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# DECREASE OF THE RESIDUAL STRENGTH DURING CREEP

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In the article safety factors of the material under instantaneous and rheological static loadings have been described utilizing the idea of a damage parameter. Two safety factors have been introduced: the instantaneous factor related to the load level (named stress safety factor), and the time-dependent one, connected with the lifetime (named time safety factor). Damage development process under rheological conditions causes decrease of the rupture strength and implies an interrelation of these two safety factors. This relation has been analyzed for uniaxial as well as biaxial stress conditions, by constructing curves of constant safety in the meaning of the factors mentioned.

## Notation

$\sigma_e$ service stress,
$\sigma^*$ residual strength,
$\sigma_1, \sigma$ principal stress,
$\sigma_e$ equivalent stress,
$\sigma_i, \sigma_{ie}$ service effective stress,
$s_e$ dimensionless applied stress,
$s^*$ dimensionless residual strength,
$s_1, s_2$ dimensionless principal stresses,
$s_{ie}$ dimensionless effective applied stress,
$R$ ultimate strength,
$t_e$ operation time,
$t^*$ time to failure at constant level of applied stresses,
$\tau_e$ dimensionless operation time,
$\tau^*$ dimensionless time-to-rupture,
$\omega$ continuity parameter,
$W_t$ time safety factor,
$W_\sigma$ stress safety factor.

## 1. Introduction

Determination of the dangerous state and the related notion of safety factor for a selected member of the given structure is, in general, an extremely complex problem depending on a number of service conditions. Knowledge of the safety margin at various working conditions is of fundamental importance for predicting the lifetime of the structure. In particular, it concerns the specific service conditions such as those existing during operation at elevated temperature. Those conditions lead to damages in the internal structure of the material and shorten the period of safe operation.

Generally, the safety of the structure is determined by evaluating the quantities representing the "distance" from the state regarded as unsafe (dangerous).

It should be noted, that the applied stress and the service conditions can be considered on the grounds of the phenomena occurring at the level of a material point, of a cross-section or of the whole structure under consideration. In this paper the phenomena occurring at the material point level are examined.

Appearance of decohesion in a portion of the considered structure has been assumed as the dangerous state. It is characterized by the boundary value of function $\omega(\sigma, t)$ which is a measure of degradation of internal structure of the material. This function, called the "continuity parameter", has been introduced by KACHANOV [4] as the ratio of the cross-sectional area of damages of an elementary volume of the material to the total cross-sectional area subject to loading.

For the damage determined in such a way, the particular values of $\omega$ have the following meaning:

$\omega = 0$ corresponds to the undamaged material;

$\omega = 1$ corresponds to the totally damaged material;

$0 < \omega < 1$ characterizes the intermediate damage state.

Therefore, from the point of view of physics, the parameter $\omega$ is a relative measure of the density of microdamages in the elementary cross-sectional area of the element; from the point of view of mathematics, parameter $\omega$ is the surface density of discontinuity of the material at a given point, when the volume of the element is tending to zero.

Description of local effects in the body by a single scalar function $\omega(\sigma, t)$ "washes away" the local effect in the whole volume of the structure considered. It amounts to the assumption that damages are isotropic, and this is the assumption used in Continuum Damage Mechanics.

Basing on the assumptions of the Continuum Damage Mechanics, an attempt has been undertaken to answer the question how the change of strength of the material is influenced by its history of loading. This strength changes due to the development of the damages and, at any moment of the operation period, it is characterized by the value of stress $\sigma^*$. Stress $\sigma^*$ is, in fact, the residual strength which at the beginning of the service is equal to the ultimate strength $R$ reduced by the damage growth. As a consequence, shortening of the lifetime of the material is observed.

## 2. Safety factors

Let us introduce the factors characterizing the distance from the dangerous state, i.e. the state when $\omega = 1$ at a point of the body. These factors will be called the safety factors. In the case of statical loadings of short duration it is sufficient to use one of those factors as the ratio of the applied stress to the failure stress (for example: ultimate strength $R$) which will be called the stress safety factor,

$$W_\sigma = \frac{\sigma_e}{R}, \tag{2.1}$$

where $\sigma_e$ is the applied stress, and $R$ is the ultimate strength.

However, for prolonged loadings it is necessary to introduce a time-dependent factor as the ratio of the operation time at a fixed applied stress $\sigma_e$ to the time when the dangerous state has been reached. This factor will be called the time safety factor,

$$W_t = \frac{t_e}{t^*}, \tag{2.2}$$

where $t_e$ is operation time, and $t^*$ is time to failure at constant $\sigma_e$.

During the service a development of damages of the material occurs, which can be described by the function $\omega(\sigma, t)$. These damages cause a change in the residual strength and, therefore, it is necessary to modify the stress safety factor to the form

$$W_\sigma = \frac{\sigma}{\sigma^*}, \tag{2.3}$$

where $\sigma^*$ – true value of the residual strength at time $t_e$.

The classical stress safety factor in the form (2.1) is used to describe such processes in which the changes of material features and load in time can be neglected, and thus it corresponds to the factor given by (2.3) when $t_e = 0$ and $\sigma^* = R$.

The stress history presented in Fig. 1a is accompanied by a corresponding history of decrease of the residual strength and, consequently, by the dependence of $W_\sigma - W_t$ on time, schematically shown in Fig. 1b.

Estimation of the true value of residual strength $\sigma^*$ is connected with estimation of the range of damages caused by additional loadings at any

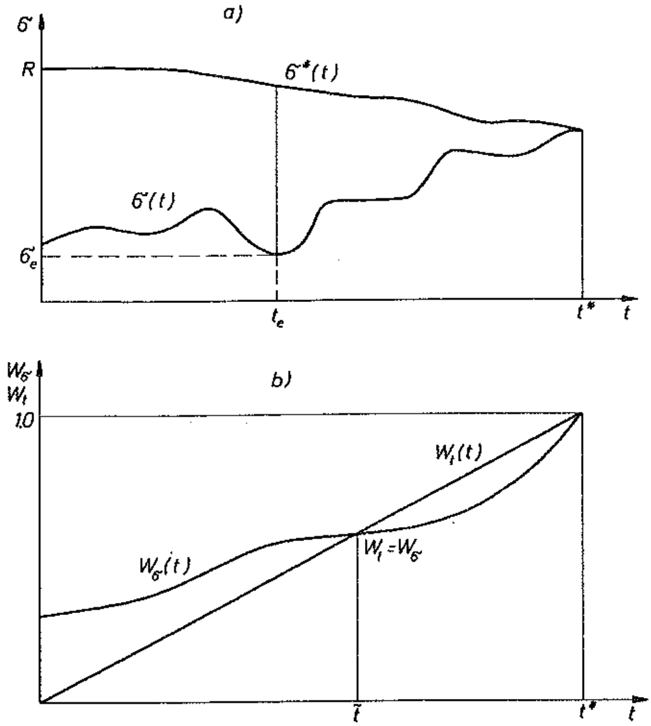



FIG. 1. Stress history accompanied by the corresponding residual strength decrease history (a) and dependence of $W_\sigma - W_t$ on time (b).

operation time $t_e$. This is possible if proper damage growth law has been formulated. This law would describe both the rheological damage development and the immediate damages independent of time.

To simplify our further considerations, let us assume a load which produces a state of steady stress and the creep of constant rate.

Let us also assume that a chosen program of the loading produces stress $\sigma_{e1}$ in the material.

If this stress will be kept at the same level during the whole operation period, then rupture characterized by $\omega = 1$ (Program 1a in Fig. 2) occurs after the time $t_1^*$ (see point $C$ in Fig. 2). If the applied stress $\sigma_{e1}$ is equal to the ultimate strength $R$, then the time to rupture is zero. On the other hand, at any moment $t_{e1}$ the applied stress can be increased to such a level $\sigma^*(t_{e1}, \sigma_{e1})$ at which the rupture occurs (Program 1b) (see point $A$ in Fig. 2). The value of $\sigma^*(t_{e1}, \sigma_{e1})$ is lower than $R$; its decrease is caused by development of damages during operation of the material. Points $A$ and $C$ belong to two usually different curves. Point $C$ belongs to the curve of time-dependent strength $t^*(\sigma_e)$, point $A$ – to the curve of residual strength

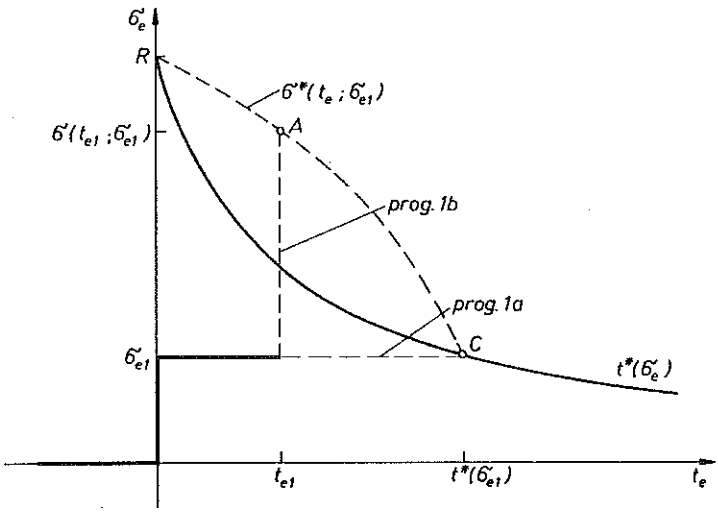



FIG. 2. Program of the service stress $\sigma_{e1}$ in the material.

$\sigma^*(t_{e1}, \sigma_{e1})$. The curve of residual strength does not depend on $t_e$ only, but also on the whole loading history represented by the stress $\sigma_{e1}$.

Similar case is shown in Fig. 3 but here the true service stress $\sigma_{e2}$ is higher than the stress $\sigma_{e1}$. Consequently, point $C$ has been moved to the position $t^*(\sigma_{e2})$ on the time-dependent curve, and point $A$ lies on another curve of the residual strength $\sigma^*(t_e, \sigma_{e2})$.

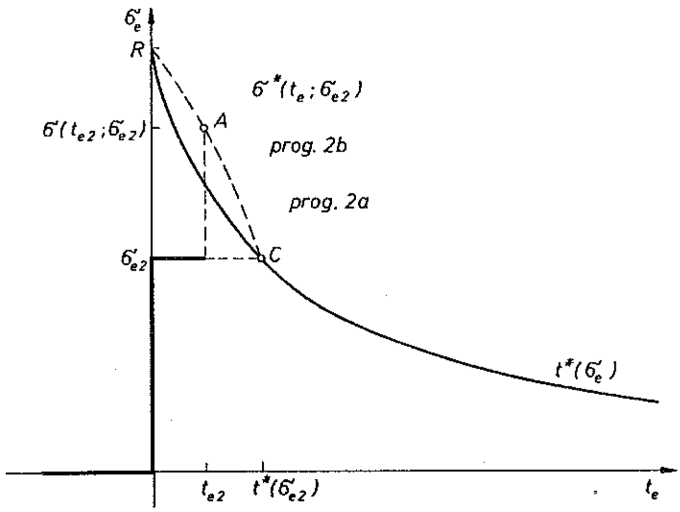



FIG. 3. Program of the service stress $\sigma_{e2}$ in the material.

For both applied stresses $\sigma_{e1}$ and $\sigma_{e2}$ point $C$ lies at the point of intersection of the two curves: the time-dependent strength and the residual strength. It means that the rupture under creep conditions at a constant stress can be understood as a decrease of residual strength to the level of this strength [5], [6].

Basing on definitions (2.2) and (2.3), Fig. 4 shows the changes of the time safety factor $W_t$ as a function of the applied time $t_e$ for different values of service stresses $\sigma_{e1}$ and $\sigma_{e2}$. In Fig. 5 the changes of the stress safety factor versus the stresses applied, for three selected operation times $t_{e1}$, $t_{e2}$ and $t_e = 0$, are presented.

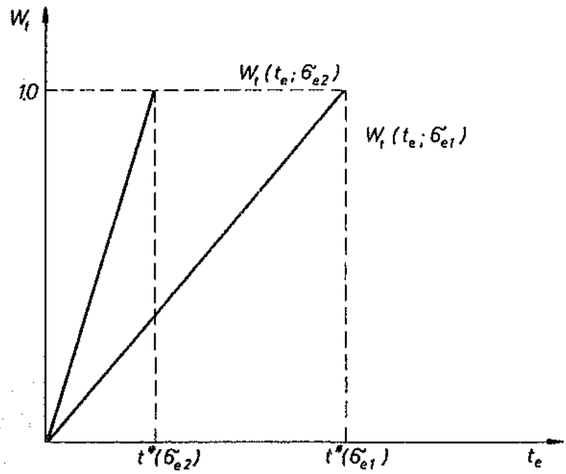



FIG. 4. Changes of time safety factor $W_t$ versus applied time $t_e$ for different values of service stresses.

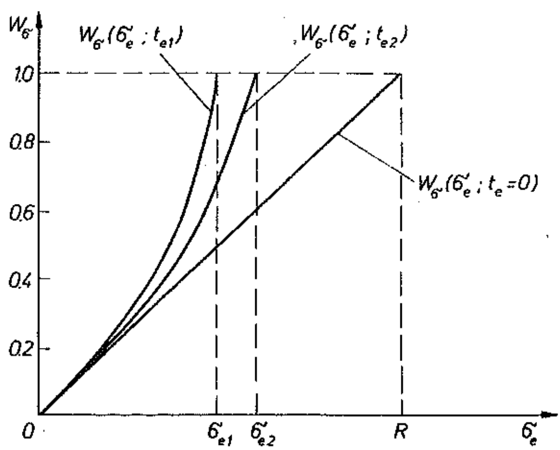



FIG. 5. Changes of stress safety factor versus applied stresses at three selected operation times.

The diagrams of $W_\sigma(\sigma_e)$ and $W_t(t_e)$ shown in Figs. 4 and 5 are superimposed sections of surfaces $W_\sigma(\sigma_e, t_e)$ and $W_t(\sigma_e, t_e)$ on planes $W_\sigma - \sigma_e$

and $W_t - t_e$, respectively. Further, we deal with determination of the curve constituting the intersection of these two safety surfaces, according to the condition

$$W_\sigma = W_t.$$

## 3. Safety at uniaxial stress

### *3.1. Material service conditions according to Kachanov's theory*

Let us assume the law of damage evolution in the form

$$\frac{d\omega}{dt} = A\left(\frac{\sigma_{\mathrm{eq}}}{1-\omega}\right)^m, \tag{3.1}$$

where $A$ and $m$ are material constants, and $\sigma_{\mathrm{eq}}$ is the equivalent stress proposed in [3]:

$$\sigma_{\mathrm{eq}} = \alpha\sigma_1 + (1-\alpha)\sigma_i. \tag{3.2}$$

In Eq. (3.2) $\sigma_1$ means the positive principal stress, and $\sigma_i$ – the effective stress used by von Mises.

Parameter $\alpha$ lies within the range of $<0,1>$, and its value depends on type of material and on its behavior during the damage process at multiaxial stress state.

At uniaxial stress $\sigma_{\mathrm{eq}} = \sigma_1$ which will be denoted below by $\sigma$.

From Eq. (3.1) we can calculate the time after which rupture under the applied stress $\sigma_e$ occurs. This time, calculated from the condition $t^*(\sigma = \sigma_e;\ \omega = 1)$, is

$$t^*(\sigma_e) = \frac{1}{A(m+1)\sigma_e^m}. \tag{3.3}$$

Let us introduce the reference time $t_R^*$

$$t_R^* = \frac{1}{A(m+1)R^m},$$

where $R$ is the ultimate strength for $t_e = 0$.

Now, relation (3.3) can be written as

$$t^*(\sigma_e) = \frac{t_R^* R^m}{\sigma_e^m}.$$

Introducing the dimensionless stress in the form

$$s = \frac{\sigma}{R}, \tag{3.4}$$

we obtain

$$t^*(s_e) = \frac{t_R^*}{s_e^m}. \tag{3.5}$$

Let us now introduce the dimensionless time related to the time $t_R^*$,

$$\tau = \frac{t}{t_R^*}. \tag{3.6}$$

Thus, the dimensionless time to rupture is

$$\tau^* = \frac{t^*}{t_R^*}.$$

Using Eq. (3.5) we obtain

$$\tau^* = \frac{1}{s_e^m}. \tag{3.7}$$

Assuming the symbols according to (3.4) and (3.6), the safety factors from Eqs. (2.2) and (2.3) are

$$W_t = \frac{\tau_e}{\tau^*}, \tag{3.8}$$

$$W_\sigma = \frac{s_e}{s^*}. \tag{3.9}$$

Using (3.7), the time-safety factor $W_t$ can be written as

$$W_t = \tau_e s_e^m. \tag{3.10}$$

Now, let us increase the load to get the stress $s^*$ which produces rupture at a fixed applied time $\tau_e$. Since according to the idea suggested by Kachanov, increase of the load does not imply damages, thus the curve of residual strength overlaps that of the time-dependent strength. Thus, $s^*$ can be evaluated from the condition

$$\tau_e(s^*) = \tau^*(s^*). \tag{3.11}$$

Introducing (3.7) to the right-hand side of (3.11), we obtain

$$\tau_e = \frac{1}{s^{*m}}$$

or

$$s^* = \tau_e^{-1/m} .$$

According to (3.9), the stress safety factor $W_\sigma$ is

$$W_\sigma = s_e / \tau_e^{-1/m}$$

or

$$W_\sigma^m = \tau_e s^m . \tag{3.12}$$

Comparison of (3.12) and (3.10) yields

$$W_\sigma^m = W_t .$$

Since both safety factors are not greater than unity, for $m > 1$ we have $W_t \leq W_\sigma$. It follows from the above that, if we apply the theory of accumulation of damages proposed by Kachanov, i.e. if only rheological damages are taken into account, then the material safety is always controlled by the applied strength but not by the operation time. On the other hand, the operation time determines the safety under creep conditions. Therefore, it is necessary to modify Eq. (3.1) in order to take into account the real decrease of the residual strength.

### *3.2. Modified theory of damages*

Let us assume the Law of Cumulation of Damages in the form proposed by CHRZANOWSKI and MADEJ in [2]:

$$\frac{d\omega}{dt} = A_0 \left( \frac{\sigma_{\text{eq}1}}{1-\omega} \right)^{m_0} \frac{d\sigma_{\text{eq}1}}{dt} + A \left( \frac{\sigma_{\text{eq}2}}{1-\omega} \right)^{m} , \tag{3.13}$$

where $A_0$, $A$, $m_0$, $m$ are material constants, $\sigma_{\text{eq}1}$ and $\sigma_{\text{eq}2}$ are equivalent stresses presented by Eq. (3.2). The stresses $\sigma_{\text{eq}1}$ and $\sigma_{\text{eq}2}$ differ from each other by the factor $\alpha$ which can be different in both terms of Eq. (3.13).

The Law of Cumulation of Damages assumed above accounts for the interaction of instantaneous processes linked with the increase of the load, as well as with the rheological damages developing under steady load during the operation.

At uniaxial stress when $\sigma_{\text{eq}1} = \sigma_{\text{eq}2} = \sigma$, Eq. (3.13) assumes the form:

$$\frac{d\omega}{dt} = A_0 \left( \frac{\sigma}{1-\omega} \right)^{m_0} \frac{d\sigma}{dt} + A \left( \frac{\sigma}{1-\omega} \right)^{m} . \tag{3.14}$$

Introducing dimensionless variables according to (3.4) and (3.6) and the notation

$$A_0 = \frac{1}{R^{m_0+1}},$$

equation (3.14) can be written as

$$\text{(3.15)} \qquad \frac{d\omega}{d\tau} = \left(\frac{s}{1-\omega}\right)^{m_0} \frac{ds}{d\tau} + \frac{1}{m+1}\left(\frac{s}{1-\omega}\right)^{m}.$$

In order to determine the curve of time-dependent strength $\tau^*(s_e)$ and the curve of residual strength $s^*(\tau_e, s_e)$. Equation (3.15) will be integrated for the following stages of the loading program shown in Figs. 2 and 3:

1) $ds \geq 0$ and $0 \leq s \leq s_e$ for $\tau = 0$,
2) $s = s_e$ for $0 < \tau \leq \tau_e$,
3) $ds \geq 0$ and $s_e \leq s \leq s^*$ for $\tau = \tau_e$.

For the first stage (instantaneous load), damages are described by the first term of Eq. (3.15) only. After integrating the equation

$$\text{(3.16)} \qquad \frac{d\omega}{ds} = \left(\frac{s}{1-\omega}\right)^{m_0}$$

with the initial condition $\omega(s=0)$, we obtain

$$\omega_1 \stackrel{\text{df}}{=} \omega(s_e, \tau = 0) = 1 - \left(1 - s_e^{m_0+1}\right)^{\frac{1}{m_0+1}}.$$

In the limiting case when $s_e = 1$ (i.e. $\sigma = R$) we obtain $\omega_1 = 1$.

At the second stage, because of $ds = 0$, Eq. (3.15) is reduced to the form:

$$\frac{d\omega}{d\tau} = \frac{1}{m+1}\left(\frac{s}{1-\omega}\right)^{m}.$$

After integration with the initial condition $\omega(\tau = 0) = \omega_1$, we obtain

$$\omega_2 \stackrel{\text{df}}{=} \omega(s_e, \tau = \tau_e) = 1 - \left[\left(1 - s_e^{m_0+1}\right)^{\frac{m+1}{m_0+1}} - \tau_e s_e^m\right]^{\frac{1}{m+1}}.$$

In the particular case when $\omega_2 = 1$, we have

$$\text{(3.17)} \qquad \tau^* = \frac{1}{s_e^m}\left(1 - s_e^{m_0+1}\right)^{\frac{m+1}{m_0+1}}.$$

This is the equation of the curve of time-dependent strength $\tau^*(s)$. Finally at the third stage, after integrating Eq. (3.16) and taking into account the damages cumulated during the first and second stage (i.e. with the initial condition for $\omega = \omega_2$), we obtain a formula for failure stresses $s^*$ in the form

$$s^* = \left\{ \left[ \left(1 - s_e^{m_0+1}\right)^{\frac{m+1}{m_0+1}} - \tau_e s_e^m \right]^{\frac{m_0+1}{m+1}} + s_e^{m_0+1} \right\}^{\frac{1}{m_0+1}} . \tag{3.18}$$

This equation describes the curve of residual strength $s^*(\tau_e, s_e)$ for the given initial load $s_e$. Now, we will use expressions (3.17) and (3.18) to calculate the values of safety factors (3.8) and (3.9) as functions of strength and the time applied.

$$W_t = s_e^m \tau_e \left(1 - s_e^{m_0+1}\right)^{-\frac{m+1}{m_0+1}} , \tag{3.19}$$

$$W_\sigma = s \left\{ \left[ \left(1 - s_e^{m_0+1}\right)^{\frac{m+1}{m_0+1}} - \tau_e s_e^m \right]^{\frac{m+1}{m_0+1}} + s_e^{m_0+1} \right\}^{-\frac{1}{m+1}} . \tag{3.20}$$

Knowledge of the safety factors enables us to estimate the material safety at any point of the body, for each value of loading and at any moment of the operation. It allows us also to determine both the possible increase of the load up to the residual strength and the possible life-time at a fixed stress. Therefore, the safety factors make it possible to estimate the life-time of the structure and the possibility of increasing its load.

For the particular case when $m = m_0$, expressions (3.17) and (3.18) have a simpler form,

$$\tau^* = \frac{1}{s_e^m} \left(1 - s_e^{m+1}\right) ,$$

$$s^* = \left(1 - \tau_e s_e^m\right)^{\frac{1}{m+1}} .$$

Basing on the above expressions, diagrams of functions $\tau^*(s_e)$ and $s^*(\tau_e)$ can be drawn. These diagrams are shown in Fig. 6.

In Fig. 7 diagrams of the function $s^*(s_e)$ for different operation times $\tau_e$ are shown.

In the case of $m_0 = m$, Eqs. (3.19) and (3.20) are reduced to the form

$$W_t = s_e^m \tau_e \left(1 - s_e^{m+1}\right)^{-1} , \tag{3.21}$$

$$W_\sigma = s_e \left(1 - \tau_e s_e^m\right)^{-\frac{1}{m+1}} . \tag{3.22}$$

Let us consider the case when the state of safety in time and stress version are equal, i.e. $W_t = W_\sigma$.

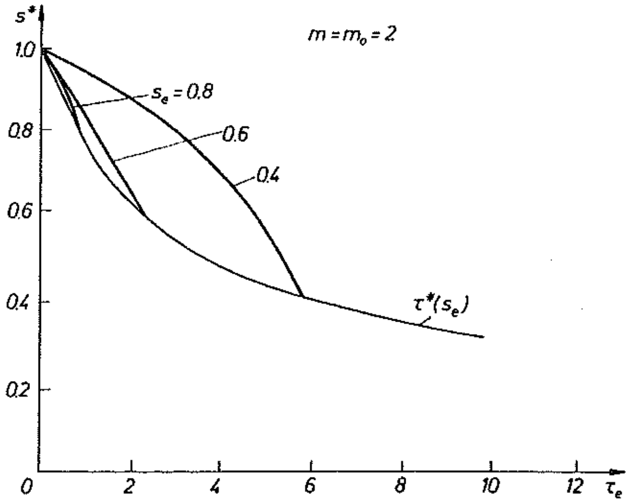



FIG. 6. Diagrams of functions $\tau^*(s_e)$ and $s^*(\tau_e)$.

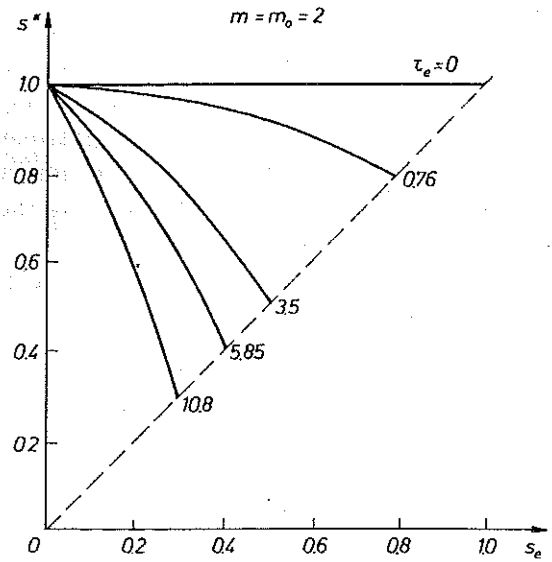



FIG. 7. Diagrams of the function $s^*(s_e)$ for different times used.

Comparing expressions (3.21) and (3.22) we obtain

$$s_e^{m-1}\tau_e\left(1-\tau_e s_e^m\right)^{-\frac{1}{m+1}} = 1 - s_e^{m+1}.$$

The curves $s_e(\tau_e)$ satisfying the condition of equality of safety factors $W_t = W_\sigma$ have been shown in Fig. 8 for different values of material constant $m$. The curves of time-dependent strength $\tau^*(s_e)$ are drawn in light lines.

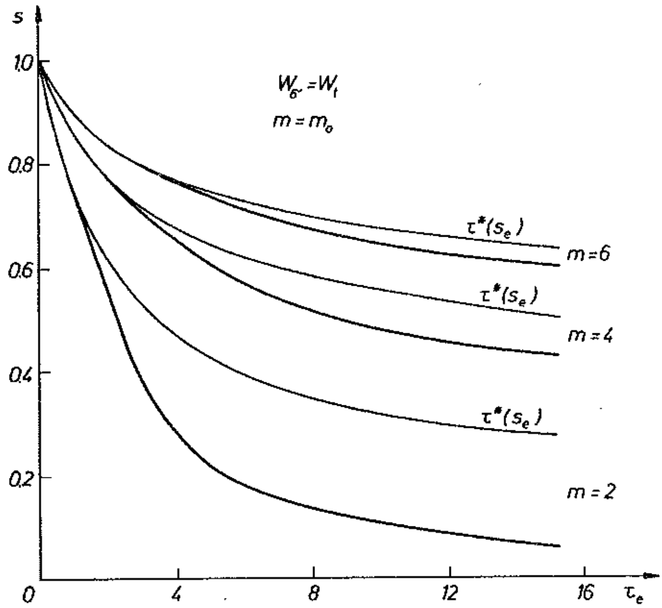



FIG. 8. The curves $s_e(\tau_e)$ satisfying the condition of equality of safety factors $W_t = W_\sigma$ for different values of material constant $m$.

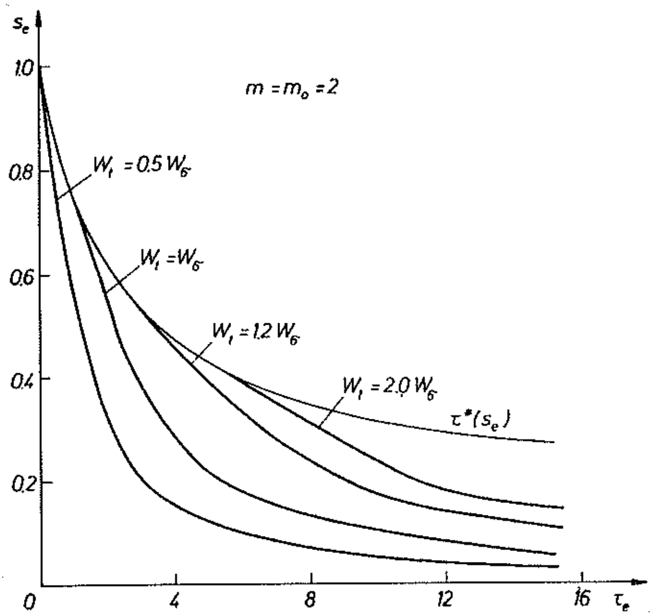



FIG. 9. Curves $s_e(\tau_e)$ for various ratios of stress and time safety factors.

Knowing the curves of constant safety we can estimate the relation between the stress-dependent and time-dependent safety factors for any applied stress, at any moment of operation. The points lying above the curve of constant safety, but below the curve of time-dependent strength (also satisfying condition $W_t = W_\sigma$), represent such cases for which the stress safety factor is greater than the time safety factor. In practice it means that, at a fixed moment of operation, shorter service periods correspond to the stress values lying above the curve of constant safety. Also, at fixed values of applied stress it is easier to increase the load for the applied periods placed to the left of the curve of constant safety than for the periods lying to the right of that curve.

In Fig. 9 some curves $s_e(\tau_e)$ have been shown for various ratios of stress and time safety factors.

As it follows from the above considerations, once the curve of constant safety is known, we can establish which of the two parameters: applied stress or time of its application, affects the material safety more, i.e. which of them determines the further operation.

## 4. Safety at creep under multiaxial stress

For multiaxial stress the Law of Cumulation of Damages (3.13) has been assumed in the form

$$\frac{d\omega}{dt} = A_0 \left(\frac{\sigma_i}{1-\omega}\right)^{m_0} \frac{d\sigma_i}{dt} + A\left(\frac{\sigma_1}{1-\omega}\right)^{m}. \tag{4.1}$$

It is equivalent to the assumption $\alpha = 0$ in Eq. (3.2) for $\sigma_{\text{eq1}}$ and $\alpha = 1$ for $\sigma_{\text{eq2}}$.

In the case of polycrystalline materials subject to uniaxial stress, change of load causes changes of both the time-to-rupture and its nature. For small loads with long time-to-rupture, the microcracks within the material run along the grain boundaries.

For high loads with shorter time-to-rupture, the damages within the material occur owing to the slip on the planes crossing the grains or blocks of crystals. This affects the behavior of the material according to various criteria at the three-dimensional state of stresses. Development process of the intercrystalline damages is mainly affected by the value of the highest principal stress. Therefore, the Clebsch – Rankine theory is the most suitable safety theory in this case. Process of intercrystalline damages is mainly determined by the kind of slip which is similar to that observed during plastic flow. Therefore, the von Mises theory is generally used as the stress condition and safety theory. Behavior of the material according to one of the theories mentioned corresponds to the limit values of parameter $\alpha$. Actual behavior of the material is the result of simultaneous appearance of both kinds of damage. For many materials important for their technical applications, change of types of damage depending on the value of the load applied can be observed. This change causes transformation of isochronous damage curves described in [1] using the Law of Damage Kinetics in the form (4.1).

Similarly to the case of uniaxial stress, the following dimensionless variables have been introduced:

$s_1 = \frac{\sigma_1}{R}$ – dimensionless maximum principal stress ($s_1 \geq s_2$),

$s_i = \frac{\sigma_i}{R}$ – dimensionless intensity of stresses.

For fixed values of $s_i$, expression $s_2(s_1)$ describes an ellipse.

Figure 10 shows the stresses $s_2$ and $s_1$ for fixed values of $s_i$ equal to 0.2 ... 1.0.

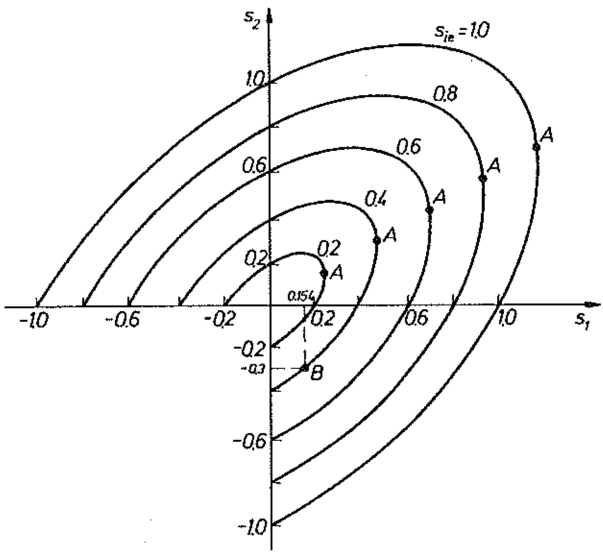



FIG. 10. Stresses $s_1$, $s_2$ for fixed $s_i$.

Furthermore, we will assume $\tau$ and $\tau^*$ as dimensionless equivalents of times $t$ and $t^*$, and $s_i^*$ – as a dimensionless equivalent of the stress $\sigma_i^*$.

With these notations Eq. (4.1) can be written as

$$\text{(4.2)} \qquad \frac{d\omega}{d\tau} = \left(\frac{s_i}{1-\omega}\right)^{m_0} \frac{ds_i}{d\tau} + \frac{1}{m+1}\left(\frac{s_1}{1-\omega}\right)^{m},$$

and the safety factors given by Eq. (2.1)–(2.2) will be

$$\text{(4.3)} \qquad W_t = \frac{\tau_e}{\tau^*} \qquad \text{at fixed } s_{ie},$$

$$(4.4) \qquad W_\sigma = \frac{s_{ie}}{s_i^*} \qquad \text{at fixed } \tau_e .$$

In order to find functions $\tau^*(s_{1e}, s_{2e})$ and $s_i^*(s_{1e}, s_{2e}, \tau_e)$ and, consequently, functions $W_\sigma(s_{1e}, s_{2e}, \tau_e)$ and $W_t(s_{1e}, s_{2e}, \tau_e)$, Eq. (4.2) is integrated for the consecutive stages of loading program:

1) $ds_i \geq 0$ and $0 \leq s_i \leq s_{ie}$ for $\tau = 0$,
2) $s_i = s_{ie}$ for $0 < \tau \leq \tau_e$,
3) $ds_i \geq 0$ and $s_{ie} \leq s_i \leq s_i^*$ for $\tau = \tau_e$.

During the first stage, the damages are described by the first term of Eq. (4.2) only. So, by integrating the equation:

$$\frac{d\omega}{ds_i} = \left(\frac{s_i}{1-\omega}\right)^{m_0}$$

under the initial condition $\omega(s_i = 0)$, we have

$$\omega_1 = \omega(s_{ie}, \tau = 0) = 1 - \left(1 - s_{ie}^{m_0+1}\right)^{\frac{1}{m_0+1}} .$$

For the limiting case when $s_i = 1$ we obtain $\omega_1 = 1$.

During the second stage, since $ds_i = 0$, we integrate Eq. (4.2) reduced to the form

$$\frac{d\omega}{d\tau} = \frac{1}{m+1}\left(\frac{s_1}{1-\omega}\right)^m ,$$

under the initial condition $\omega(\tau = 0) = \omega_1$, and obtain

$$\omega_2 = \omega(s_{1e}, \tau = \tau_e) = 1 - \left[\left(1 - s_{ie}^{m_0+1}\right)^{\frac{m+1}{m_0+1}} - \tau_e s_{1e}^m\right]^{\frac{1}{m+1}} .$$

In the case when $\omega_2 = 1$, we have

$$\tau^* = \frac{1}{s_{1e}^m}\left(1 - s_{ie}^{m_0+1}\right)^{\frac{m+1}{m_0+1}} .$$

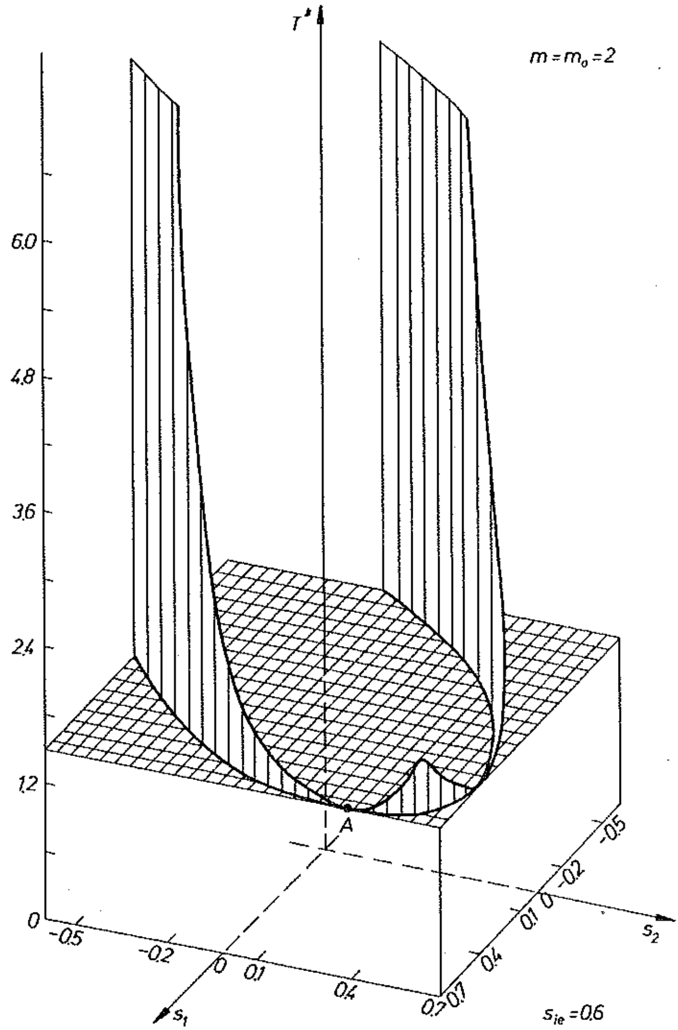
Thus, the time-to-rupture $\tau^*$ depends on both $s_i$ and the higher one of the principal stresses $s_1$. As a result, we obtain a surface of time-dependent strength instead of the curve. This surface is shown in Fig. 11 for fixed stresses $s_1$ and $s_2$ forming the same ellipse $s_{ie} = \text{const}$.

If $s_2$ increases from $s_i$ to a point $A$ at which $\dfrac{ds_2}{ds_1} = \infty$ (see Fig. 10), then $\tau^*$ decreases from $\infty$ at $s_2 = -s_{ie}$, reaching its minimum at point $A$. Further increase of $s_2$ causes time-to-rupture increase up to the value of $\tau^*(s_1 = s_2)$. From Fig. 11 it is seen that time-to-rupture is shorter for higher $s_{ie}$.

During the third stage, after integrating of Eq. (4.2) and taking into account the damages initiated during the stages 1 and 2 (i.e. for initial condition $\omega = \omega_2$), we obtain the expression for the failure stress $s_i^*$ in the form:

$$s_i^* = \left\{ \left[ \left(1 - s_{ie}^{m_0+1}\right)^{\frac{m+1}{m_0+1}} - \tau_e s_{1e}^{m} \right]^{\frac{m_0+1}{m+1}} + s_{ie}^{m_0+1} \right\}^{\frac{1}{m_0+1}} .$$

This equation describes the change of residual strength $s_i^*$ vs. time for a given fixed effective applied stress $s_{ie}$ at extreme value of principal stress $s_{1e}$.




[FIG. 11a]

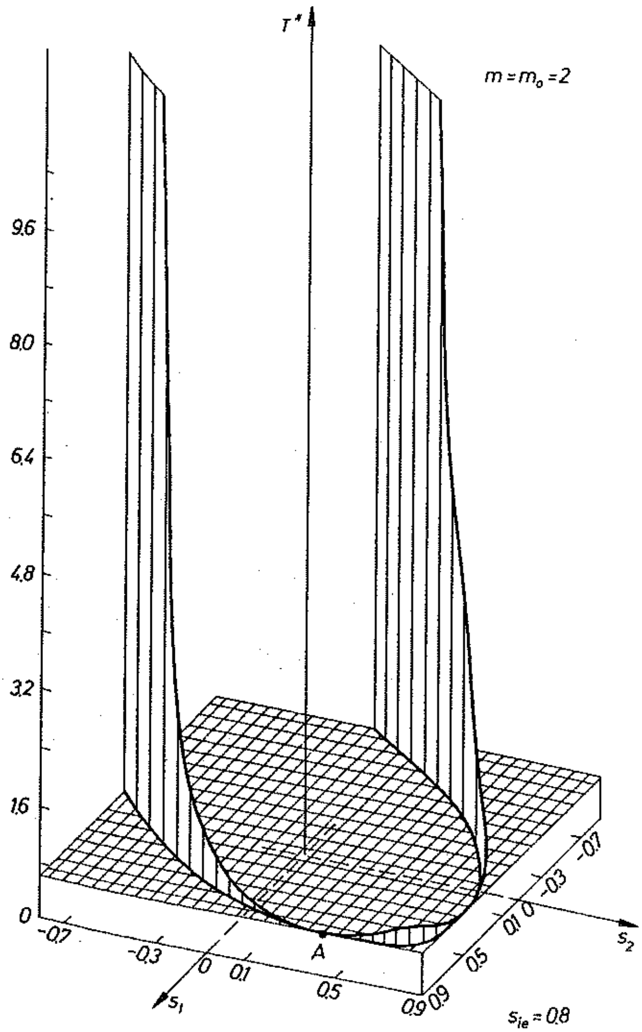
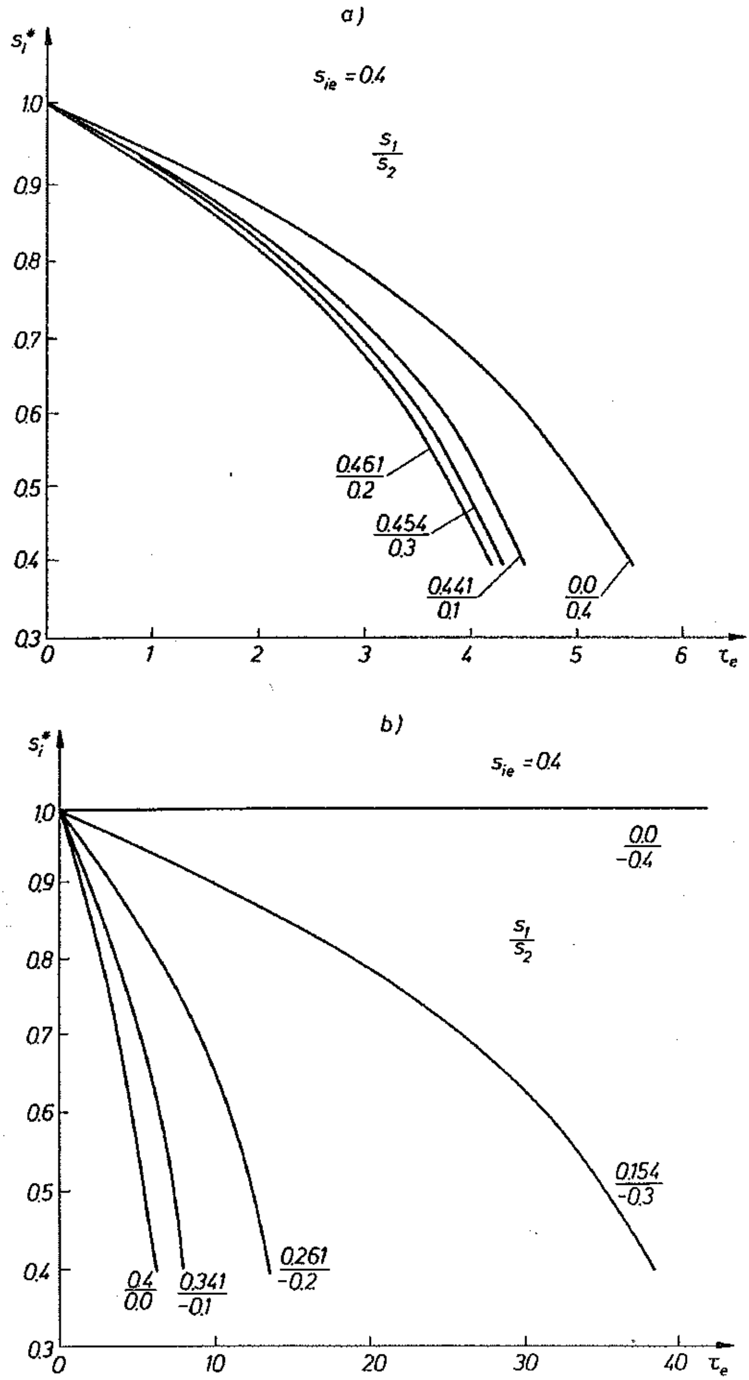


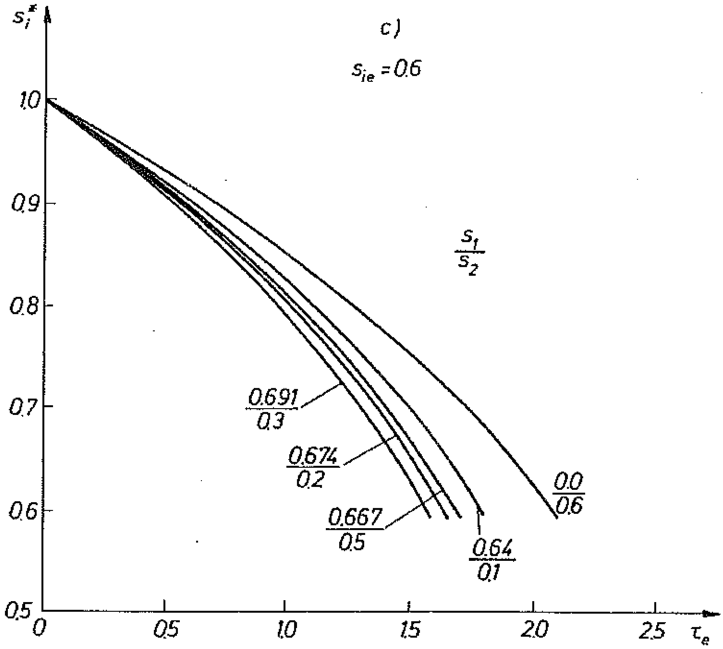

FIG. 11. Surface of time-dependent strength for fixed stresses $s_1$ and $s_2$ forming the same ellipse $s_{ie}$ = const.

Figure 12a shows curves of residual strength for various combinations of stresses $s_1$ and $s_2$ which correspond to the points in the Fig. 10 lying on the ellipse $s_{ie} = 0.4$ within the first quadrant of the coordinate system. Similarly, Fig. 12b shows the curves of residual strength for points lying on the same ellipse within the fourth quadrant of that system. Figs. 12c and d correspond to the effective applied stress $s_{ie} = 0.6$. Similar curves of residual strength can be created for other values of $s_{ie}$.

Figure 13 presents changes of stress and time safety factors determined by means of Eqs. (4.3) and (4.4), for a fixed level of the loads $s_i$ versus the


a)
$s_i^*$
$s_{ie}=0.4$
$\frac{s_1}{s_2}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.461 / 0.2
0.454 / 0.3
0.441 / 0.1
0.0 / 0.4
0
1
2
3
4
5
6
$\tau_e$
b)
$s_i^*$
$s_{ie}=0.4$
0.0 / -0.4
$\frac{s_1}{s_2}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.154 / -0.3
0.261 / -0.2
0.4 / 0.0
0.341 / -0.1
0
10
20
30
40
$\tau_e$

[FIG. 12 a,b]



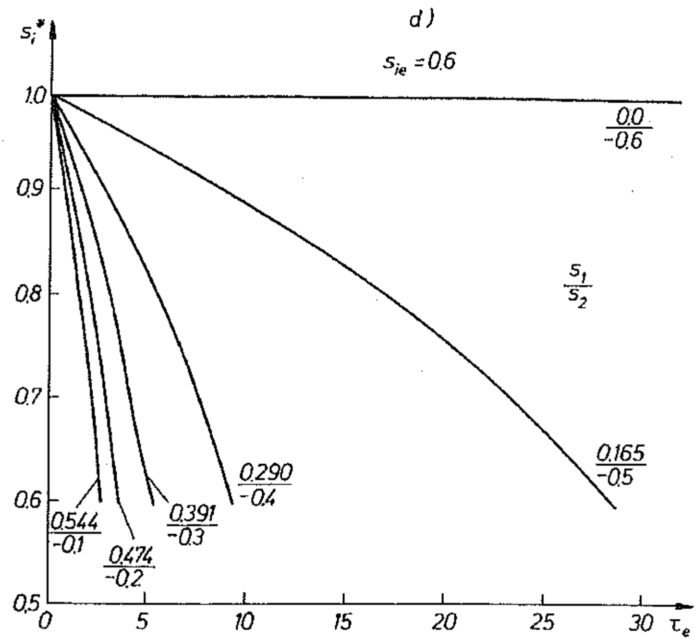
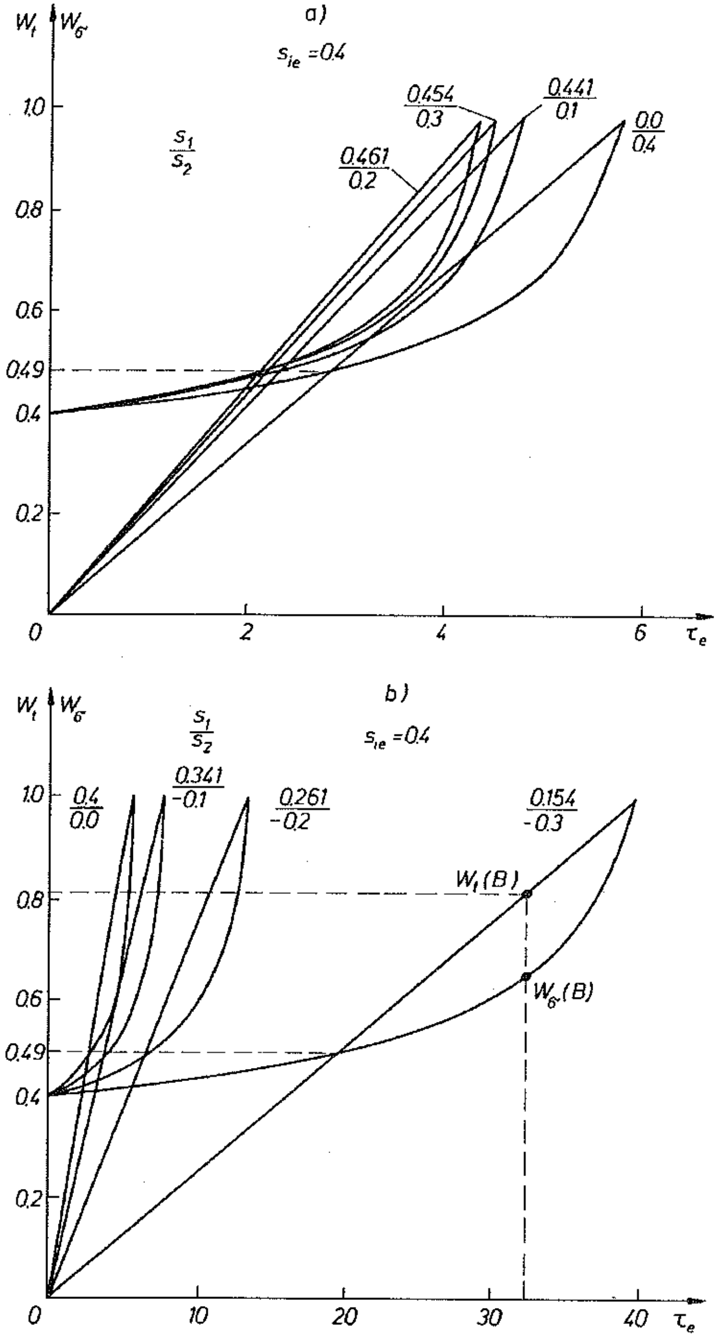




FIG. 12. Curves of residual strength for various combinations of stresses $s_1$ and $s_2$ corresponding to $s_{ie} = 0.4$ and $s_{ie} = 0.6$.



a)
$s_{ie}=0.4$
$W_t$ $W_\sigma$
$\frac{s_1}{s_2}$
0.454/0.3
0.441/0.1
0.461/0.2
0.0/0.4
1.0
0.8
0.6
0.49
0.4
0.2
0
2
4
6
$\tau_e$
b)
$s_{ie}=0.4$
$W_t$ $W_\sigma$
$\frac{s_1}{s_2}$
0.4/0.0
0.341/-0.1
0.261/-0.2
0.154/-0.3
$W_t(B)$
$W_\sigma(B)$
1.0
0.8
0.6
0.49
0.4
0.2
0
10
20
30
40
$\tau_e$


[Fig. 13 a,b]

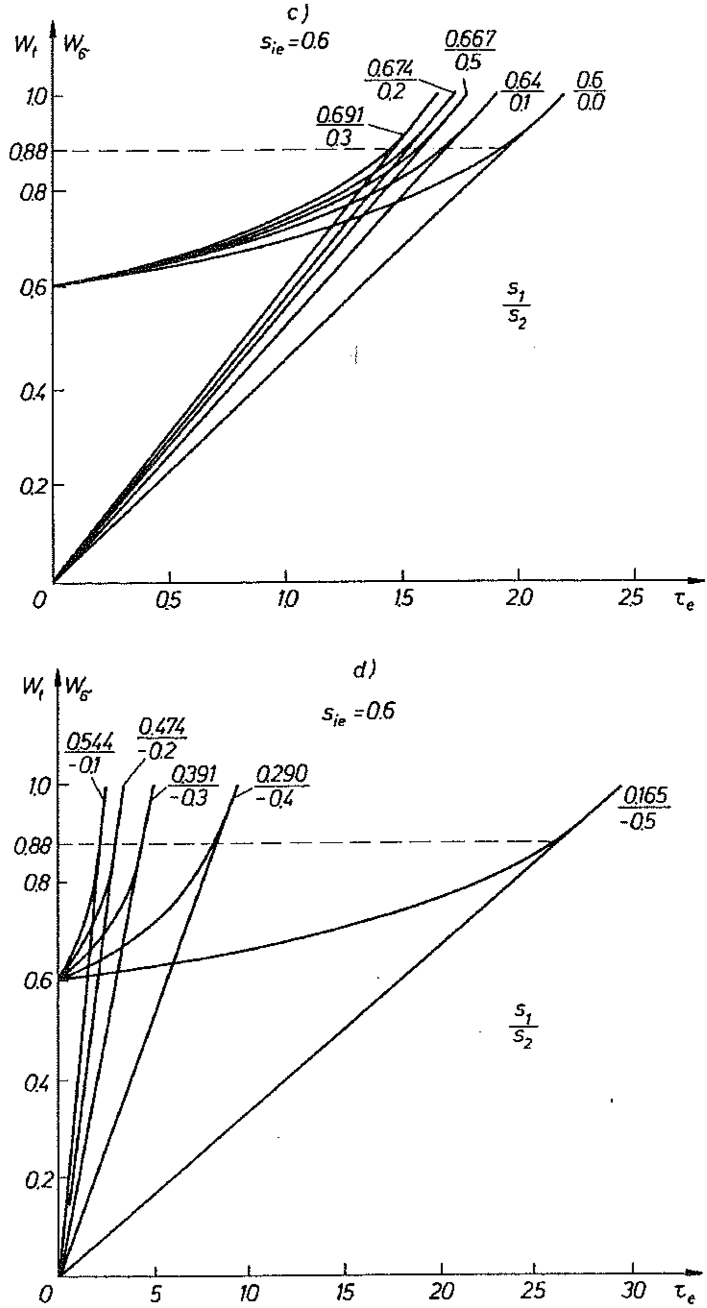



FIG. 13. Changes of stress and time safety factors for fixed level of the loads $s_i$ versus operation time $\tau_e$.

operation time $\tau_e$.

Intersection points of the curves $W_\sigma(\tau_e)$ and $W_t(\tau_e)$ for different ratios $s_1/s_2$ and at fixed $s_{ie}$ determine such combinations of the stresses $s_1$ and $s_2$ when $W_\sigma = W_t$. For various combinations of $s_1$ and $s_2$ these points lie on the same line. When $s_{ie}$ increases, this line moves towards the line $W_\sigma = W_t = 1$.

Beginning with $W_\sigma = W_t \neq 1$, the time safety factor is higher than the stress-dependent one, i.e. the safety in the time version is lower than that in stress version.

Figure 14 shows the values of applied stresses $s_{ie}$ for which the factors of time- and stress-dependent strength are equal. The diagrams $W_t = W_\sigma(s_{ie})$ have been drawn for various but fixed values $m = m_0$.

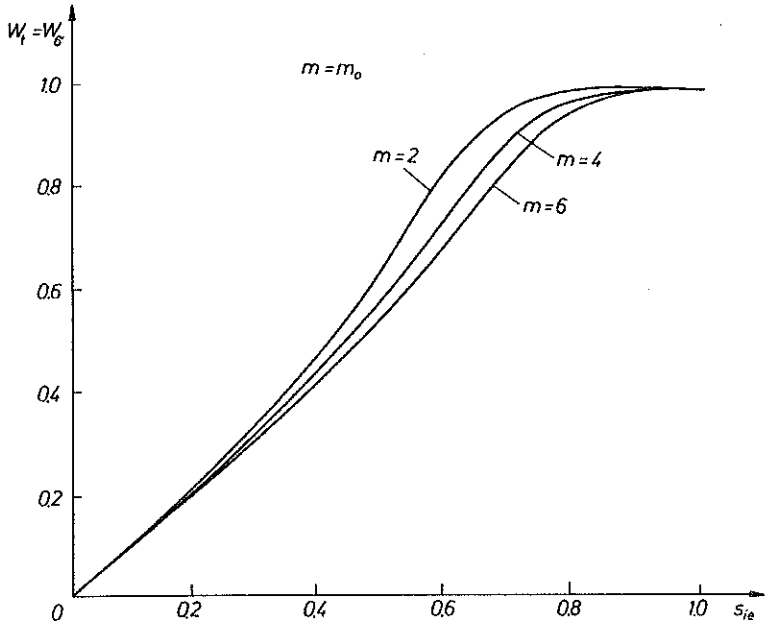



FIG. 14. Diagrams $W_t = W_\sigma(s_{ie})$ for various values of $m = m_0$.

As it follows from Fig. 14, the influence of the material constants on the applied stresses for which $W_t = W_\sigma$, is insignificant at very low and at very high applied stresses. For stresses $s_{ie}$ from within the range 0.2 ... 0.8, growth of the material constant causes, that the same safety states are reached at higher applied stress $s_{ie}$.

Figure 15 shows the same correlations at a fixed material constant $m$ and variable $m_0$.

From the diagrams in Fig. 15 it is seen that growth of the material constant $m_0$ reduces the applied stress $s_{ie}$ at which the same safety state in

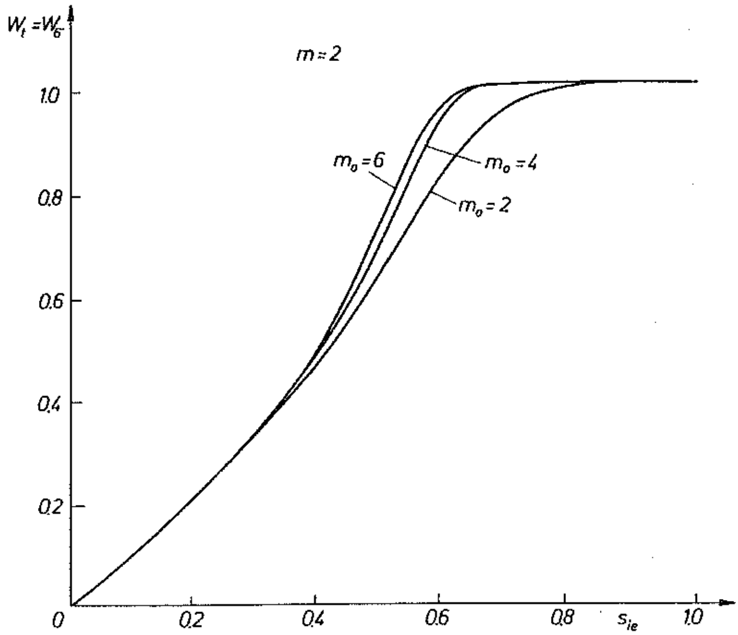



FIG. 15. Diagrams $W_t = W_\sigma(s_{ie})$ for fixed material constant $m$ and variable $m_0$.

time and stress version is reached. Moreover, changes of material constant $m_0$ are significant only for a narrow range of the stresses applied.

Practically, if the safety factors are known, we are able to predict the life-time of the material according to its way of loading. Let us assume that stresses at a point of the body are, for example, $s_1 = 0.154$, $s_2 = -0.3$, what corresponds to point $B$ of the ellipse $s_{ie} = 0.4$ in Fig. 10. For the time $\tau_e = 32.5$, the stress safety factor is $W_\sigma = 0.65$. It means that 65% of the stress (which is residual strength for the time considered), have been reached. This stress is $s_i^* = 0.62$. So, if at the considered time the stresses are changed in such a way that their combination corresponds to the point of ellipse $s_{ie} = 0.62$, then this time is the time to rupture. At $\tau_e = 32.5$ the time safety factor is $W_t = 0.82$. It means that, at a fixed initial level of stress, the time elapsed afterwards constitutes 18% of the whole time used (see Fig. 13b).

## 5. Summary

From the considerations presented above it follows that introduction of two different safety factors enables us to estimate the degree of material exertion at a point during creep. This allows us to predict the period of safe operation according to the value of the stress applied and its duration.

The Law of Cumulation of Damages in the form of (3.13) can be successfully used to describe the damaged material safety, both in the time meaning and the strength meaning. It allows us to estimate the decrease of residual strength during operation of the material depending on the load history.

The presented analysis of material safety has been carried out at the level of a material point. In the next step, such analysis should be generalized to any engineering structure for which the level of loading and its time of action are the main independent variables.

## References


1. M. Chrzanowski and J. Madej, *Isochronous creep rupture curves in plane stress*, Mech. Res. Comm., **7**, 1, 47–48, 1980.
2. M. Chrzanowski and J. Madej, *A CDM motivated safety factor for creeping structures in plane stress condition*, V-th. Int. Conf. Creep of Mat., USA, 447–452, 1992.
3. D. Hayhurst *et al., Development of continuum damage in the creep rupture of notched bars*, Phil. Tran. R. Soc. London, **A311**, 103–129, 1984.
4. L.M. Kachanov, *Time to failure in creep conditions*, Izv. AN, SSSR, OTN, 26–31, 1958.
5. A. Litewka and J. Morzynska, *Theoretical end experimental study of fracture of a damaged solid*, Res. Mech., **3**, 27, 1989.
6. R.K. Penny, *The usefulness of engineering damage parameters during creep*, Metals and Mat., **8**, 2, 278–283, 1974.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