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ON DYNAMIC PROPERTIES OF TWO MODELS OF BEAM ON NONLINEAR FOUNDATION SUBJECTED TO MOVING LOAD(*)

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The subjects of consideration are the infinite Bernoulli – Euler and Timoshenko models of a beam resting on a nonlinear visco-elastic foundation. The beams are subjected to a distributed inertialess loading constant in a given sector and moving with a constant velocity. Nonlinearity of the foundation was assumed in the form of a piece-wise linear characteristic of elasticity. Results obtained for the Bernoulli – Euler beam have been compared to the corresponding results obtained for a Timoshenko beam. The considered problems may find some applications in modern transportation systems.

1. Introduction

The subjects of consideration are infinite Bernoulli - Euler and Timoshenko models of a beam resting on inertialess nonlinear visco-elastic foundation. The beams are subjected to a distributed inertialess loading, constant in a given sector and moving with a constant velocity. Stationary vibrations of beams in coordinates connected with the moving load are described by parabolic or hyperbolic differential equations of fourth order. Nonlinearity of the foundation was assumed in the form of a piece-wise linear characteristic of elasticity. To determinate the solutions, the approximate method has been applied using the analytical solutions of linear approximations. The results obtained for the Bernoulli - Euler beam have been compared to the corresponding results for a Timoshenko beam.

The analytical solution for the linear case and load described by a Heaviside function was obtained in [1]. An extensive survey of papers devoted to this type of problems can be found in a few papers, e.g. in [3]. An

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approximate procedure, like that used in [2] and [4], is proposed for the solution of this problem.

2. Differential equations of motion of the beams and formulae for bending moments and shear forces

The motion of the Timoshenko beam written in the fixed Cartesian coordinate system $\bar{0}\bar{x}_1\bar{y}$, in which $\bar{0}\bar{x}_1$ is the axis of the beam and the $\bar{0}\bar{y}$-axis is directed downwards, is described by the partial differential equations

$$\begin{gathered}\kappa AG\left(\frac{\partial^2 y}{\partial \bar{x}_1^2}-\frac{\partial \psi}{\partial \bar{x}_1}\right)-\varrho\frac{\partial^2 y}{\partial t^2}+p=0,\\ EI\frac{\partial^2\psi}{\partial \bar{x}_1^2}+\kappa AG\left(\frac{\partial y}{\partial \bar{x}_1}-\psi\right)-\varrho r^2\frac{\partial^2\psi}{\partial t^2}-N\frac{\partial y}{\partial \bar{x}_1}+m=0,\end{gathered}\tag{2.1}$$

with the following notation:

- y displacement in $\bar{y}$ direction,
- ψ angle of rotation of the beam due to pure shear,
- EI flexural rigidity,
- κ the Timoshenko shear coefficient,
- A cross-sectional area with moment of inertia I,
- ϱ constant linear density,
- N constant tensile force,
- t time,
- $m = m(\bar{x}_1, t)$ external continuously distributed loading moment,
- $p = p(x_1, t)$ external continuously distributed load,
- $r = (I/A)^{0.5}$.

The motion of the Bernoulli - Euler beam is described by the partial differential equation

$$EI\frac{\partial^4 y}{\partial \bar{x}_1^4}-N\frac{\partial^2 y}{\partial \bar{x}_1^2}+\varrho\frac{\partial^2 y}{\partial t^2}+\frac{\partial m}{\partial \bar{x}_1}-p=0.\tag{2.2}$$

It is assumed that

$$p = p_v - p_0 \qquad \text{and} \qquad m = m_v - m_0,$$

where p_v, m_v – given moving continuously distributed forces and moments, p_0, m_0 – loadings resulting from the reaction of the beam foundation.

The assumed nonlinear characteristics of the foundation are

$$(2.3)\qquad \begin{aligned} p_0 &= c_p y + b_p \frac{\partial y}{\partial t} + p_0^*, \\ m_0 &= c_m \psi + b_m \frac{\partial \psi}{\partial t} + m_0^*, \end{aligned}$$

where c_p, c_m, b_p, b_m – coefficients of elasticity and damping of the linear characteristic of foundation, p_0^*, m_0^* – nonlinear terms.

Introducing the Cartesian coordinate system $0x_1y$ related to the moving load by

$$\bar{y} = y \qquad \text{and} \qquad \bar{x}_1 = x_1 + vt,$$

where v denotes the constant velocity of the loading motion, under the assumption that the solutions are stationary in the rectangular coordinate systems $0x_1y$ connected with the loading front, the equations of motion of the beam (2.1) are as follows

$$(2.4)\qquad \begin{aligned} &\kappa AG\left(\frac{d^2y}{dx_1^2} - \frac{d\psi}{dx_1}\right) - \varrho v^2 \frac{d^2y}{dx_1^2} + p_v - c_p y + b_p v \frac{dy}{dx_1} - p_0^* = 0, \\ &EI\frac{d^2\psi}{dx_1^2} + \kappa AG\left(\frac{dy}{dx_1} - \psi\right) - \varrho r^2 v^2 \frac{d^2\psi}{dx_1^2} - N\frac{dy}{dx_1} + m_v - c_m\psi \\ &\qquad + b_m v \frac{d\psi}{dx_1} - m_0^* = 0, \end{aligned}$$

while the equation of motion of the beam (2.2) takes the form

$$(2.5)\qquad \begin{aligned} &EI\frac{d^4y}{dx_1^4} + b_m v\frac{d^3y}{dx_1^3} + (\varrho v^2 - N - c_m)\frac{d^2y}{dx_1^2} - b_p v\frac{dy}{dx_1} \\ &\qquad + c_p y - \frac{dm^*}{dx_1} + p_0^* = p_v - \frac{dm_v}{dx_1}. \end{aligned}$$

Furthermore, like in [1], new dimensionless variables are introduced

$$x = x_1/r \qquad \text{and} \qquad u = y/y_s,$$

where $y_s > 0$ is a given value, and the following dimensionless coefficients are introduced:

$$V = (v/r)(\varrho/c_p)^{0.5}$$

$$\begin{aligned} V_1 &= (\kappa AG/c_p)^{0.5}/r, & V_2 &= (EA/c_p)^{0.5}/r, \\ b &= 0.5b_p(c_p\varrho)^{-0.5}, & B &= 0.5b_m(c_p\varrho)^{-0.5}/r^2, \\ S &= N/(r^2c_p), & C &= c_m/(r^2c_p). \end{aligned}$$

Similarly, the dimensionless loadings can be expressed as follows:

$$\bar{p}_v = p_v/p_s, \qquad \bar{m}_v = m_v/(p_s r),$$
$$\bar{p}_0^* = p_0^*/p_s, \qquad \bar{m}_0^* = m_0^*/(p_s r),$$

in which

$$p_s = c_p y_s.$$

Assuming

$$\bar{m}_0^* = 0,$$

eliminating ψ from the equations of the Timoshenko beam (2.4) and denoting $(') = d/dx$, we obtain

$$\begin{aligned}(2.7)\qquad (V^2 - V_1^2)(V^2 - V_2^2)u^{(4)} - 2V\left[B(V^2 - V_1^2) + b(V^2 - V_2^2)\right]u'''\\ + \left[V^2(V_1^2 + 1 + 4bB + C) - (S + C)V_1^2 - V_2^2\right]u''\\ -2V\left[b(V_1^2 + C) + B\right]u' + (V_1^2 + C)u + (V^2 - V_2^2)\bar{p}_0^{*''} - 2BV\bar{p}_0^{*'}\\ +(V_1^2 + C)\bar{p}_0^* = -V_1^2\bar{m}_v' + (V^2 - V_2^2)\bar{p}_v'' - 2BV\bar{p}_v' + (V_1^2 + C)\bar{p}_v.\end{aligned}$$

The equation of the Bernoulli - Euler beam (2.5) takes the form

$$(2.8)\qquad V_2^2u^{(4)} + 2VBu''' + (V^2 - S - C)u'' - 2Vbu' + u + \bar{p}_0^* - \bar{m}_0^{*'} = \bar{p}_v - \bar{m}_v'.$$

Dividing both sides of Eq. (2.7) by $(V_1^2 + C)$ we obtain for $V_1^2 \to \infty$

$$(2.9)\qquad \left(1 - \frac{V^2}{V_2^2}\right)V_2^2u^{(4)} + 2VBu''' + (V^2 - S - C)u'' - 2Vbu' + u + \bar{p}_0^* = \bar{p}_v - \bar{m}_v'.$$

By comparing Eqs. (2.8) with (2.9), it can be proved that for $V^2 \ll V_2^2$

$$1 - \frac{V^2}{V_2^2} \approx 1,$$

and under the assumption (2.6), Eq. (2.9), resulting from the equations of Timoshenko beam, becomes identical with the Bernoulli - Euler beam equation (2.8).

Bending moments and shear forces in a Timoshenko beam are described by the formulae

$$M = -EI\frac{d\psi}{dx_1} \qquad \text{and} \qquad T = \kappa AG\left(\frac{dy}{dx_1} - \psi\right),$$

while in the case of a Bernoulli – Euler beam, these formulae are

$$M = -EI\frac{d^2y}{dx_1^2} \qquad \text{and} \qquad T = \frac{dM}{dx_1} + N\frac{dy}{dx_1} - m.$$

For the dimensionless variables x and u, the dimensionless moments and forces are introduced

$$\overline{M} = M/M_0 \qquad \text{and} \qquad \overline{T} = T/T_0, \tag{2.10}$$

where

$$M_0 = V_2^2 p_s r^2 \qquad \text{and} \qquad T_0 = V_1^2 p_s r.$$

Then for the case of a Timoshenko beam, formulae (2.10) take the form

$$\overline{M} = \left[(V^2 - V_1^2)u'' - 2bVu' + u - \bar{p}_v + \bar{p}_0^*\right] V_1^{-2},$$

$$\overline{T} = u' - \bar{\psi},$$

where $\bar{\psi} = (r/y_s)\psi$ is expressed by the formula

$$\bar{\psi} = \Big\{D(V^2)u''' - 2V\left[B(V^2 - V_1^2) + b(V^2 - V_2^2)\right]u'' + \Big[V^2(1 + 4bB)$$
$$- V_1^2(V_1^2 - S) - V_2^2\Big]u' + 2BV(\bar{p}_v - \bar{p}_0^* - u) + (V^2 - V_2^2)(\bar{p}_0^* - \bar{p}_v)'\Big\}$$
$$\times \left[V_1^2(V_1^2 + C)\right]^{-1}.$$

For the Bernoulli – Euler beam formulae (2.10) take the form

$$\overline{M} = -u'',$$

$$\overline{T} = \left[-V_2^2u''' - 2VBu'' + (S + C)u' - \bar{m}_v\right] V_1^{-2}.$$

The expressions presented enable us to calculate the bending moment and shear force for a known solution $u = u(x)$.

3. Solutions for the linear case and analysis of nonlinear equations of beams

When condition (2.6) is fulfilled, Eqs. (2.7) and (2.8) can be written as follows

$$F[u(x)] + f[u(x)] = g_m\left[\bar{m}_v(x)\right] + g_p\left[\bar{p}_v(x)\right], \tag{3.1}$$

where F, f, g_m and g_p denote the differential operators F, g_m, g_p – linear, and f – nonlinear. Operator f has the form

$$f[u] = g_p\left[\bar{p}_0^*(u, u')\right].$$

For a Timoshenko beam we obtain the equation

$$\begin{aligned} F[u(x)] = D(V^2)u^{(4)} - 2V[B(V^2 - V_1^2) + b(V^2 - V_2^2)]u''' \\ +[V^2(V_1^2 + 1 + 4bB + C) - (S + C)V_1^2 - V_2^2]u'' \\ -2V[b(V_1^2 + C) + B]u' + (V_1^2 + C)u, \end{aligned}$$

$$\begin{aligned} g_m\left[\bar{m}_v(x)\right] &= -V_1^2\bar{m}_v', \\ g_p\left[\bar{p}_v(x)\right] &= (V^2 - V_2^2)\bar{p}_v'' - 2BV\bar{p}_v' + (V_1^2 + C)\bar{p}_v, \\ D(V^2) &= (V^2 - V_1^2)(V^2 - V_2^2). \end{aligned}$$

For a Bernoulli – Euler beam, the equation has the form

$$F[u(x)] = V_2^2u^{(4)} + 2VBu''' + (V^2 - S - C)u'' - 2Vbu' + u,$$

$$g_m\left[\bar{m}_v(x)\right] = -V_1^2\bar{m}_v', \qquad g_p\left[\bar{p}_v(x)\right] = \bar{p}_v.$$

In a thorough analysis, certain non-linearity has been taken into account. Taking into consideration the bilinear character of elasticity and linear damping, the nonlinear characteristic of foundation in the formula (2.3) takes the form

$$p_0 = b_p\frac{\partial y}{\partial t} + \begin{cases} p_n + \xi c_p(y_n - y) & \text{for} \quad y \le y_n, \\ c_p y & \text{for} \quad y > y_n, \end{cases}$$

where $y_n < 0$, ξ are given values, and

$$p_n = c_p y_n < 0.$$

Coefficient ξ describes the inclination of the piece-wise linear characteristic of elasticity. For $\xi = 1$ the bilinear character of the curve disappears. Thus the nonlinear component in Eq. (3.1) is expressed by the formula

$$\bar{p}_0^* = (1 - \xi)(u_n - u)\mathbf{1}(u_n - u), \tag{3.2}$$

where $\mathbf{1}(z)$ is the Heaviside function, and

$$u_n = y_n/y_s.$$

Let us assume that

$$\bar{m}_v \equiv 0 \qquad \text{and} \qquad \bar{p}_v = q_0 \mathbf{1}(-x),$$

where $q_0 = q/p_s$.

For the linear case we assume $\bar{p}_0^* \equiv 0$.

Taking into account that the linear approximation of Eq. (3.1) can be written in the form

$$F[u(x)] = g_m [\bar{m}_v(x)] + g_p [\bar{p}_v(x)], \tag{3.3}$$

the solution $u_\infty(x)$ satisfying the boundary conditions of an infinite beam is obtained by means of the method of Fourier and Laplace transforms and the convolution theorem. This solution for both types of beams is expressed by the formula

$$u_\infty(x) = q_0 \mathbf{1}(-x) + \sum_{j=1}^{4} \mathbf{1}(-x \operatorname{sgn} n_j) b_j h_j(x),$$

where $n_j = \operatorname{Re} s_j$, s_j are roots of the characteristic polynomial, and b_j are constants.

Functions $h_j(x)$ have the forms:

$$\exp(n_j x) \qquad \text{for} \qquad \operatorname{Im} s_j = 0$$

or

$$\exp(n_j x)\cos(k_j x) \qquad \text{and} \qquad \exp(n_j x)\sin(k_j x)$$

for

$$k_j = |\operatorname{Im} s_j| > 0.$$

For the moving load of constant value in the given sector $x \in < x_a;\ x_b >$

$$\bar{p}_v(x) = q_0 \{\mathbf{1}[-(x - x_b)] - \mathbf{1}[-(x - x_a)]\},$$

the solution $u_0(x)$ takes the form

$$u_0(x) = u_\infty(x - x_b) - u_\infty(x - x_a). \tag{3.4}$$

The approximate solution of the nonlinear equation (3.1) was obtained by the method applied in [4] to the Bernoulli-Euler beam. This method was generalized for the case of a Timoshenko beam in [2]. The stationary solution of nonlinear equation (3.1) is approximated by the successive terms of the functional series $u_k(x)$, $k = 0, 1, 2, \ldots$. The function $u_0(x)$ is described by Eq. (3.4). To determine the successive functions $u_k(x)$, the series of linear differential equations is used

$$F[\Delta u_{k+1}(x)] = g_p\left[\bar{p}_{vk}(x)\right], \qquad k = 0, 1, 2, \ldots, \tag{3.5}$$

where

$$\Delta u_{k+1}(x) = u_{k+1}(x) - u_0(x),$$
$$g_p\left[\bar{p}_{vk}(x)\right] = -f[u_k(x)],$$
$$\bar{p}_{vk}(x) \approx -\bar{p}_0^*\left[(u_k(x), u_k'(x)\right].$$

Equations (3.5) are similar to Eq. (3.3). The approximation results from the fact that the form $\bar{p}_{vk}(x)$ should enable us to apply the known analytical solutions of a linear differential equation. According to that $\bar{p}_{vk}(x)$ is taken in the form of a finite sum of dimensionless moving continuous loadings of constant values in the given sections. It allows us to use the solutions of the form (3.4) and to apply the principle of superposition for linear differential equations (3.3). The solution was estimated according to the following criterion

$$\max_{x\in(-\infty;\infty)} |u_{k+1}(x) - u_k(x)| \le \varepsilon, \qquad k = 0, 1, 2, \ldots,$$

where $\varepsilon > 0$ is considered as the permissible error.

4. Results of the numerical analysis

The influence of nonlinearity and the velocity of the moving load, for chosen parameters characterizing the dynamical system, on the displacements, bending moments and shear forces in beams has been considered.

In Figs. 1–3 for a Bernoulli - Euler beam, and in Figs. 4–6 for a Timoshenko beam, are presented the functions of dimensionless displacements $u(x)$, bending moments $M(x)$ and shear forces $T(x)$, determined numerically for the linear (lin) and nonlinear $n - l$ cases. For the curves shown in Figs. 1–6, the following parameters have been assumed

$$\begin{gathered} V = 13, \qquad V_1 = 45, \qquad V_2 = 90, \\ q_0 = 2, \qquad b = 0.2, \\ x_a = -150, \qquad x_b = B = C = S = 0, \\ \xi = 0, \qquad u_n = -0.35 \quad \text{and} \quad \varepsilon = 0.02. \end{gathered} \tag{4.1}$$

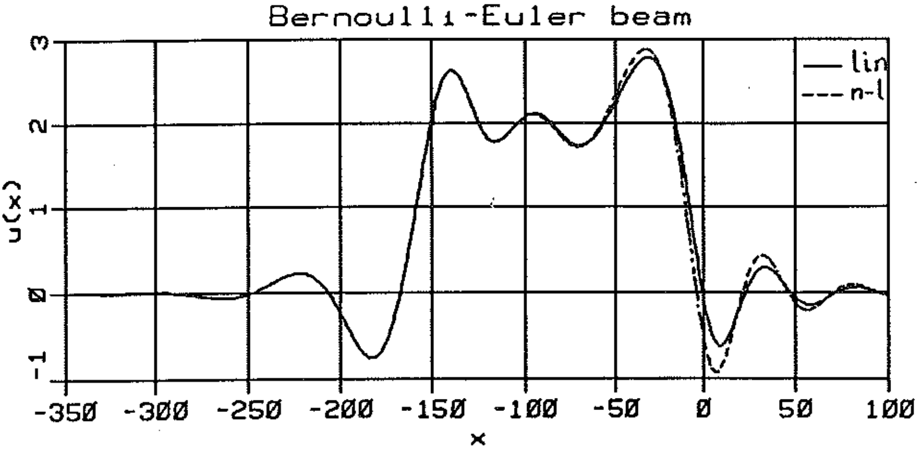

FIG. 1.

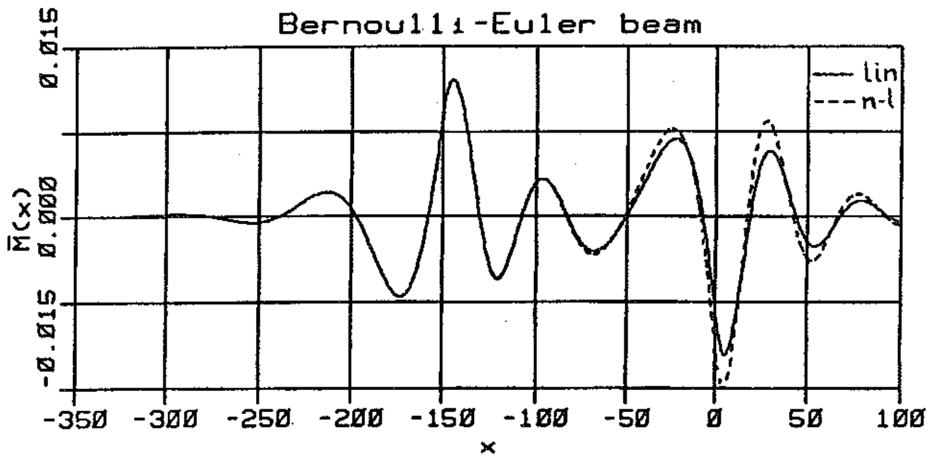

FIG. 2.

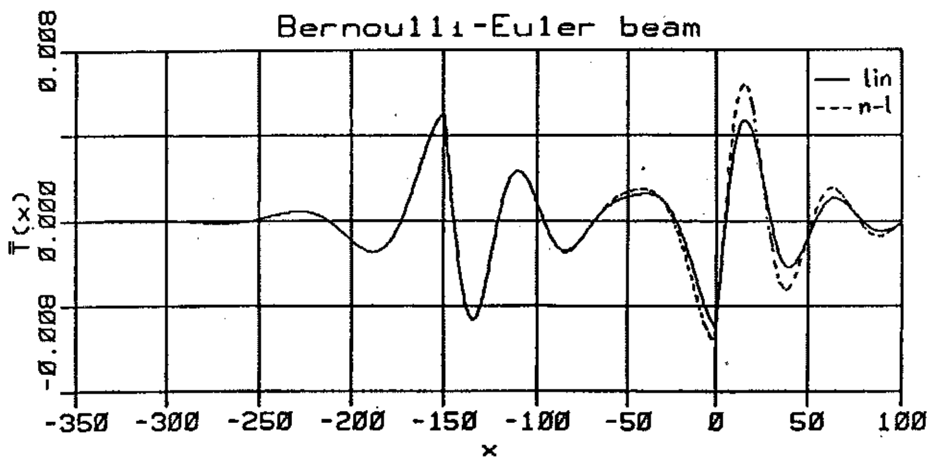

FIG. 3.

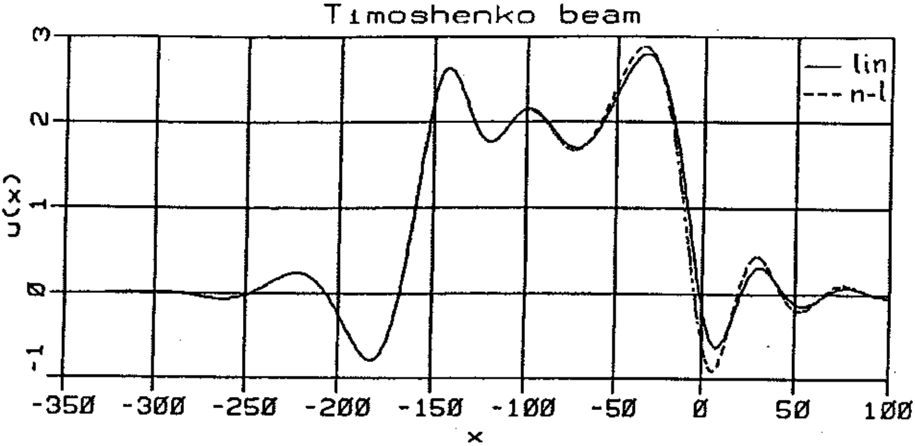

Fig. 4.

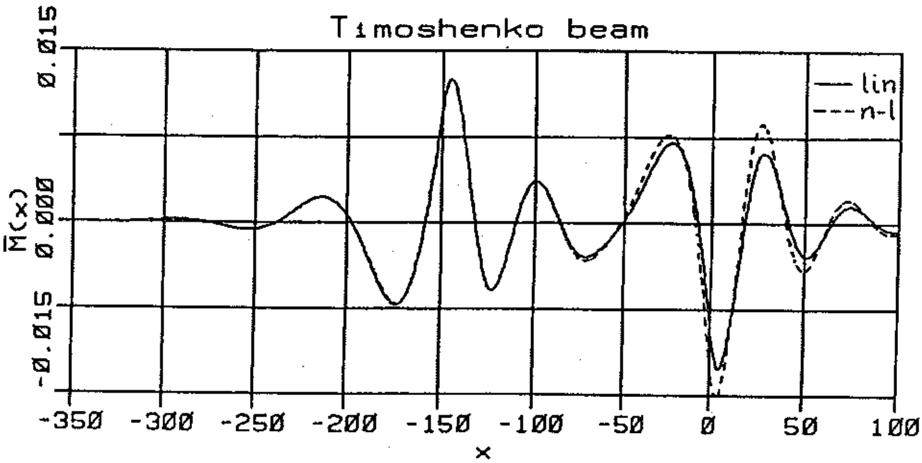

Fig. 5.

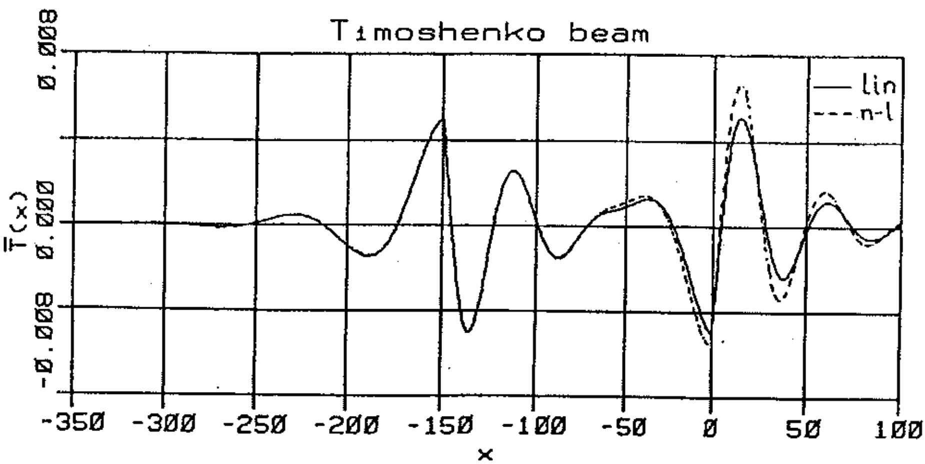

Fig. 6.

Figures 7–9 demonstrate the comparison between the functions of displacements $u_0(x)$, bending moments $M_0(x)$ and shear forces $T_0(x)$ determined for the Bernoulli–Euler and Timoshenko beam in linear cases. In Figs. 10–12 the comparison is shown between the functions of dimensionless displacements $u(x)$, bending moments $M(x)$ and shear forces $T(x)$ for the nonlinear cases. For the curves shown in Figs. 7–12, the following parameters have been assumed:

$$V = 16 \quad \text{and} \quad u_n = -0.5. \tag{4.2}$$

The remaining parameters are the same as those in Eqs. (4.1).

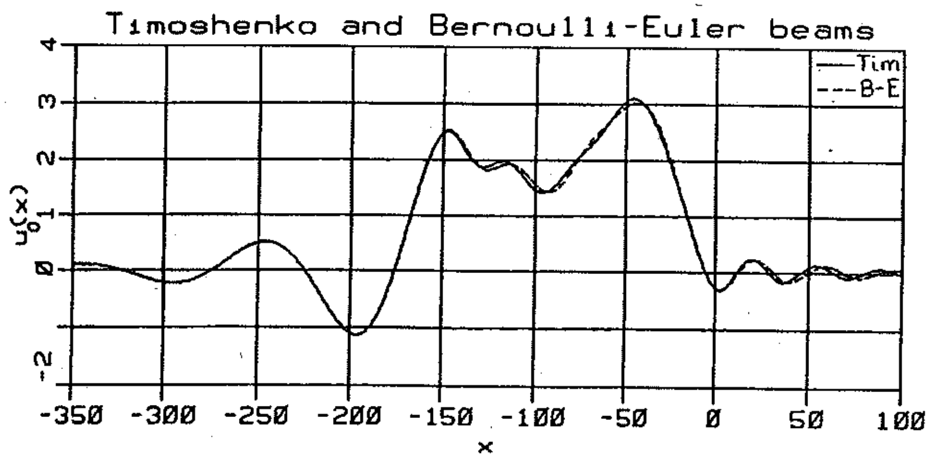

FIG. 7.

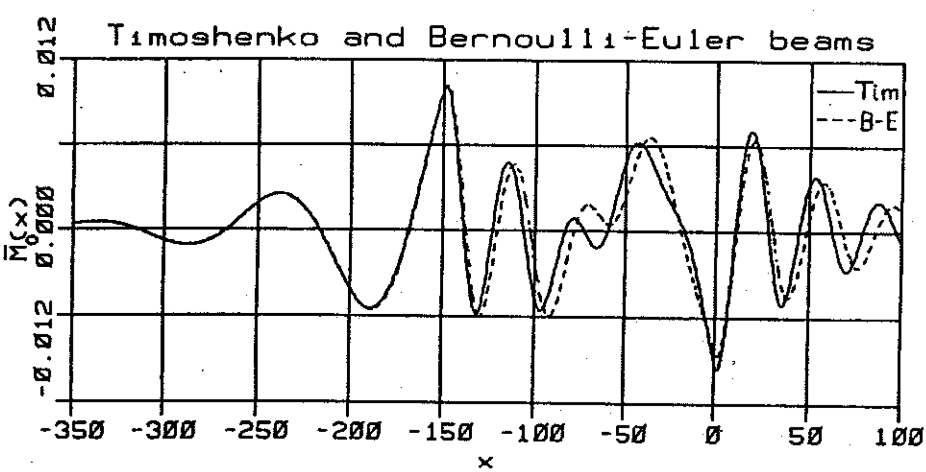

FIG. 8.

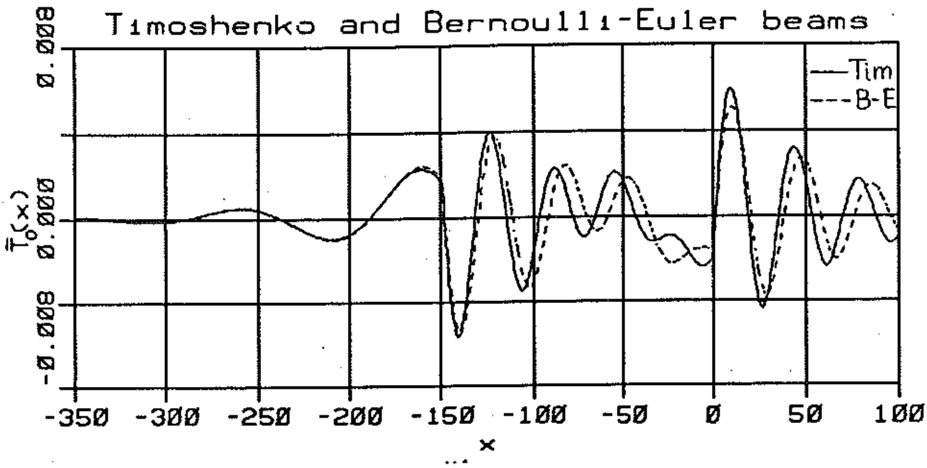

FIG. 9.

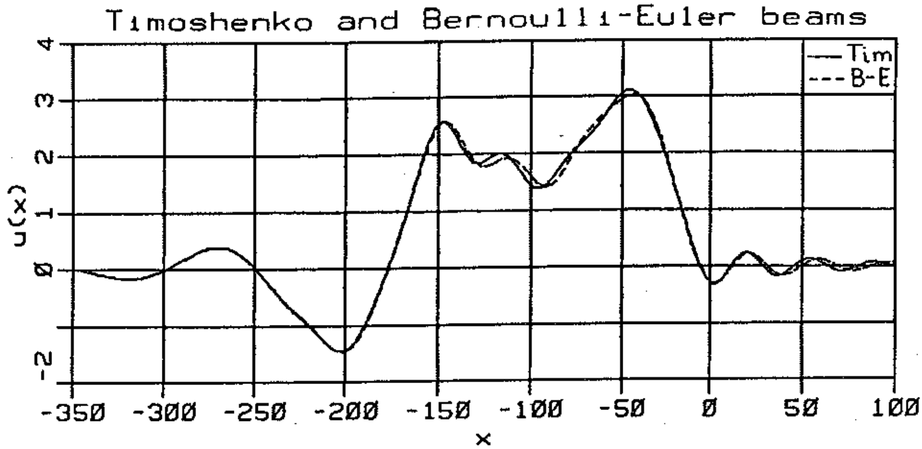

FIG. 10.

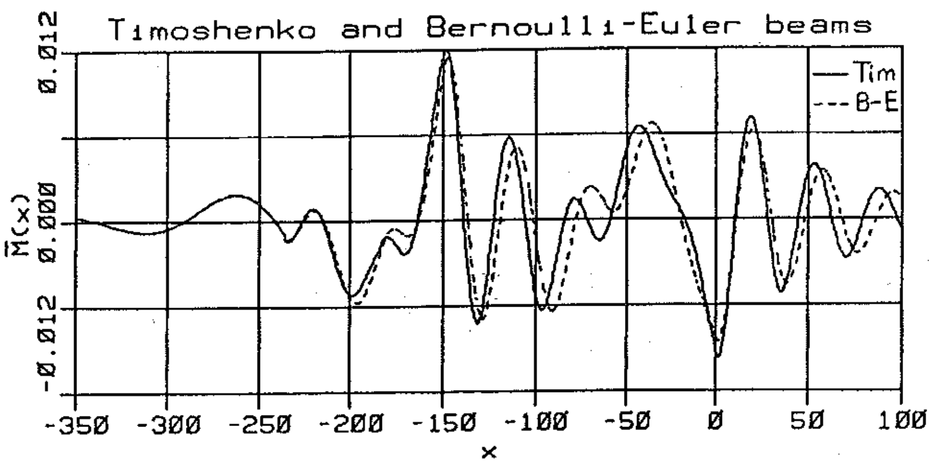

FIG. 11.

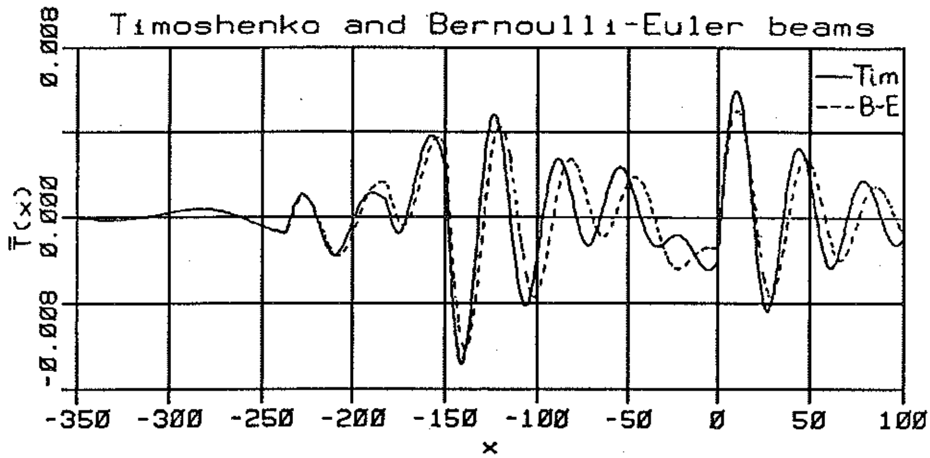

FIG. 12.

Critical dimensionless velocity for a Bernoulli - Euler beam is described by formula

$$V_{cr} = (2V_2 + S + C)^{0.5}.$$

For the parameters given by Eqs. (4.1) and (4.2), this velocity is

$$V_{cr} = 13.416.$$

5. Concluding remarks

The comparison of the numerical results obtained enables us to state that, for the set of dynamical parameters under consideration, the dynamical displacements, bending moments and shear forces determined for the Bernoulli - Euler beam are close to those of the Timoshenko beam. Differences appear principally in the wave-lengths in front of the load. The influence of nonlinearity is manifested by increasing amplitudes of dynamic parameters of motion: in front of the load for smaller velocities, and behind the load for greater values of velocity of the moving load.

The method presented takes into account nonlinearities of the foundation and enables us to investigate the influence of parameters of the mechanical system on the wave propagation in a beam under moving load. The knowledge of the properties of solution enables us to control the system and minimize the noise generation. The considered problems concerning the mechanical models may find applications in modern transportation systems.

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