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ANALYSIS OF DYNAMIC INTERACTION BETWEEN THE CONTINUOUS STRING AND MOVING OSCILLATOR

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The paper presents the analysis of dynamic interaction between a discrete oscillator of two degrees of freedom and a finite continuous string suspended on rigid and visco-elastic concentrated supports. This oscillator moves along the string with a constant sub-critical or super-critical speed. Dynamic transverse displacements of the string as well as the contact force between the oscillator and the string are determined using the d'Alembert solutions of the wave motion equations. Such approach leads to a system of algebraic and ordinary differential equations with a "shifted" argument which are solved numerically in an appropriate sequence. From the performed comparison of numerical results for the sub-critical oscillator speeds with analogous results obtained by means of the finite element method it follows, that the method based on the d'Alembert solutions in the form of travelling waves seems to be more accurate. Moreover, the approach proposed in the paper is numerically very efficient which makes it advantageous for investigations of more complex systems.

1. Introduction

The problem of dynamic interaction between continuous strings and moving oscillators belongs to the wide group of vibration phenomena of one-dimensional continuous media under moving loads [1]. These phenomena can be often observed in practice in the cases of rope drives, fast funicular or cable railways, ski lift installations, vibrations of bridges as well as in the case of vibrations of wires, rods, strips and other structures during technological processes of rolling and drawing. Because of their great practical importance, these vibrations have been considered so far by many authors applying theoretical, numerical and experimental approaches and using various methods, in which the usual objects of considerations were infinite and finite strings and beams, [1–11]. One of the most typical examples of the phenomenon mentioned above are transverse vibrations of the catenary suspensions excited by pantographs of electric locomotives. Then, for analytical and numerical investigations, the catenary suspension is modelled by a set of finite continuous strings, but the pantograph is represented by a moving dynamic oscillator of two or three degrees of freedom [8–11]. Vibration analyses of such systems were performed in [10] using the separated variable solutions together with the Ritz method, or in [8, 9, 11], where the finite element method was used. Both the approaches lead to solutions in the form of a series of standing harmonic waves of frequencies equal to successive natural frequencies of the considered system. In order to obtain results which would be more or less reliable from the physical viewpoint, the mentioned methods make it necessary to take into account at least the first 200–300 eigenmodes of vibrations without securing the required numerical accuracy. This is particularly true in cases of greater frequencies of external excitation and for speeds of the moving load approaching the transverse wave propagation velocity in the string. Moreover, because of the discretized mechanical models of many degrees of freedom applied, these methods are also troublesome from the viewpoint of computer effort.

In order to avoid the difficulties mentioned above, in the paper an alternative approach is proposed. For the considered string system excited by the moving oscillator the d'Alembert solutions of the string motion equations are sought in the form of travelling waves. This method was applied in [7] for the analysis of transverse vibrations of a continuous finite string excited by a moving force.

2. Assumptions

The subject of considerations of the paper are transverse vibrations of a continuous finite homogeneous string suspended at its ends on rigid supports, and in the middle of its length – by means of the concentrated visco-elastic support, Fig. 1. This support consists of lumped mass m_0, two massless springs of stiffnesses k_1 and k_2, and of two viscous dampers of constant coefficients c_1 and c_2. This string is excited to transverse "small" vibrations by the moving dynamic oscillator of two degrees of freedom. This oscillator moves along the string in the positive direction of the spatial coordinate axis $\tilde{x}$ with constant velocity $\tilde{v}$. The oscillator lumped mass m_2 slides along the string. However, the lumped mass m_1 is connected with the moving base and the mass m_2 by means of springs of stiffnesses r_1 and r_2, respectively, as well as by means of viscous dampers of constant coefficients

d_1 and d_2. On the mass m_1 an arbitrary force $\tilde{F}(t)$ is imposed, where t denotes time. Moreover, the oscillator can be also excited kinematically by the moving base motion $\tilde{w}_e(t)$ perpendicular to the $\tilde{x}$ axis, Fig. 1. The identical mechanical system was also investigated in [8], where the finite element method was used.

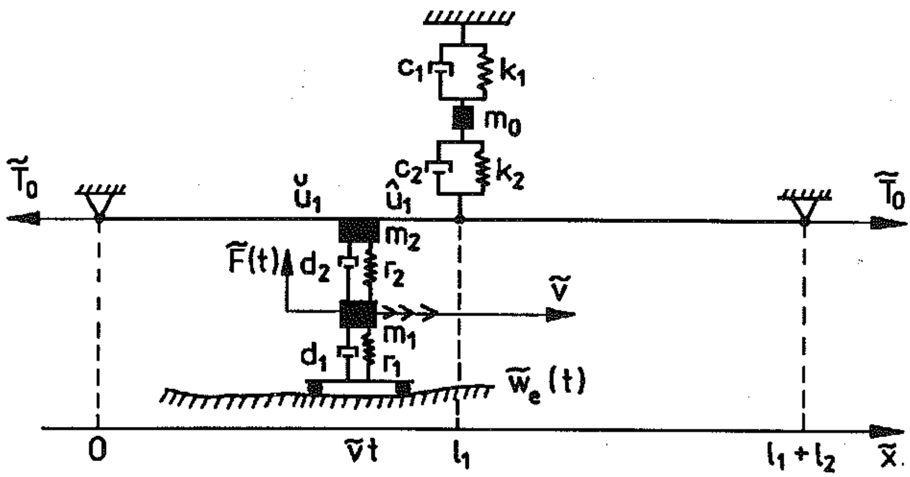

FIG. 1. Finite continuous string excited by the moving oscillator.

3. Formulation of the problem

For "small" transverse vibrations of the string occurring in one plane classical wave equations are applied as the equations of motion

$$\mu \frac{\partial^2 \tilde{u}_i(\tilde{x},t)}{\partial t^2} - \tilde{T}_0 \frac{\partial^2 \tilde{u}_i(\tilde{x},t)}{\partial \tilde{x}^2} = 0, \qquad i = 1,2, \tag{3.1}$$

where $\tilde{T}_0$ denotes the tensile force in the string, μ is the string mass density and subscripts i denote parts of the string on the left and right-hand side of the visco-elastic support, respectively, Fig. 1. Let us introduce the following dimensionless quantities:

$$x = \frac{\tilde{x}}{l_s}, \quad \tau = \frac{at}{l_s}, \quad \lambda_i = \frac{l_i}{l_s}, \quad u_i(x,\tau) = \frac{\tilde{u}_i(\tilde{x},t)}{u_s}, \quad w_j(\tau) = \frac{\tilde{w}_j(t)}{u_s}, \tag{3.2}$$

and

$$v = \frac{\tilde{v}}{a}, \qquad i = 1, 2, \qquad j = 0, 1, 2,$$

where $a = \sqrt{\tilde{T}_0/\mu}$, l_i are lengths of the two parts of the string, Fig. 1, $\tilde{w}_j(t)$, $j = 0,1,2$, are the support and oscillator mass displacements, and l_s [m] and u_s [m] are arbitrary values.

Upon an introduction of the moving spatial coordinate axis $\xi = x - v\tau$ together with (3.2) one obtains the motion equations (3.1) in the following form:

$$\frac{\partial^2 u_i(\xi,\tau)}{\partial \tau^2} - 2v\frac{\partial^2 u_i(\xi,\tau)}{\partial\xi\,\partial\tau} - (1-v^2)\frac{\partial^2 u_i(\xi,\tau)}{\partial \xi^2} = 0, \qquad i = 1,2. \tag{3.3}$$

The above equations are solved under homogeneous initial conditions

$$\begin{aligned} &u_i(\xi,\tau) = 0, \qquad \frac{\partial u_i(\xi,\tau)}{\partial \tau} = 0, \qquad i = 1,2,\\ &w_j(\tau) = 0, \qquad \frac{dw_j(\tau)}{d\tau} = 0, \qquad j = 0,1,2, \quad \text{for} \quad \tau = 0, \end{aligned} \tag{3.4}$$

which means that no motion of the system is assumed before the oscillator approaches the string. Equations (3.3) are solved also under the following boundary conditions:

$$\begin{aligned} &u_1(\xi,\tau) = 0 \qquad \text{for} \quad \xi = -v\tau,\\ &u_2(\xi,\tau) = 0 \qquad \text{for} \quad \xi = \lambda_1 + \lambda_2 - v\tau, \end{aligned}$$

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \ddot{w}_0(\tau) \\ \ddot{u}_2(\xi,\tau) \end{bmatrix} + \begin{bmatrix} C_{12} & -C_2 \\ -C_2 & C_2 \end{bmatrix}\begin{bmatrix} \dot{w}_0(\tau) \\ \dot{u}_2(\xi,\tau) \end{bmatrix}$$
$$+ \begin{bmatrix} K_{12} & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{bmatrix} w_0(\tau) \\ u_2(\xi,\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ T\left(\dfrac{\partial u_2(\xi,\tau)}{\partial\xi} - \dfrac{\partial u_1(\xi,\tau)}{\partial \xi}\right) \end{bmatrix},$$
$$u_1(\xi,\tau) = u_2(\xi,\tau) \qquad \text{for} \quad \xi = \lambda_1 - v\tau,$$

$$P(\tau) \doteq T\left(\frac{\partial \hat{u}_j(\xi,\tau)}{\partial \xi} - \frac{\partial \check{u}_j(\xi,\tau)}{\partial\xi}\right), \qquad \check{u}_j(\xi,\tau) = \hat{u}_j(\xi,\tau), \tag{3.5}$$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{w}_1(\tau) \\ \ddot{w}_2(\tau) \end{bmatrix} + \begin{bmatrix} D_{12} & -D_2 \\ -D_2 & D_2 \end{bmatrix}\begin{bmatrix} \dot{w}_1(\tau) \\ \dot{w}_2(\tau) \end{bmatrix}$$
$$+ \begin{bmatrix} R_{12} & -R_2 \\ -R_2 & R_2 \end{bmatrix}\begin{bmatrix} w_1(\tau) \\ w_2(\tau) \end{bmatrix} = \begin{bmatrix} F(\tau) \\ -P(\tau) \end{bmatrix}, \qquad \text{for} \quad \xi = 0, \quad j = 1,2,$$

(3.5) [cont.]

$$\begin{bmatrix} M_0 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_1 \end{bmatrix} \begin{bmatrix} \ddot{w}_0(\tau) \\ \ddot{w}_2(\tau) \\ \ddot{w}_1(\tau) \end{bmatrix} + \begin{bmatrix} C_{12} & -C_2 & 0 \\ -C_2 & C_2 + D_2 & -D_2 \\ 0 & -D_2 & D_{12} \end{bmatrix} \begin{bmatrix} \dot{w}_0(\tau) \\ \dot{w}_2(\tau) \\ \dot{w}_1(\tau) \end{bmatrix}$$

$$+ \begin{bmatrix} K_{12} & -K_2 & 0 \\ -K_2 & K_2 + R_2 & -R_2 \\ 0 & -R_2 & R_{12} \end{bmatrix} \begin{bmatrix} w_0(\tau) \\ w_2(\tau) \\ w_1(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ T\left(\dfrac{\partial u_2(\xi,\tau)}{\partial \xi} - \dfrac{\partial u_1(\xi,\tau)}{\partial \xi}\right) \\ F(\tau) \end{bmatrix},$$

$$\text{for} \quad \xi = 0 \quad \text{and for} \quad \tau = \frac{\lambda_1}{v}, \quad \text{i.e.} \quad \text{for} \quad \xi = \lambda_1 - v\tau = 0.$$

Here

$$j = 1 \quad \text{for} \quad 0 < \tau < \frac{\lambda_1}{v} \qquad \text{and} \qquad j = 2 \quad \text{for} \quad \frac{\lambda_1}{v} < \tau < \frac{\lambda_1 + \lambda_2}{v},$$

$$M_k = \frac{m_k}{m_s}, \qquad k = 0,\ 1,\ 2,$$

$$C_{12} = \frac{(c_1 + c_2)l_s}{am_s}, \quad C_2 = \frac{c_2 l_s}{am_s}, \quad K_{12} = \frac{(k_1 + k_2)l_s^2}{a^2 m_s}, \quad K_2 = \frac{k_2 l_s^2}{a^2 m_s},$$

$$D_{12} = \frac{(d_1 + d_2)l_s}{am_s}, \quad D_2 = \frac{d_2 l_s}{am_s}, \quad R_{12} = \frac{(r_1 + r_2)l_s^2}{a^2 m_s}, \quad R_2 = \frac{r_2 l_s^2}{a^2 m_s},$$

$$T_0 = \tilde{T}_0 \frac{l_s}{a^2 m_s}, \qquad T = T_0(1 - v^2),$$

$$F(\tau) = (\tilde{F}(t) + r_1 \tilde{w}_e(t) + d_1 \dot{\tilde{w}}_e(t)) \frac{l_s^2}{a^2 m_s u_s},$$

m_s [kg] is an arbitrary value, superscripts ˆ and ˇ denote the parts of the j-th string segment located in front and behind the moving oscillator, respectively. The last boundary condition $(3.5)_8$ corresponds to the time instant when the oscillator approaches the visco-elastic support.

Solutions of the equations of motion (3.3) are sought in the general form of the d'Alembert solutions,

$$(3.6) \quad u_i(\xi,\tau) = f_i(\tau - \tau_{0i} - \xi - v\tau + \xi_{0i}) + g_i(\tau - \tau_{0i} + \xi + v\tau - \xi_{0i}), \quad i = 1,2.$$

Functions f_i and g_i, $i = 1, 2$, represent transverse waves propagating in the string towards the right and left, respectively, Fig. 1, as a result of interaction with the moving oscillator. These functions are continuous and equal to zero for negative arguments. Similarly as in [7], they are determined by the boundary conditions and the assumed initial conditions. In arguments of these functions τ_{0i} denote dimensionless time shifts after which, at the beginning of the process, a front of perturbation arrives to the first cross-section ξ_{0i} of the i-th string segment, $i = 1, 2$. If the oscillator moves with a speed smaller than the transverse wave propagation velocity, i.e. with the so-called "sub-critical speed" $\tilde{v} < a$ (or $v < 1$), then the front of perturbation propagates as a transverse wave. However, for the so-called "super-critical speed" of the oscillator when $\tilde{v} > a$ (or $v > 1$), the front of perturbation propagates with the oscillator. Consequently, forms of the d'Alembert solutions for sub- and super-critical speeds of the oscillator are different, and thus it is necessary to solve this problem separately for these two characteristic cases.

3.1. The sub-critical case ($v < 1$)

In this case the front of perturbation in the string propagates with the transverse wave propagation velocity a, and interaction with the moving oscillator is a source of two travelling waves of different lengths. The "shorter" wave propagates in front of the oscillator towards the right, but the "longer" wave propagates behind the oscillator towards the left, Fig. 1. Thus, for the considered case, the d'Alembert solutions are assumed in the following form:

$$
\begin{aligned}
&u_1(\xi,\tau) = f_1(\tau - v\tau - \xi) + g_1(\tau + v\tau + \xi) \\
&\qquad\qquad\qquad\qquad \text{for} \quad -v\tau \le \xi \le \lambda_1 - v\tau, \\
&u_2(\xi,\tau) = f_2(\tau - v\tau - \xi) + g_2(\tau + v\tau + \xi - 2\lambda_1) \\
&\qquad\qquad\qquad\qquad \text{for} \quad \lambda_1 - v\tau \le \xi \le \lambda_1 + \lambda_2 - v\tau.
\end{aligned}
\tag{3.7}
$$

By substituting (3.7) into (3.5) and denoting in each equation the largest argument by z, one obtains the following system of algebraic and ordinary differential equations with a "shifted" argument for functions f_i and g_i, $i = 1, 2$:

$$
\begin{aligned}
&g_2(z) = -f_2(z - 2\lambda_2), \\
&g_1(z) = -f_1(z - 2\lambda_1) + f_2(z - 2\lambda_1) + g_2(z - 2\lambda_1), \\
&\begin{bmatrix} M_1(1+v^2) & 0 \\ 0 & M_2(1+v^2) \end{bmatrix}
\begin{bmatrix} w_1''\left(\dfrac{z + 2\lambda_1\eta}{1+v}\right) \\ \breve{g}_j''(z) \end{bmatrix} \\
&\qquad + \begin{bmatrix} D_{12}(1+v) & -D_2(1+v) \\ -D_2(1+v) & (2T_0 + D_2)(1+v) \end{bmatrix}
\begin{bmatrix} w_1'\left(\dfrac{z + 2\lambda_1\eta}{1+v}\right) \\ \breve{g}_j'(z) \end{bmatrix}
\end{aligned}
\tag{3.8}
$$

(3.8) [cont.]

$$+\begin{bmatrix} R_{12} & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} w_1\left(\frac{z+2\lambda_1\eta}{1+v}\right) \\ \breve{g}_j(z) \end{bmatrix}$$

$$= \left\{ \begin{array}{c} F\left(\frac{z+2\lambda_1\eta}{1+v}\right) + D_2(1-v)f_j'\left(\frac{1-v}{1+v}(z+2\lambda_1\eta)\right) \\ 2T_0(1+v)g_j'(z) - M_2(1-v)^2 f_j''\left(\frac{1-v}{1+v}(z+2\lambda_1\eta)\right) \\ +R_2 f_j\left(\frac{1-v}{1+v}(z+2\lambda_1\eta)\right) \\ -D_2(1-v)f_j'\left(\frac{1-v}{1+v}(z+2\lambda_1\eta)\right) - R_2 f_j\left(\frac{1-v}{1+v}(z+2\lambda_1\eta)\right) \end{array} \right\},$$

$$f_1(z) = -g_1(z),$$

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_0''(z) \\ f_2''(z) \end{bmatrix} + \begin{bmatrix} C_{12} & -C_2 \\ -C_2 & 2T + C_2 \end{bmatrix} \begin{bmatrix} w_0'(z) \\ f_2'(z) \end{bmatrix} + \begin{bmatrix} K_{12} & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} w_0(z) \\ f_2(z) \end{bmatrix} = \begin{bmatrix} C_2 g_2'(z) + K_2 g_2(z) \\ 2T f_1'(z) - C_2 g_2'(z) - K_2 g_2(z) \end{bmatrix},$$

$$\begin{bmatrix} M_1(1-v^2) & 0 \\ 0 & M_2(1-v^2) \end{bmatrix} \begin{bmatrix} w_1''\left(\frac{z}{1-v}\right) \\ \hat{f}_j''(z) \end{bmatrix}$$

$$+\begin{bmatrix} D_{12}(1-v) & -D_2(1-v) \\ -D_2(1-v) & (2T_0+D_2)(1-v) \end{bmatrix} \begin{bmatrix} w_1'\left(\frac{z}{1-v}\right) \\ \hat{f}_j'(z) \end{bmatrix}$$

$$+\begin{bmatrix} R_{12} & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} w_1\left(\frac{z}{1-v}\right) \\ \hat{f}_j(z) \end{bmatrix}$$

$$= \left\{ \begin{array}{c} F\left(\frac{z}{1-v}\right) + D_2(1+v)g_j'\left(\frac{1+v}{1-v}z - 2\lambda_1\eta\right) \\ 2T_0(1-v)f_j'(z) - M_2(1+v)^2 g_j''\left(\frac{1+v}{1-v}z - 2\lambda_1\eta\right) \\ +R_2 g_j\left(\frac{1+v}{1-v}z - 2\lambda_1\eta\right) \\ -D_2(1-v)g_j'\left(\frac{1-v}{1+v}z - 2\lambda_1\eta\right) - R_2 g_j\left(\frac{1-v}{1+v}z - 2\lambda_1\eta\right) \end{array} \right\},$$

where

$$j = 1, \qquad \eta = 0 \qquad \text{for} \qquad 0 < \tau < \frac{\lambda_1}{v},$$

$$j = 2, \qquad \eta = 1 \qquad \text{for} \qquad \frac{\lambda_1}{v} < \tau < \frac{\lambda_1 + \lambda_2}{v},$$

and for z corresponding to $\tau = \lambda_1/v$ we obtain

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_0 \end{bmatrix} \begin{bmatrix} w_1''(z+\lambda_1) \\ f_2''(z) \\ w_0''(z+\lambda_1) \end{bmatrix} + \begin{bmatrix} D_{12} & -D_2 & 0 \\ -D_2 & D_2 + C_2 + 2T & -C_2 \\ 0 & -C_2 & C_{12} \end{bmatrix} \begin{bmatrix} w_1'(z+\lambda_1) \\ f_2'(z) \\ w_0'(z+\lambda_1) \end{bmatrix}$$

$$+ \begin{bmatrix} R_{12} & -R_2 & 0 \\ -R_2 & R_2 + K_2 & -K_2 \\ 0 & -K_2 & K_{12} \end{bmatrix} \begin{bmatrix} w_1(z+\lambda_1) \\ f_2(z) \\ w_0(z+\lambda_1) \end{bmatrix}$$

$$= \begin{bmatrix} F(z+\lambda_1) + D_2 g_2'(z) + R_2 g_2(z) \\ 2T f_1'(z) - M_2 g_2''(z) - (D_2 + C_2) g_2'(z) - (R_2 + K_2) g_2(z) \\ C_2 g_2'(z) + K_2 g_2(z) \end{bmatrix}.$$

Superscripts ˆ and ˇ denote respectively the "shorter" and "longer" waves generated directly by the oscillator. However, functions f_i and g_i without these superscripts describe the waves reflected from the supports. The above system of equations is solved numerically in the presented order, where for the differential equations $(3.8)_{3,5,6,7}$ the Newmark method is used. By solving Eqs. (3.8) together with (3.7) one obtains, for a considered time span, the dynamic response of the system in the form of transverse deflections of arbitrary cross-sections of the string, displacements of the support and oscillator masses as well as the dynamic contact force between the string and the oscillator.

3.2. *The super-critical case* $(v > 1)$

If the oscillator speed is greater than the transverse wave propagation velocity a, there are no perturbations in the string in front of the oscillator, and behind it two waves of different lengths propagate in two directions. The "shorter" wave propagates towards the right and the "longer" wave

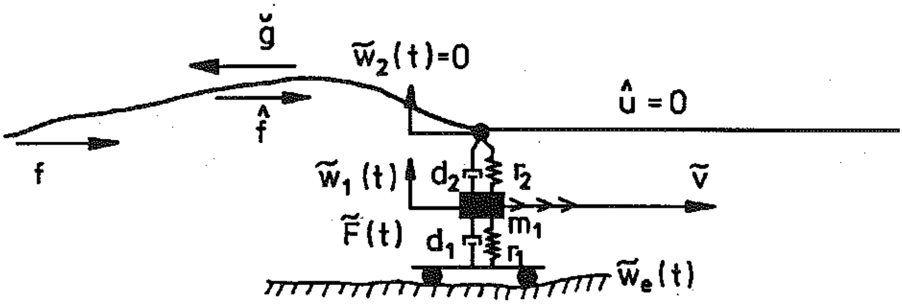

FIG. 2. Waves generated by the oscillator moving with the super-critical speed.

propagates towards the left, Fig. 2. Thus, in this case we assume the undeflected string in front of the oscillator and a zero deflection of its contact point with the string. Hence, for $\hat{u}_j(\xi,\tau)=0$, $j=1,2$, and $w_2(\tau)=0$, the boundary condition $(3.5)_7$ reduces to the simplified relations

$$
(3.9)\qquad \begin{aligned} M_1 \ddot{w}_1(\tau) + D_{12}\dot{w}_1(\tau) + R_{12}w_1(\tau) &= F(\tau),\\ D_2\dot{w}_1(\tau) + R_2 w_1(\tau) &= P(\tau). \end{aligned}
$$

As it was mentioned above, the front of perturbation in the string propagates with the velocity equal to the oscillator speed $\tilde{v}$, and for the "super-critical case" the d'Alembert solutions of Eqs. (3.3) are assumed in the following form:

$$
(3.10)\qquad \begin{aligned} u_1(\xi,\tau) &= f_1(\tau - v\tau - \xi) + g_1(\tau + v\tau + \xi)\\ &\qquad \text{for} \quad -v\tau < \xi < \lambda_1 - v\tau,\\ u_2(\xi,\tau) &= f_2\left(\tau - v\tau - \xi + \lambda_1\left(1 - \frac{1}{v}\right)\right)\\ &\qquad + g_2\left(\tau + v\tau + \xi - \lambda_1\left(1 + \frac{1}{v}\right)\right)\\ &\qquad \text{for} \quad \lambda_1 - v\tau < \xi < \lambda_1 + \lambda_2 - v\tau. \end{aligned}
$$

Substitution of solutions (3.10) into (3.5) and (3.9) leads to the following relations for functions f_i and g_i, $i = 1, 2$:

$$
(3.11)\qquad g_2(z) = \begin{cases} 0 & \text{for} \quad v\tau \le \lambda_1 + \lambda_2,\\ -\hat{f}_2(2\lambda_2 - z) & \text{for} \quad \dfrac{\lambda_1+\lambda_2}{v} < \tau \le \lambda_1 + \lambda_2,\\ -f_2(z - 2\lambda_2) & \text{for} \quad \tau > \lambda_1 + \lambda_2, \end{cases}
$$

(3.11) [cont.]

$$g_1(z) = \begin{cases} 0 & \text{for } v\tau \le \lambda_1, \\ -\hat{f}_1(2\lambda_1 - z) + f_2\left(z - \lambda_1\left(1 + \frac{1}{v}\right)\right) & \\ \qquad + g_2\left(z - \lambda_1\left(1 + \frac{1}{v}\right)\right) & \text{for } \frac{\lambda_1}{v} < \tau \le \lambda_1, \\ -f_1(z - 2\lambda_1) + f_2\left(z - \lambda_1\left(1 + \frac{1}{v}\right)\right) & \\ \qquad + g_2\left(z - \lambda_1\left(1 + \frac{1}{v}\right)\right) & \text{for } \tau > \lambda_1, \end{cases}$$

$$\breve{g}'_j(z) = \begin{cases} \dfrac{1}{2T_0(1+v)} P\left(\dfrac{z + \eta\lambda_1\left(1 + \frac{1}{v}\right)}{1+v}\right) & \text{for} \quad v\tau \le \sum_{i=1}^{j} \lambda_i, \\ 0 & \text{for} \quad v\tau > \sum_{i=1}^{j} \lambda_i, \end{cases}$$

where

$$P\left(\frac{z + \eta\lambda_1\left(1 + \frac{1}{v}\right)}{1+v}\right) = P(\kappa) = D_2(1+v)w'_1(\kappa) + R_2 w_1(\kappa),$$

$$\kappa = \frac{z + \eta\lambda_1\left(1 + \frac{1}{v}\right)}{1+v},$$

$$M_1(1+v)^2 w''_1(\kappa) + D_{12}(1+v)w'_1(\kappa) + R_{12}w_1(\kappa) = F(\kappa),$$

$$\hat{f}'_j(z) = \begin{cases} \dfrac{1}{2T_0(1-v)} P\left(\dfrac{z + \eta\lambda_1\left(1 - \frac{1}{v}\right)}{v-1}\right) & \text{for} \quad v\tau \le \sum_{i=1}^{j} \lambda_i, \\ 0 & \text{for} \quad v\tau > \sum_{i=1}^{j} \lambda_i, \end{cases}$$

here

$$P\left(\frac{z + \eta\lambda_1\left(1 - \frac{1}{v}\right)}{v-1}\right) = P(\chi) = D_2(v-1)w'_1(\chi) + R_2 w_1(\chi),$$

(3.11) [cont.]

$$\chi = \frac{z + \eta\lambda_1\left(1 - \frac{1}{v}\right)}{v - 1},$$

$$M_1(v-1)^2 w_1''(\chi) + D_{12}(v-1)w_1'(\chi) + R_{12}w_1(\chi) = F(\chi),$$

and

$$j = 1, \qquad \eta = 0 \qquad \text{for} \qquad 0 < \tau < \frac{\lambda_1}{v},$$

$$j = 2, \qquad \eta = 1 \qquad \text{for} \qquad \frac{\lambda_1}{v} < \tau < \frac{\lambda_1 + \lambda_2}{v},$$

$$f_1(z) = -g_1(z),$$

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_0''\left(z + \frac{\lambda_1}{v}\right) \\ f_2''(z) \end{bmatrix} + \begin{bmatrix} C_{12} & -C_2 \\ -C_2 & 2T + C_2 \end{bmatrix} \begin{bmatrix} w_0'\left(z + \frac{\lambda_1}{v}\right) \\ f_2'(z) \end{bmatrix}$$
$$+ \begin{bmatrix} K_{12} & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} w_0\left(z + \frac{\lambda_1}{v}\right) \\ f_2(z) \end{bmatrix} = \begin{bmatrix} C_2 g_2'(z) + K_2 g_2(z) \\ \mathfrak{A} - C_2 g_2'(z) - K_2 g_2(z) \end{bmatrix},$$

where

$$\mathfrak{A} = \begin{cases} 0 & \text{for} \quad v\tau \le \lambda_1, \\ -2T\hat{f}_1'\left(\lambda_1\left(1 - \frac{1}{v}\right) - z\right) & \text{for} \quad \frac{\lambda_1}{v} < \tau \le \lambda_1, \\ +2Tf_1'\left(z - \lambda_1\left(1 - \frac{1}{v}\right)\right) & \text{for} \quad \tau > \lambda_1. \end{cases}$$

Determination of the dynamic response of the system reduces to successive solution of Eqs. (3.11) together with Eqs. (3.10) in a way similar to that used in the sub-critical case.

4. Numerical examples

Numerical calculations were performed for the following parameters of the system described above:

$$l_1 = l_2 = 100\,[\text{m}], \qquad \tilde{T}_0 = 10000\,[\text{N}],$$

$$\mu = 0.89\,[\text{kg/m}], \qquad k_1 = k_2 = 1454\,[\text{N/m}],$$

$$c_1 = c_2 = 0, \qquad r_1 = 0, \qquad r_2 = 2800\,[\text{N/m}], \qquad m_1 = 20\,[\text{kg}].$$

The constant force $\tilde{F}(t) = \tilde{F}_0 = 90\,[\text{N}]$ was applied to the oscillator, and the kinematic external excitation $\tilde{w}_e(t)$ was neglected.

4.1. Results for the sub-critical case $v < 1$)

In this case the system was investigated for $m_0 = 0$, $m_2 = 16$ [kg], $d_1 = 40$ [Ns/m], $d_2 = 25$ [Ns/m] and for three values of the oscillator speed $v = 0.40$, 0.75 and $v = 0.95$. The results for $v = 0.75$ are presented in Figs. 3a and 3b. Fig. 3a shows deflections of the string for successive time instants 1–9, i.e. before (1–6) and after (7–9) the oscillator has left the string. In this figure one can observe transverse wave generation by the oscillator and wave reflections from the supports. However, in Fig. 3b there is shown the history of the string deflection at its instantaneous contact point with the oscillator mass as well as the history of the dynamic contact force P between the string and the oscillator. These histories are presented in the string length domain, which for constant v corresponds also to the time domain. At the beginning of the process, i.e. before the oscillator has reached the visco-elastic support, the string deflections and the contact force history are rather of a quasi-static character. The contact force for $l < 100$ [m] is almost constant and equal to the constant external excitation force $\tilde{F}_0$. An approach of the oscillator on the visco-elastic support ($l \to 100$ [m]) causes rapid changes of the string deflection and a "peaky" increase of the contact force. Then, for $l > 100$ [m] the system is excited to transient vibrations, and when the oscillator reaches the rigid support, i.e. for $l \to 200$ [m], the great peak values of the contact force occur.

For the remaining values of v analogous results were obtained. String deflections for successive time instants 1–9 for $v = 0.40$ and $v = 0.95$ are presented in Figs. 4a and 4b, respectively. From the three investigated values of v, the smallest string deflections and local gradients of deflections were obtained for $v = 0.40$. The case $v = 0.95$, however, is characterized by the greatest values of the string deflections and local deflection gradients. This fact is also confirmed by string deflections at the point of instantaneous contact with the oscillator, which are shown in Figs. 5a and 5b. However, the influence of the oscillator speed v is particularly essential for the contact force history. For $v = 0.95$ greater peak values were obtained than for $v = 0.75$. For $v = 0.40$, during "slower travel" along the string, the oscillator meets the waves reflected from the visco-elastic support and from the right-hand rigid support closer to its "starting point", i.e. the left-hand rigid support. First time the oscillator meets the wave reflected from the visco-elastic support at about 1/3 of the string length, Figs. 4a and 5a. However, for $v = 0.75$ and

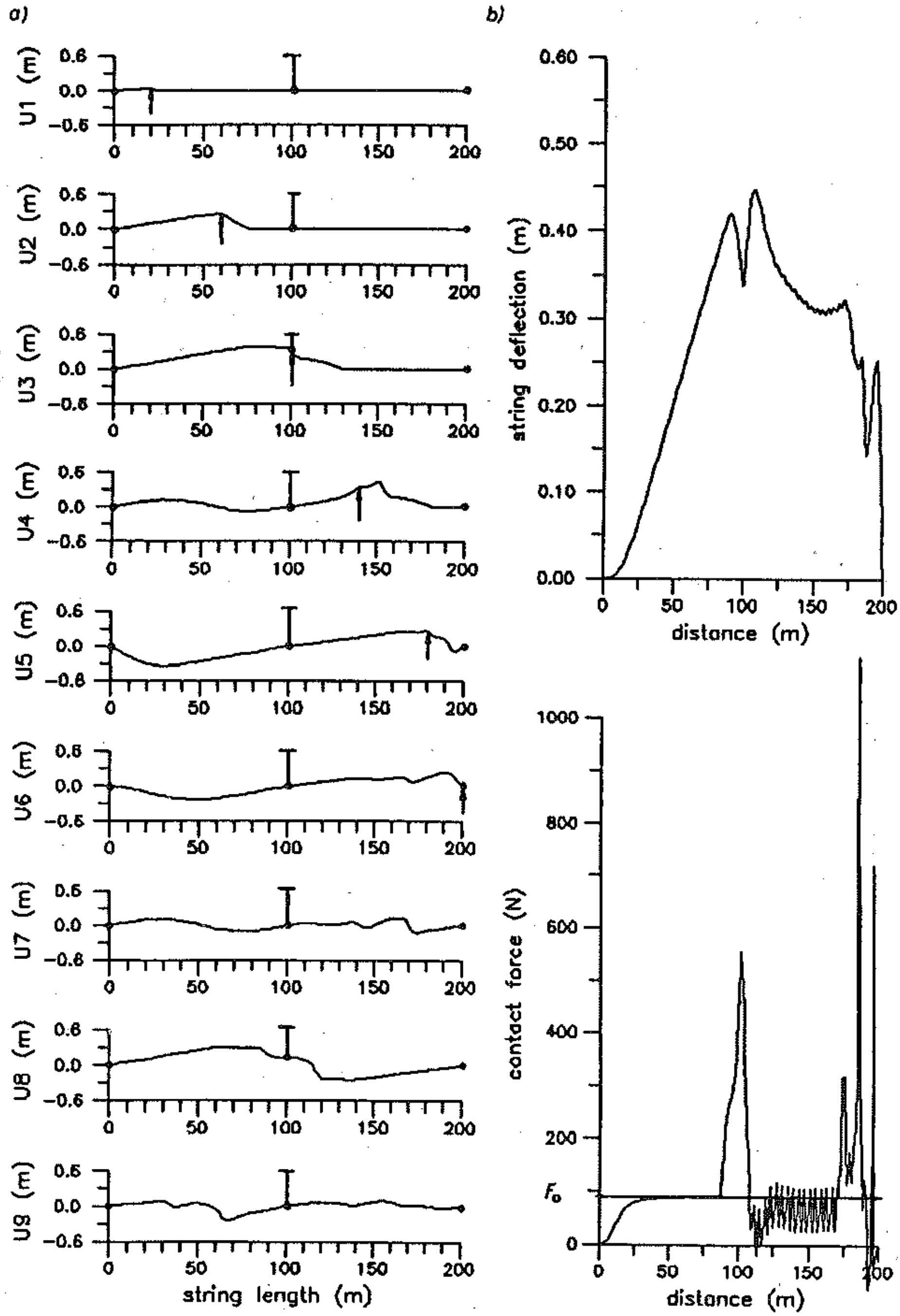

FIG. 3. Dynamic response of the system for $v = 0.75$ in the form of: string deflections for successive time instants (a), string deflection at its contact point with the oscillator, contact force between the string and the oscillator (b).

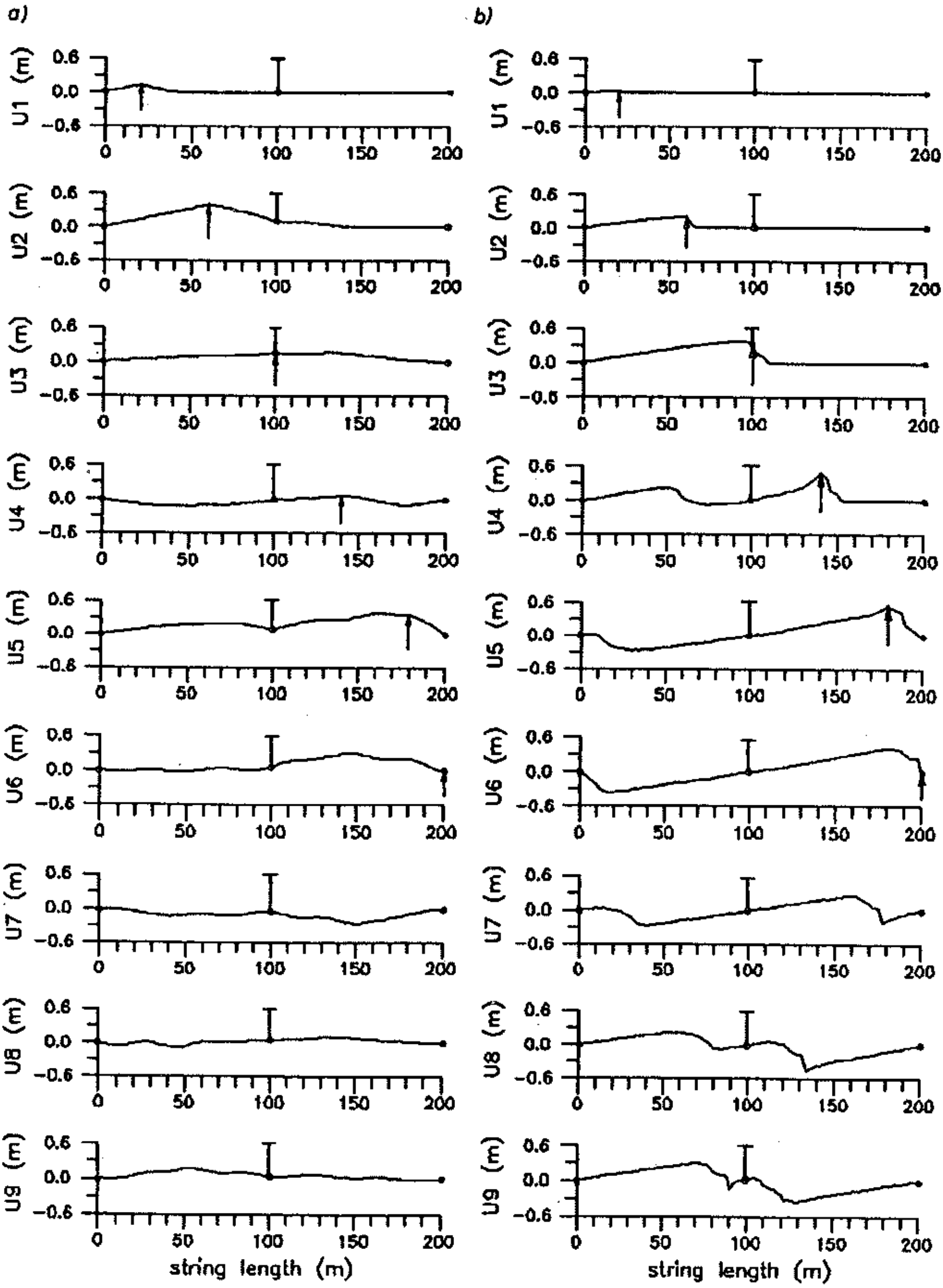

FIG. 4. Deflections of the string for successive time instants for $v = 0.40$ (a) and for $v = 0.95$ (b).

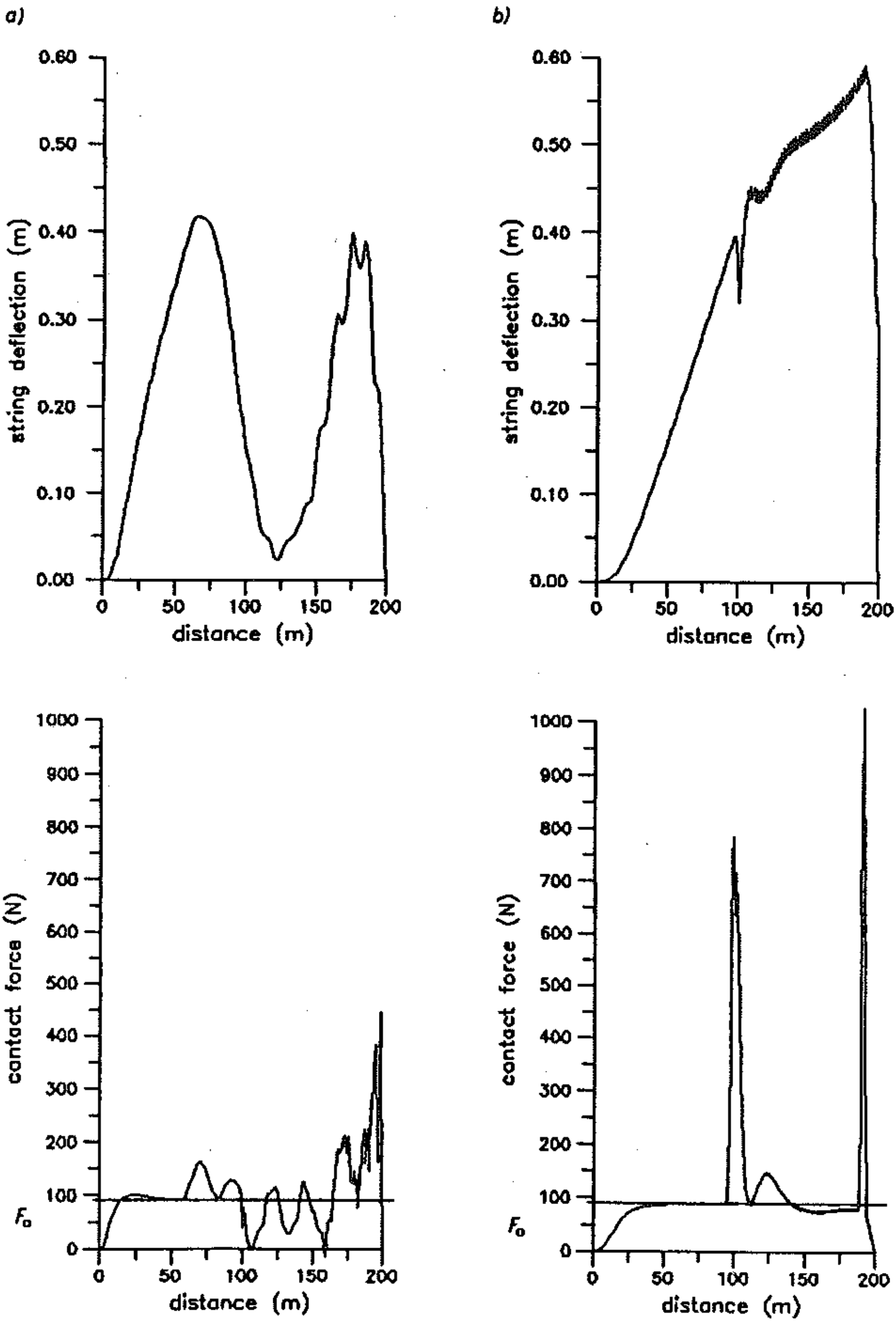

FIG. 5. String deflections at its contact point with the oscillator and histories of the contact force for $v = 0.40$ (a) and $v = 0.95$ (b).

$v = 0.95$, during very fast motion with speeds close to the critical speed, the oscillator meets the waves reflected from the visco-elastic support and from the right-hand rigid support very close to them, just before approaching these supports, Figs. 3, 4b and 5b. As it follows from Figs. 3b and 5b, for $v = 0.75$ and $v = 0.95$ only the waves reflected from the right-hand rigid support essentially influence the contact force values. However, the waves reflected from the visco-elastic support considerably perturb the contact force history only for $v = 0.40$, Fig. 5a.

The results concerning deflection of the string at its contact point with the oscillator and the contact force P for $v = 0.40$ and $v = 0.95$ were compared with analogous results obtained in [8] for the identical mechanical system, where the string was represented by the finite element model. The forms of respective curves in [8] are similar to those in Fig. 5, but in cases of the local extreme values significant differences take place – reaching about 20 – 60%. Using the proposed wave method, greater absolute peak values are obtained, in particular for $v = 0.95$. This fact can be explained by an application of the modal transformation method for the finite element model. This leads to superposition of the harmonic standing waves in the string which results in numerical "smoothing out" of all the resultant peak values. Analogous differences of the results appeared in [7], where the proposed wave method was compared with the Fourier method for an identical string excited by a moving force. Moreover, as it follows from [8], in order to obtain a solution, which was convergent and more or less reliable from the physical viewpoint, for the considered case it was necessary to take into account at least 200 eigenmodes of vibrations. On the other hand, for $v = 0.95$ the "sharpest" peak of the string deflection at its contact point with the oscillator was obtained for 10 eigenmodes taken into account, while further increase in number of the eigenmodes did not improve the accuracy of the solution. From the performed comparison it follows that the results obtained using the wave method proposed in the paper seem to be more accurate than the analogous results following from the finite element approach.

4.2. Results for the super-critical case ($v > 1$)

This case is not of practical importance, but it can be used for theoretical purposes in order to demonstrate possibilities of the wave method applied. In the super-critical case numerical calculations were performed for $m_0 = 10$ [kg], $m_2 = 0$ and for $d_1 = d_2 = 0$. Figure 6 presents the string deflection

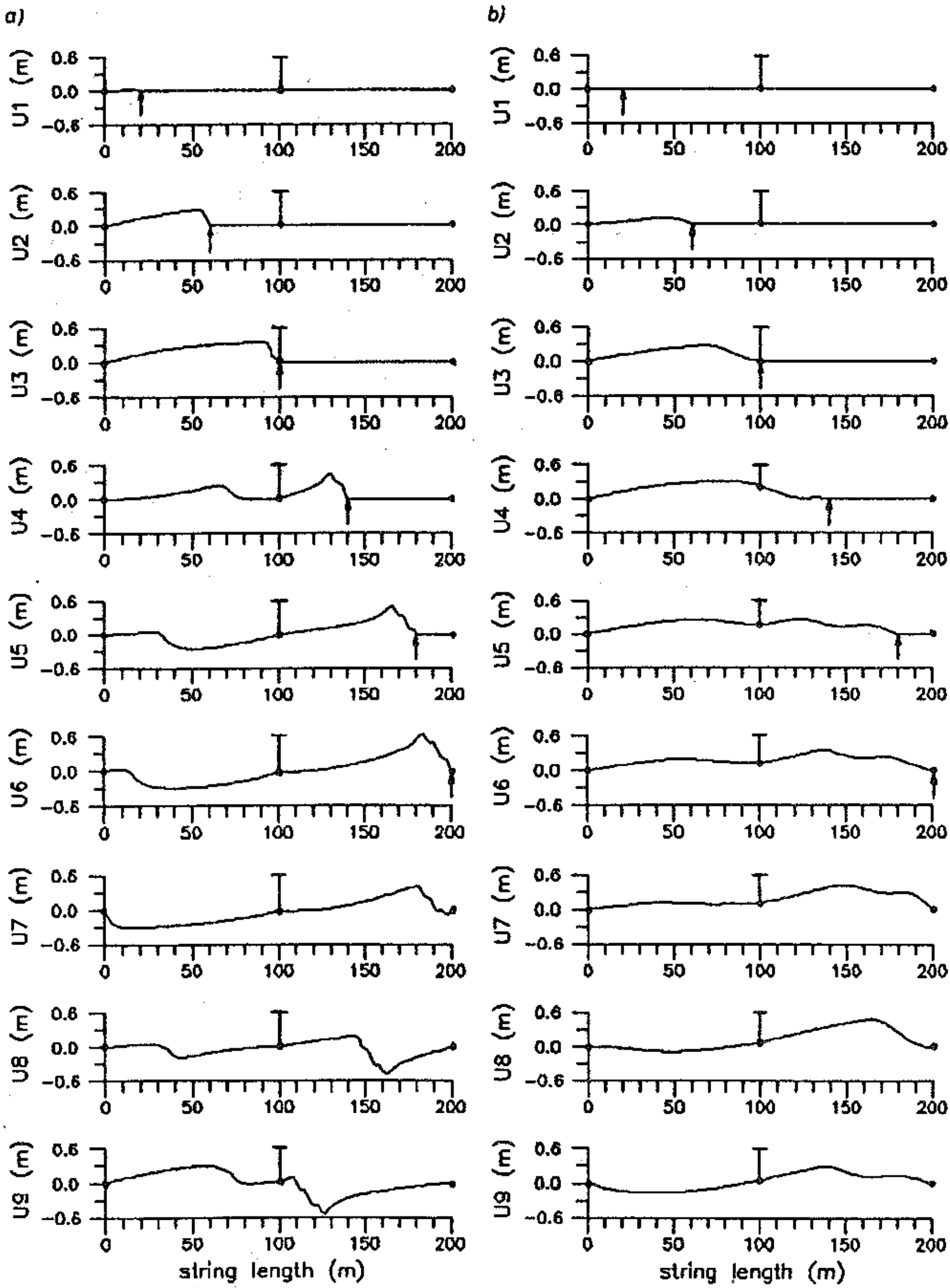

FIG. 6. Deflections of the string for successive time instants for $v = 1.1$ (a) and for $v = 1.6$ (b).

curves for successive time instants 1–9, i.e. before and after the oscillator leaves the string, for $v = 1.1$, Fig. 6a, and for $v = 1.6$, Fig. 6b. Greater string deflections as well as greater local deflection gradients take place for $v = 1.1$, i.e. for the oscillator speed closer to the critical speed $v = 1$. Similar effects were obtained in the sub-critical case for $v = 0.95$, Fig. 4b, and for $v = 0.75$, Fig. 3a. But the histories of the dynamic contact force P are of a completely different character than those in the sub-critical case, Fig. 7. Since for $v > 1$ there is no deflection at the contact point of the string with the oscillator, motion of the oscillator mass m_1 reduces to vibrations of the system of one degree of freedom excited by the constant force $\tilde{F}_0$. Thus, for $d_1 = d_2 = 0$ histories of the contact force in Fig. 7 have sinusoidal forms of the mean value $\tilde{F}_0$ and of amplitudes equal to $\tilde{F}_0$, which is in agreement with the fundamental theory of vibrations [2]. Certainly, the number of vibration cycles of these histories depends on the physical time period of the oscillator travel along the string. Number of these cycles for $v = 1.1$ is greater (about 2.6 cycles, Fig. 7a) than for $v = 1.6$ (about 1.8 cycles, Fig. 7b).

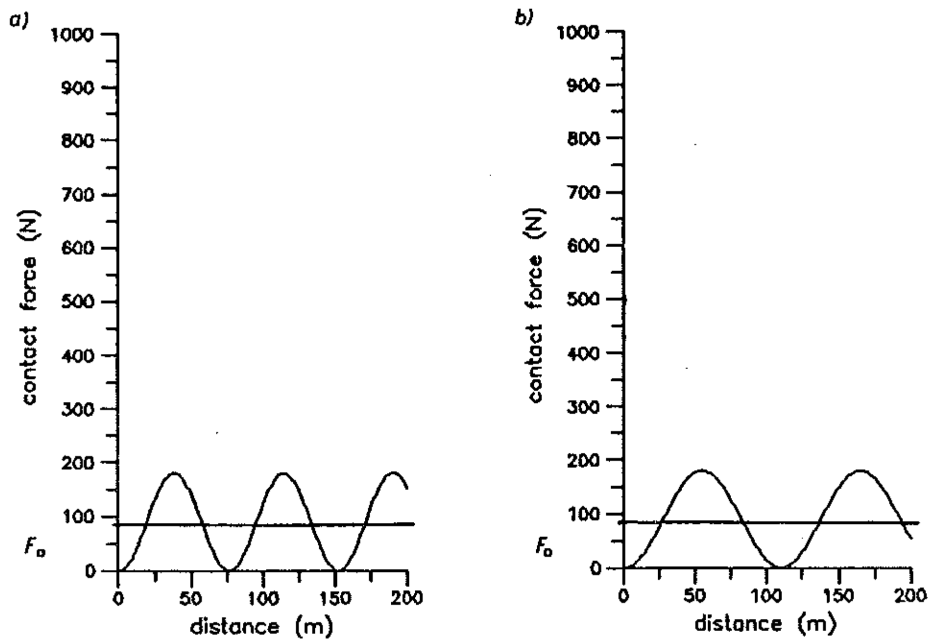

FIG. 7. Dynamic contact force between the string and the oscillator for $v = 1.1$ (a) and for $v = 1.6$ (b).

5. Final remarks

In the paper the one-dimensional elastic wave propagation theory was applied to the transverse vibration analysis of the finite string excited by the moving dynamic oscillator of two degrees of freedom. The wave motion equations for the string were solved using the d'Alembert solutions. Determination of these solutions by the boundary conditions led to appropriate systems of algebraic and ordinary differential equations with a "shifted" argument. Successive solution of these equations enables us to obtain the dynamic response of the system for various oscillator speeds, i.e. both for smaller (sub-critical) and for greater (super-critical) values than the transverse wave propagation velocity in the string.

From the results of calculations performed for sub-critical speeds compared with analogous results obtained using the finite element method it follows that for the oscillator speed closer to the critical speed, greater discrepancies of results took place – reaching 20 – 60%. Application of the finite element method resulted in "smoothing out" of the peak values. Thus, one can conclude that the wave approach based on the d'Alembert solution appears to be more reliable.

For the sub-critical speeds, all approaches of the oscillator on the string supports cause great peak values of the dynamic contact force between the oscillator and the string. However, for the super-critical speeds the contact force values depend only on parameters of the oscillator and on its external excitation. In such a case successive approaches of the oscillator on the supports do not influence the oscillator-string contact force history.

The method presented in the paper is characterized by a great numerical efficiency and a clear mathematical description of the wave effects. These advantages make the proposed approach particularly convenient for dynamic investigations of more complex systems, first of all for the studies of interaction of the electric locomotive pantograph with the railway catenary suspension.

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