5. The plate reinforced with four fibre families forming rectangular nets in two layers

The instance indicated in the heading may refer to a plate reinforced with two rectangular nets of fibres parallel to the axes of a reference system and which are disposed at the bottom and the top layers of the plate, respectively (see Fig.4).

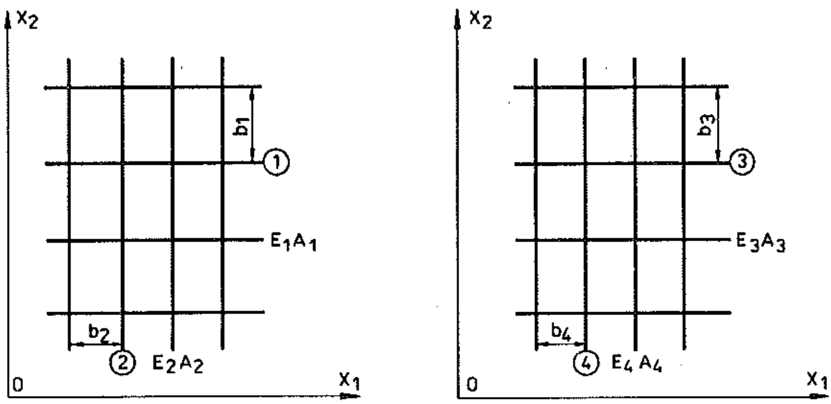

FIG. 4.

Hence in this case we have

$$S_1^1 = 1, \qquad S_2^1 = 0, \qquad S_1^2 = 0, \qquad S_2^2 = 1,$$

$$S_1^3 = 1, \qquad S_2^3 = 0, \qquad S_1^4 = 0, \qquad S_2^4 = 1.$$

One obtains the equation of an orthotropic plate

$$D_{11} w_{,1111} + 2D_{12} w_{,1122} + D_{22} w_{,2222} = p + \frac{Eh^2}{2} (\lambda_1 \zeta_1 + \lambda_3 \zeta_3) \overset{\circ}{\varepsilon}{}^*_{11,11} + (\lambda_2 \zeta_2 + \lambda_4 \zeta_4) \overset{\circ}{\varepsilon}{}^*_{22,22} - \overset{\circ}{M}_{ij,ij}, \tag{5.1}$$

with the notation

$$D_{11} = D + D_{11}^{\mathrm{I}}, \qquad D_{12} = D + \nu D_{12}^{\mathrm{I}}, \qquad D_{22} = D + D_{22}^{\mathrm{I}}; \tag{5.2}$$

$$D_{11}^{\mathrm{I}} = \frac{Eh^2}{4} \left[\lambda_1 \zeta_1^2 + \lambda_3 \zeta_3^2 - \frac{(\lambda_1 \zeta_1 + \lambda_3 \zeta_3)^2 [1 + (1-\nu^2)(\lambda_2 + \lambda_4)]}{1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + (1-\nu^2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)} \right], \tag{5.3}$$

(5.3) [cont.]
$$D^{\mathrm{I}}_{12} = \frac{Eh^2}{4}\frac{(\lambda_1\zeta_1+\lambda_3\zeta_3)(\lambda_2\zeta_2+\lambda_4\zeta_4)}{1+\lambda_1+\lambda_2+\lambda_3+\lambda_4+(1-\nu^2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)},$$
$$D^{\mathrm{I}}_{22} = \frac{Eh^2}{4}\left[\lambda_2\zeta_2^2+\lambda_4\zeta_4^2 - \frac{(\lambda_2\zeta_2+\lambda_4\zeta_4)^2[1+(1-\nu^2)(\lambda_1+\lambda_3)]}{1+\lambda_1+\lambda_2+\lambda_3+\lambda_4+(1-\nu^2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)}\right].$$

Let us suppose that the plate is subjected to the temperature field

$$\theta = \theta_0 + \Delta\theta\frac{x_3}{h}, \tag{5.4}$$

and, that the plate was prestressed. Let $\overset{\circ}{S}{}^1_1$, $\overset{\circ}{S}{}^2_1$, $\overset{\circ}{S}{}^3_1$, $\overset{\circ}{S}{}^4_1$ denote prestressing forces in typical fibres of each family, respectively. Further, let us assume that a temperature of the fibre is equal to that of the matrix, i.e.

$$\theta^r = \theta_0 + \Delta\theta\frac{z_r}{h}, \tag{5.5}$$

then

$$\begin{aligned}
\overset{\circ}{S}{}^1_{11} &= \frac{E_1A_1}{b_1}\overset{\circ}{\varepsilon}{}^1 = Eh\lambda_1\alpha_1\theta^1 - \overset{\circ}{S}{}^1_1 b_1,\\
\overset{\circ}{S}{}^2_{11} &= 0,\\
\overset{\circ}{S}{}^3_{11} &= \frac{E_3A_3}{b_3}\overset{\circ}{\varepsilon}{}^3 = Eh\lambda_3\alpha_3\theta^3 - \overset{\circ}{S}{}^3_1 b_3,\\
\overset{\circ}{S}{}^4_{11} &= 0, \qquad \overset{\circ}{S}{}^1_{22} = 0,\\
\overset{\circ}{S}{}^2_{22} &= \frac{E_2A_2}{b_2}\overset{\circ}{\varepsilon}{}^2 = Eh\lambda_2\alpha_2\theta^2 - \overset{\circ}{S}{}^2_1 b_2,\\
\overset{\circ}{S}{}^3_{22} &= 0,\\
\overset{\circ}{S}{}^4_{22} &= \frac{E_4A_4}{b_4}\overset{\circ}{\varepsilon}{}^4 = Eh\lambda_4\alpha_4\theta^4 - \overset{\circ}{S}{}^4_1 b_4
\end{aligned} \tag{5.6}$$

and $\overset{\circ}{S}{}^r_{12} = 0$ for each r.

Next we calculate

$$\begin{aligned}
\overset{\circ}{N}_{11} &= \frac{Eh}{1-\nu}\alpha\theta_0 + \overset{\circ}{S}{}^1_{11} + \overset{\circ}{S}{}^3_{11},\\
\overset{\circ}{N}_{22} &= \frac{Eh}{1-\nu}\alpha\theta_0 + \overset{\circ}{S}{}^2_{22} + \overset{\circ}{S}{}^4_{22}, \qquad \overset{\circ}{N}_{12} = 0;
\end{aligned} \tag{5.7}$$

$$
(5.8)\quad
\begin{aligned}
\overset{\circ}{M}_{11} &= (1+\nu)D\alpha\frac{\Delta\theta}{h} + \overset{\circ}{S}{}^{1}_{11}z_1 + \overset{\circ}{S}{}^{3}_{11}z_3\,, \\
\overset{\circ}{M}_{22} &= (1+\nu)D\alpha\frac{\Delta\theta}{h} + \overset{\circ}{S}{}^{2}_{22}z_2 + \overset{\circ}{S}{}^{4}_{22}z_4\,, \quad \overset{\circ}{M}_{12} = 0\,;
\end{aligned}
$$

$$
(5.9)\quad
\begin{aligned}
\overset{\circ}{\varepsilon}{}^{*}_{11} &= \frac{1}{Eh}\frac{[1+(1-\nu^2)(\lambda_2+\lambda_4)]\,\overset{\circ}{N}_{11} - \nu\,\overset{\circ}{N}_{22}}{1+\lambda_1+\lambda_2+\lambda_3+\lambda_4+(1-\nu^2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)}\,, \\
\overset{\circ}{\varepsilon}{}^{*}_{22} &= \frac{1}{Eh}\frac{[1+(1-\nu^2)(\lambda_1+\lambda_3)]\,\overset{\circ}{N}_{22} - \nu\,\overset{\circ}{N}_{11}}{1+\lambda_1+\lambda_2+\lambda_3+\lambda_4+(1-\nu^2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)}\,, \\
\overset{\circ}{\varepsilon}{}^{*}_{12} &= 0\,.
\end{aligned}
$$

In the above formulae α and α_r are the linear coefficients of thermal expansion of the matrix, and of the fibre in the r-th family.

Moreover, as an example we give the formulae for the bending and twisting moments

$$
(5.10)\quad
\begin{aligned}
M_{11} &= -D_{11}w_{,11} - \nu(D+D^{\mathrm{I}}_{12})w_{,22} \\
&\qquad + \frac{Eh^2}{2}(\lambda_1\zeta_1+\lambda_3\zeta_3)\,\overset{\circ}{\varepsilon}{}^{*}_{11} - \overset{\circ}{M}_{11}\,, \\
M_{12} &= -D(1-\nu)w_{,12}\,, \\
M_{22} &= -\nu(D+D^{\mathrm{I}}_{12})w_{,11} - D_{22}w_{,22} \\
&\qquad + \frac{Eh^2}{2}(\lambda_2\zeta_2+\lambda_4\zeta_4)\,\overset{\circ}{\varepsilon}{}^{*}_{22} - \overset{\circ}{M}_{22}\,.
\end{aligned}
$$

To conclude, let us state two particular cases.

1) The reinforcement is disposed symmetrically with respect to the middle plane. In this case one has $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$, $z_3 = -z_1$, $z_4 = -z_2$ and

$$
(5.11)\quad D^{\mathrm{I}}_{11} = \frac{1}{2}Eh^3\lambda_1\zeta_1^2\,, \quad D^{\mathrm{I}}_{12} = 0\,, \quad D^{\mathrm{I}}_{22} = \frac{1}{2}Eh^3\lambda_2\zeta_2^2\,.
$$

2) The reinforcement is only in the bottom layer.

Taking $\lambda_3 = \lambda_4 = 0$ one gets

$$
(5.12)\quad
\begin{aligned}
D^{\mathrm{I}}_{11} &= \frac{1}{4}Eh^3\frac{\lambda_1\zeta_1^2(1+\lambda_2)}{1+\lambda_1+\lambda_2+(1-\nu^2)\lambda_1\lambda_2}\,, \\
D^{\mathrm{I}}_{12} &= \frac{1}{4}Eh^3\frac{\lambda_1\lambda_2\zeta_1\zeta_2}{1+\lambda_1+\lambda_2+(1-\nu^2)\lambda_1\lambda_2}\,, \\
D^{\mathrm{I}}_{22} &= \frac{1}{4}Eh^3\frac{\lambda_2\zeta_2^2(1+\lambda_1)}{1+\lambda_1+\lambda_2+(1-\nu^2)\lambda_1\lambda_2}\,.
\end{aligned}
$$

The latter case is often encountered in the structural design of reinforced concrete. It is related to the two-way reinforced slabs designed according to the phase Ia. The stiffness constants of plates with technical orthotropy have been given by many authors for a long time. In the monograph [11] KĄCZKOWSKI gives the formulae which go back to Huber:

$$
\begin{aligned}
D_{11}^{\mathrm{I}} &= \frac{Eh^3}{4(1-\nu^2)} \frac{\lambda_1 \zeta_1^2}{1+\lambda_1}, \\
D_{22}^{\mathrm{I}} &= \frac{Eh^3}{4(1-\nu^2)} \frac{\lambda_2 \zeta_2^2}{1+\lambda_2}, \qquad (5.13) \\
D_{12} &= \frac{Eh}{1-\nu^2} \sqrt{\left(\frac{h^2}{12} + \frac{\lambda_1 z_1^2}{1+\lambda_1}\right)\left(\frac{h^2}{12} + \frac{\lambda_2 z_2^2}{1+\lambda_2}\right)}.
\end{aligned}
$$

The first two of the formulae (5.13) show some similarity to the formulae $(5.12)_{1,3}$, though they are "poorer" than the latter ones (no coupling between the two directions). The formula $(5.13)_3$ is not comparable with $(5.12)_2$. More simplified formulae of this kind were given in other papers devoted to reinforced slabs (e.g.[15]).

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