ENGINEERING TRANSACTIONS • Engng. Trans. • 41, 1, 21–30, 1993
Polish Academy of Sciences • Institute of Fundamental Technological Research

FATIGUE RELIABILITY IN UNIAXIAL STATE OF STRESS PRODUCED BY SYNCHRONOUS LOADS

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The paper deals with the stress-based fatigue analysis of structural members subjected to synchronous and in-phase loads. The uniaxial state of stress produced by combined bending and tension-compression is considered. Attention is focused on the failure subregion for high cycle fatigue where the linear logarithmic models for fatigue behaviour are applicable. The relationships for calculation of fatigue reliability and number of cycles to failure are proposed. For the assessment of the probability of fatigue failure under the stress with normally distributed amplitudes of its components, the reliability index is determined.

NOTATION

c_{bt}	covariance of the amplitudes σ_b and σ_t,
D	total damage caused by N' stress cycles ($N' \leq N$),
f	safety factor,
$f_{bt}(\sigma_b, \sigma_t)$	joint probability density function of the amplitudes σ_b and σ_t,
K_b, K_t	material constants in Eqs. (2.3) and (2.4),
L_b, L_t	maximum stress amplitudes corresponding to the highest points of straight regression lines in the plots $\log \sigma_b$ vs. $\log N_b$ and $\log \sigma_t$ vs. $\log N_t$, respectively,
m_b, m_t	exponents in Eqs. (2.3) and (2.4),
M	safety margin,
N	number of stress cycles to cause failure under combined bending and tension-compression,
N_b, N_t	numbers of stress cycles to cause failure under alternate bending with a stress amplitude σ_b; and under symmetric tension-compression with a stress amplitude σ_t, respectively,
N_0	required number of stress cycles to achieve a given design life,
P_1	probability that stress cycles do not exceed the safe region,
P_2	probability that stress cycles do not exceed the failure subregion,
P_3	probability that stress cycles are within the failure subregion,
P_F	probability of fatigue failure,

R	fatigue reliability,
s_b, s_t	standard deviations of the amplitudes σ_b and σ_t,
s_μ	standard deviation of the dimensionless safety margin,
S_b, S_t	fatigue limits at alternate bending and at symmetric tension-compression,
β	reliability index,
μ	dimensionless safety margin,
$\bar{\mu}$	mean value of the dimensionless safety margin,
σ_b	amplitude of the stress component resulting from alternate bending,
σ_t	amplitude of the stress component resulting from symmetric tension-compression,
$\bar{\sigma}_b, \bar{\sigma}_t$	mean values of the amplitudes σ_b and σ_t.

1. Introduction

In fatigue design the basic variable space can be divided into the safe region and failure region [1]. When the stress amplitude under sinusoidal loading is σ_i, the safety factor can be calculated as

$$f = \frac{S_0}{\sigma_i} \tag{1.1}$$

in the safe region, and

$$f = \frac{N_i}{N_0} \tag{1.2}$$

in the failure region. In Eq. (1.1) S_0 is the fatigue limit and in Eq. (1.2) N_i is the number of cycles of stress with the amplitude σ_i to cause failure, and N_0 is the required number of stress cycles to achieve a given design life.

The fatigue reliability of a structural member is defined by [1]

$$R = 1 - P_F = P\,(M \geq 0), \tag{1.3}$$

where P_F is the probability of fatigue failure and P is the probability that the safety margin M is greater than or equal to zero. The safety margin is

$$M = S_0 - \sigma_i \tag{1.4}$$

in the safe region, and

$$M = N_i - N_0 \tag{1.5}$$

in the failure region. Instead of M, the dimensionless safety margin

$$\mu = \frac{M}{S_0} \tag{1.6}$$

or, respectively,

$$\mu = \frac{M}{N_i} \tag{1.7}$$

can be taken into consideration. From Eqs. (1.1), (1.2) and (1.4)–(1.7) it follows that

$$\mu = 1 - \frac{1}{f}. \tag{1.8}$$

When the safety margin (or the dimensionless safety margin) is linear in basic variables and the latter are normally distributed random variables with known mean values and standard deviations, the probability P_F can be determined in the standard tables of the normal distribution [1,2]. The same concept is used in this paper for an uniaxial state of a multi-component stress. As an example, the stress with two synchronous components is taken. The case of non-normally distributed variables is also considered.

Fatigue behaviour under constant amplitude stress is usually described by the linear logarithmic $S - N$ curve [2]. In the present paper an attempt is made to use such curves for the stress with more than one component. For this purpose, an adequate failure subregion for two synchronous stress components is used.

If more than one time-varying load act in combination on a structure, the knowledge of the distribution of only the maximum values of the individual loading processes gives insufficient information to evaluate the combined effect exactly [1]. However, in practical calculations real loadings are most frequently simplified in such a way that the mathematical problems connected with estimating the joint distribution function are avoided. For example, using TURKSTRA'S rule [3] the reliability of a structure is checked only at those points in time where the individual load processes reach their maximum values. In the FERRY BORGES-CASTANHETA model [4], instead of the combination of r real processes, 2^{r-1} combinations of rectangular load processes are considered.

Although the approximate methods seem to reflect the most important characteristics of load combinations, the calculation procedures based on the joint probability density function of load amplitudes and on an appropriate multi-load fatigue failure criterion might be useful. Such criteria are described e.g. in [5-7]. In the following the stress criterion [7] based on the linear logarithmic S-N curves is used. It is assumed that uniaxial state of stress at a given point of a structural member is to be estimated as a combination of the sinusoidal stress components resulting from in-phase alternate bending and symmetric tension-compression, and that the amplitudes of these components are random variables with known statistics.

2. Combined bending and tension-compression

Let us consider the case when an alternate bending moment and a symmetric push-pull force are synchronous and in-phase processes which produce, at a given point of a structure, the resultant normal stress of an amplitude

$$\sigma_r = \sigma_b + \sigma_t, \tag{2.1}$$

where σ_b and σ_t are the amplitudes of the stress components. The joint probability density function, mean values, standard deviations and covariance of the amplitudes σ_b and σ_t will be denoted $f_{bt}(\sigma_b, \sigma_t)$, $\bar{\sigma}_b$, $\bar{\sigma}_t$, s_b, s_t and c_{bt}, respectively.

As mentioned in Sec. 1, the most commonly used model for fatigue behaviour of constructional steels under constant amplitude loading is the linear logarithmic one

$$N_i \sigma_i^m = K, \tag{2.2}$$

where m and K are selected constants. Equation (2.2) is applicable for $\sigma_i \in (S_0, L]$, where L is the maximum stress amplitude corresponding to the highest point of the straight regression line in the plot $\log \sigma_i$ versus $\log N_i$. In particular, we have

$$N_b \sigma_b^{m_b} = K_b \tag{2.3}$$

for bending, and

$$N_t \sigma_t^{m_t} = K_t \tag{2.4}$$

for tension-compression. For these loads the fatigue limits and maximum stress amplitudes will be denoted S_b, S_t, L_b and L_t (Fig. 1a, b).

In order to determine the combined effect of these loads, the following conditions must be taken into account [7,8]

$$\frac{\sigma_b}{S_b} + \frac{\sigma_t}{S_t} \le 1, \tag{2.5}$$

$$\frac{\sigma_b}{S_b} + \frac{\sigma_t}{S_t} > 1, \tag{2.6}$$

$$\frac{\sigma_b}{L_b} + \frac{\sigma_t}{L_t} \le 1. \tag{2.7}$$

The condition (2.5) determines the safe region, whereas the inequality (2.6) – the region where fatigue failure occurs. The conditions (2.6) and (2.7) determine the failure subregion where the stress-based fatigue calculations may be performed (Fig. 1c).

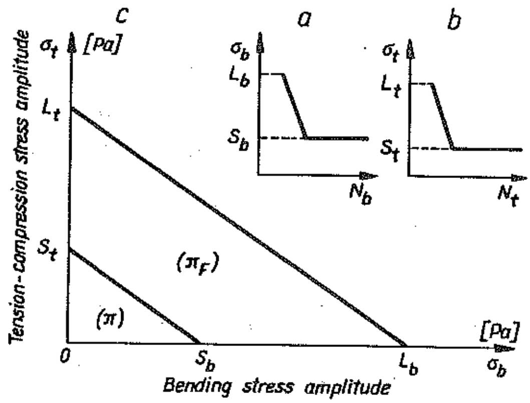

FIG. 1. Log stress amplitude vs. log cycles to failure for alternate bending (a) and for symmetric tension-compression (b). (c) – safe region (π) and failure subregion (π_F) for combined bending and tension-compression (in linear scale); (π) = triangle $0S_bS_t$, (π_F) = triangle $0L_bL_t$ – (π).

For constant amplitudes σ_b and σ_t the safety factor is

$$f = \left(\frac{\sigma_b}{S_b} + \frac{\sigma_t}{S_t}\right)^{-1} \tag{2.8}$$

in the safe region, and

$$f = \frac{N}{N_0} \tag{2.9}$$

in the failure region. N is the number of stress cycles to cause failure, which for the failure subregion can be determined by the following equation

$$\frac{1}{N} = \frac{\sigma_b^{m_b}}{K_b} + \frac{\sigma_t^{m_t}}{K_t}\,. \tag{2.10}$$

It should be noted that, according to [7], Eq. (2.10) can be used even if $\sigma_b < S_b$ and/or $\sigma_t < S_t$ provided both the conditions (2.6) and (2.7) are fulfilled.

When the amplitudes σ_b and σ_t are random Gaussian variables, the reliability index [1,2]

$$\beta = \frac{\bar{\mu}}{s_\mu} \tag{2.11}$$

can be used as a measure of safety. Here $\bar{\mu}$ and s_μ are the mean value and standard deviation of the dimensionless safety margin. The reliability index relevant to the uncertainty of fufilling the condition (2.5) can be determined by means of Eqs. (1.8), (2.8) and (2.11) as

$$\beta = \left(1 - \frac{\bar{\sigma}_b}{S_b} - \frac{\bar{\sigma}_t}{S_t}\right)\left(\frac{s_b^2}{S_b^2} + \frac{s_t^2}{S_t^2} + 2\frac{c_{bt}}{S_b S_t}\right)^{-1/2} \tag{2.12}$$

When the conditions (2.6) and (2.7) are fulfilled, the quantity μ is, according to Eqs. (1.8), (2.9) and (2.10), nonlinear in the amplitudes σ_b and σ_t, and its approximate mean value and standard deviation can be obtained by linearization. After expanding μ in Taylor series about $\bar{\sigma}_b$, $\bar{\sigma}_t$ and retaining only the linear terms, one gets

$$\bar{\mu} \cong \mu(\bar{\sigma}_b, \bar{\sigma}_t) = 1 - N_0\left(\bar{\sigma}_b^{m_b} K_b^{-1} + \bar{\sigma}_t^{m_t} K_t^{-1}\right), \tag{2.13}$$

$$\begin{aligned} s_\mu &\cong \left[s_b^2\left(\frac{\partial\mu}{\partial\sigma_b}\right)_0^2 + s_t^2\left(\frac{\partial\mu}{\partial\sigma_t}\right)_0^2 + 2c_{bt}\left(\frac{\partial\mu}{\partial\sigma_b}\right)_0\left(\frac{\partial\mu}{\partial\sigma_t}\right)_0\right]^{1/2} \\ &= N_0\left[\left(s_b m_b K_b^{-1}\bar{\sigma}_b^{m_b-1}\right)^2 + \left(s_t m_t K_t^{-1}\bar{\sigma}_t^{m_t-1}\right)^2 \right. \\ &\qquad \left. + 2c_{bt} m_b m_t K_b^{-1} K_t^{-1} \bar{\sigma}_b^{m_b-1}\bar{\sigma}_t^{m_t-1}\right]^{1/2}. \end{aligned} \tag{2.14}$$

The subscript 0 at the derivatives denotes that they are estimated at the mean.

The probability of fatigue failure for the known reliability index can be evaluated in the standard tables of the normal distribution or in the diagrams as in [2]. The index (2.12) determines the probability of fatigue failure without reference to the number of load cycles, whereas Eqs. (2.11), (2.13) and (2.14) make it possible to estimate the probability of fatigue failure during an assumed design life.

When the joint probability density function $f_{bt}(\sigma_b, \sigma_t)$ is known, the probabilities of fulfilling the conditions (2.5)–(2.7) can be calculated as (see Fig. 1c)

$$P_1 = \int_0^{S_t} d\sigma_t \int_0^{F_1} f_{bt}(\sigma_b, \sigma_t)\, d\sigma_b, \tag{2.15}$$

$$P_2 = \int_0^{L_t} d\sigma_t \int_0^{F_2} f_{bt}(\sigma_b, \sigma_t)\, d\sigma_b, \tag{2.16}$$

$$P_3 = P_2 - P_1. \tag{2.17}$$

P_1 is the probability of fulfilling the condition (2.5), P_2 is the probability of fulfilling the condition (2.7), P_3 is the probability of fulfilling both the conditions (2.6) and (2.7), and

$$F_1 = S_b\left(1-\frac{\sigma_t}{S_t}\right), \tag{2.18}$$

$$F_2 = L_b\left(1-\frac{\sigma_t}{L_t}\right). \tag{2.19}$$

Thus, the fatigue reliability is

$$R = P_1. \tag{2.20}$$

Eq. (2.20) is valid also for non-normally distributed amplitudes σ_b and σ_t.

To retain the relation between the safety margin and failure probability via reliability index for non-normally distributed variables, it is necessary to approximate the non-normally distributed variable with a normally distributed variable.

3. Number of cycles to failure

In practice, some of the conditions mentioned above do not hold true and Eqs. (2.11)–(2.14) cannot be used. In particular, in most cases the amplitudes of the stress components will be non-normally distributed and the probability P_3 will be less than unity. Therefore, this Section is concerned with the fatigue failure caused by synchronous stress components with amplitudes σ_b and σ_t distributed (normally or non-normally) over the whole region (2.7), i.e. when the probability P_2 equals unity. Then, from all the stress cycles only those fulfilling both the conditions (2.6) and (2.7) cause the damage. Thus, when the number of stress cycles over a time interval $[0, t']$ is N', the number of cycles causing the damage is

$$N_D = N'P_3. \tag{3.1}$$

According to Eq. (2.10), the partial damage due to one cycle of N_D cycles is

$$D_1 = \frac{\sigma_b^{m_b}}{K_b} + \frac{\sigma_t^{m_t}}{K_t}\,. \tag{3.2}$$

We assume that Miner's rule is applicable so that the total damage D over the time interval $[0, t']$ is

$$D = N' \left[\int_0^{L_t} d\sigma_t \int_0^{F_2} f_{bt}(\sigma_b, \sigma_t) \left(K_b^{-1}\sigma_b^{m_b} + K_t^{-1}\sigma_t^{m_t} \right) d\sigma_b - \int_0^{S_t} d\sigma_t \int_0^{F_1} f_{bt}(\sigma_b, \sigma_t) \left(K_b^{-1}\sigma_b^{m_b} + K_t^{-1}\sigma_t^{m_t} \right) d\sigma_b \right]. \tag{3.3}$$

The failure occurs when the total damage exceeds unity, hence the number of cycles to failure is

$$N = \left[\int_0^{L_t} d\sigma_t \int_0^{F_2} f_{bt}(\sigma_b, \sigma_t) \left(K_b^{-1}\sigma_b^{m_b} + K_t^{-1}\sigma_t^{m_t} \right) d\sigma_b - \int_0^{S_t} d\sigma_t \int_0^{F_1} f_{bt}(\sigma_b, \sigma_t) \left(K_b^{-1}\sigma_b^{m_b} + K_t^{-1}\sigma_t^{m_t} \right) d\sigma_b \right]^{-1}. \tag{3.4}$$

4. Example

A constructional member is subjected to in-phase bending and tension-compression with normally distributed amplitudes. It is fabricated from the steel with the following data:

$$K_b = 1.2 \cdot 10^6 S_b^{m_b}, \quad K_t = 1.1 \cdot 10^6 S_t^{m_t}, \quad S_b = 240\,\text{MPa}, \quad S_t = 180\,\text{MPa},$$

$$m_b = 11, \quad m_t = 10, \quad L_b = 370\,\text{MPa}, \quad L_t = 300\,\text{MPa}.$$

The fatigue reliability is to be verified at the points A and B where the stress statistics are (in MPa):

$$\begin{array}{llll} \text{A} & \bar{\sigma}_b = 100, & \bar{\sigma}_t = 80, & s_b = 12, \quad s_t = 10, \quad c_{bt} = 0; \\ \text{B} & \bar{\sigma}_b = 150, & \bar{\sigma}_t = 150, & s_b = s_t = 15, \quad c_{bt} = 0. \end{array}$$

From Eqs. (1.8) and (2.8) we get $\bar{\mu} = 0.14$ at the point A and $\bar{\mu} = -0.34$ at the point B. The positive value of $\bar{\mu}$ points out the possibility of using Eq. (2.12), whereas the negative one indicates that the condition (2.6) is fulfilled in the mean.

Since the condition (2.7) is at point B also fulfilled in the mean, Eqs. (2.11), (2.13) and (2.14) can be used.

Having calculated from Eq. (2.12) the value $\beta = 1.87$ at point A, one gets in the diagram [2] failure probability P_F versus quantile $u_p = -\beta$ the value $P_F = 0.04$. Hence the fatigue reliability at the point A is $R = 1-0.04 = 0.96$.

The results of determination of fatigue reliability at point B are depicted in Fig. 2.

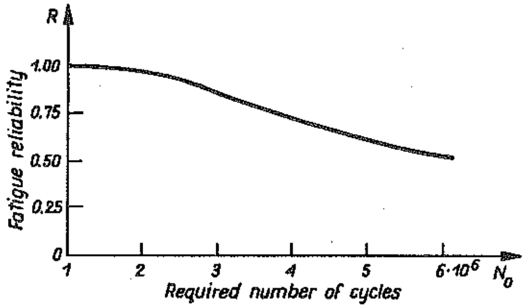

FIG. 2. Influence of duration of the required design life on the fatigue reliability of a constructional member subjected to in-phase bending and tension-compression.

$\bar{\sigma}_b = \bar{\sigma}_t = 150$ MPa, $s_b = s_t = 15$ MPa, $c_{bt} = 0$,

$S_b = 240$ MPa, $S_t = 180$ MPa, $m_b = 11$, $m_t = 10$.

5. Concluding remarks

One of the major difficulties in using reliability methods in fatigue analysis based on $S-N$ curves is that the $S-N$ data are typically from constant amplitude testing. However, the more adequate data in this respect are very unlikely to be available. Therefore, there is a great need for more random loading testing or means of extracting information about random loading from the constant amplitude data [9].

Another shortcoming of the presented calculation procedure (beyond the restrictive assumptions) follows from the fact that some important factors influencing fatigue life have not been taken into account. These are mainly stress concentrations due to discontinuities or poor workmanship, welding (residual stress under cut and stress concentration around the joint) and corrosive environment as well as stress distribution, stress ratio and mean stress. Moreover, most fatigue tests are conducted on small and simple specimens under simple loading mechanisms. Real constructional elements are in general larger and subjected to more complicated stress field. For these reasons, Eqs. (2.3) and (2.4) should be adjusted, e.g. by certain scale, stress concentration and ratio and/or environmental conditions factors [2].

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Received May 18, 1992.