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NUMERICAL SOLUTION TO THE VARIATIONAL PROBLEM OF SEISMOSCOPY

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The method for solving the variational problem of seismoscopy, submitted in the present paper, offers various possibilities for verifying the structure and the properties of a medium. This qualitatively new approximate method for solving numerically such a problem is reduced, by appropriate discretization, to an operation on graphs. An algorithm for seeking in those graphs for paths of minimum length has been worked out. It is adapted to the structure of the graphs and, therefore, is effective, if the way of discretization involves graphs with a number of vertices amounting to some millions. The method is illustrated by a simple but not trivial example, in which analytical and numerical results are compared.

1. Introduction

The quantity to be recorded is the time necessary for a wave to travel the distance from the point of excitation to that of reception. Some attempts have been made by authors interested in such problems to use the results of measurements made for many such pairs of points for verifying the assumed distribution of the propagation velocity of a disturbance over the region considered. The application range of the above method is wide and includes identification of cavities and reservoirs of water, exploration of salt deposits or inspecting the state of dams and supports in mines. It may also be used for ultrasonic examination of machine elements to control their state of stress. The present paper suggests an algorithm for numerical solution of the following problem. For a prescribed distribution of the propagation velocity of a wave, find the time necessary for that wave to travel the distance between a definite pair of points. By appropriate discretization the problem can be reduced to that of seeking in a graph for paths of minimum length. The main difficulties which must be overcome are those of constructing a network of points and handling a graph with a very large number (of some millions) of vertices. The novelty of the paper consists in graphs being used as a means for solving the problem, the construction of a discrete network and modification of the algorithm for determining the shortest possible path. The algorithm thus obtained for solving the problem is essentially new as compared with the existing ones. The equations which were integrated in former attempts to solve numerically the problem considered [2, 3] were those of the seismic ray propagation from the point of excitation at a definite angle. In this case there appears a separate problem, which is that of reaching the point of reception, which has already been determined. The approximation error made at each computation step may result in considerable deformation of the ray, including formation of a loop. As a result, the algorithms hitherto obtained were time-consuming and of poor accuracy. The algorithm proposed in the present paper is free of those drawbacks.

2. Formulation of the problem

Let us consider the following plane problem. It is assumed that a wave is propagating in the plane

$$X = \{(x, y):\ 0 \le x \le x_c,\ 0 \le z \le z_c\}. \tag{2.1}$$

The time needed by a wave generated at a certain instant of time at a point N to reach the point of reception O is determined as a minimum value of the following curvilinear integrals taken along all the possible curves connecting the point N with the point O

$$t_{NO} = \min_{\overline{NO}} \int_N^O \frac{ds}{v(x, z)}, \tag{2.2}$$

where $v(x, z)$ expresses the velocity distribution of the wave. This is a typical variational problem of finding the minimum of a curvilinear integral, the integration curve being unknown.

3. A discrete model

3.1. *The discrete network*

Analytical solution of the problem (2.2) is possible only for a few simple velocity distributions $v(x, z)$. In the general case we must have recourse to numerical solution methods. The present numerical solution method is based on discretization of the problem. It is assumed that the wave issuing from a prescribed point of a discrete network may propagate only to points in a certain neighbourhood of that point. The problem of choice of the type of network is fundamental. A rectangular network does not satisfy the fundamental condition that increased network density should result in better accuracy. Let us take as an example the case of constant velocity in the region considered and let the wave propagation between vertices of the same basic rectangle be admissible. Then, in the case illustrated in Fig. 1, the shortest path between the points N and O is independent of the degree of density of the network. It follows that the network must be more complicated. Its structure is shown in Fig. 2. The entire region is divided into deltoids and triangles according to the diagram presented.

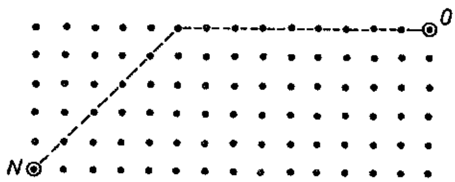

FIG. 1.

The discrete network is composed of:

1) the vertices of all the deltoids and triangles,

2) the points of division of the sides of the deltoids and triangles.

A fragment of the network illustrating its structure is represented in Fig. 2. The network for the example discussed is shown in Fig. 11. A network constructed in such a manner is sufficiently irregular and enables an approximation to a curve to be improved by increasing the density of the network. It should also be observed that this densification can be achieved in two ways, that is by increasing the number of layers or by dividing the sides. An increase in number of layers is essential, above all, for the approximation to the velocity distribution in the region. Together with simultaneous densification of subdivision of sides, it influences the accuracy of approximation to a curve.

3.2. *Approximation to the propagation process of a wave*

It is assumed that a wave may propagate between points of the discrete network described above if the following rules are satisfied.

Starting out from a given point, it may pass along a rectilinear path to any point belonging to a deltoid or a triangle (mentioned in Sec. 3.1) to which the initial point belongs and which does not lie on the same side, and

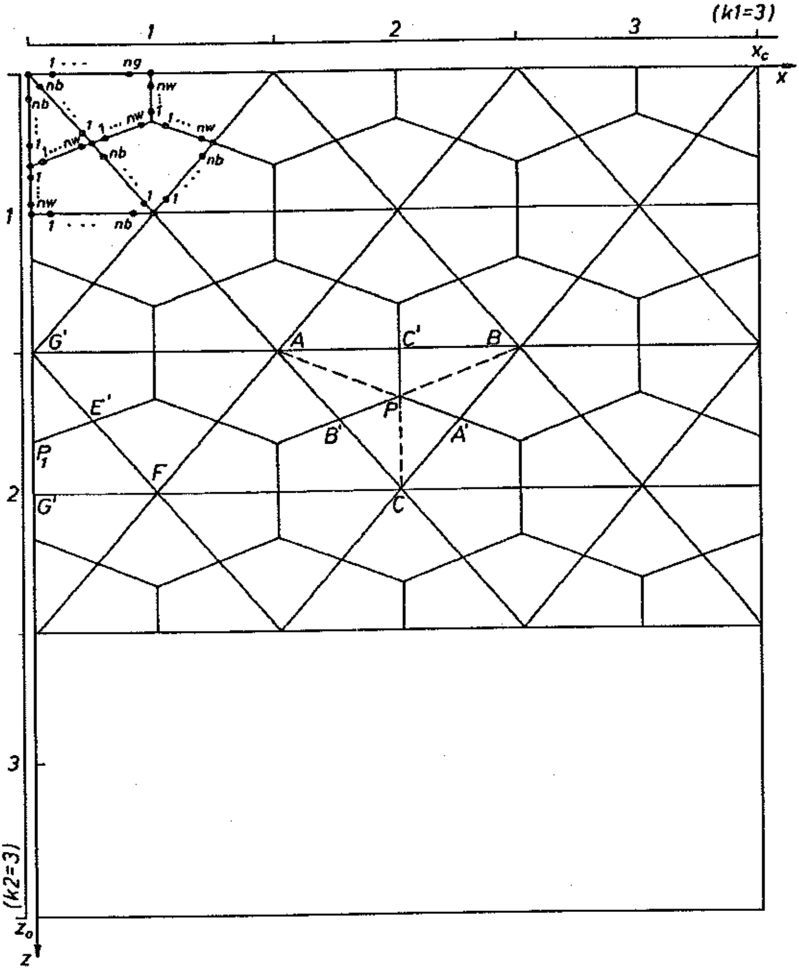

FIG. 2.

to the neighbouring points of those edges on which the initial point lies. An example is given in Fig. 3.

• To each deltoid and triangle an inverse wave velocity is assigned. This correspondence is a consequence of the approximation to the velocity.

• The time necessary for a wave to cover the distance between two points satisfying the above rules is equal to the product of the geometrical distance between them and the inverse of the velocity corresponding to the given deltoid, if the segment of the curve between those two points lies inside a deltoid or a triangle.

If this segment lies on the boundary between two deltoids or between a deltoid and a triangle, the time mentioned is equal to the product of

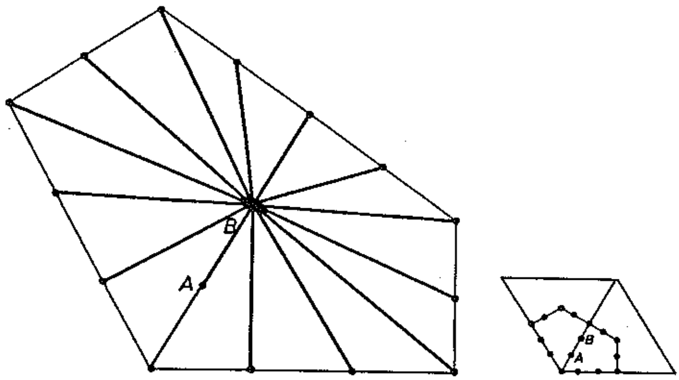

FIG. 3.

the geometrical distance and the arithmetic mean of the inverse velocities corresponding to the given deltoids or a deltoid and a triangle. For the sides coinciding with the boundary the inverse velocity is a prescribed value.

From the point of view of approximation to the variational problem (2.2), the above assumptions are equivalent to that of approximating the integration curve by segments, and they establish the rules for determining the integration function for each segment.

3.3. The graph problem corresponding to the original problem

The original problem consists, with the discrete approximation explained above, in finding a sequence of points

$$x_0 = N, \qquad x_1, \ldots, x_n = O \tag{3.1}$$

such that a wave may travel the distance between any pair of neighbouring points x_i, x_{i+1} $(i = 1, \ldots, n-1)$, according to the rules explained in Sec. 3.2 and ensuring that the sum of times of travel for consecutive segments $[x_1, x_{i+1}]$ $(i = 1, \ldots, n-1)$ is minimum. There is an obvious analogy to the problem of finding the shortest path in the graph. The realization of this idea will furnish a tool for solving the problem which occurs as a result of discrete approximation to the original problem. The number of vertices in the graph may be considerable and is estimated at some millions, which means that the algorithms as yet available for determining the shortest path, are useless. The algorithm which is to be submitted here takes into consideration the special features of the graph obtained, thus enabling us to determine simultaneously all the paths for the expected number of points.

A non-oriented graph assigned to a discrete approximation of the original problems can be determined as follows:

- a vertex of a graph is assigned to each point of the discrete network;
- two vertices of the graph are interconnected by an edge, if and only if direct passage of a wave between the relevant points is admissible, in agreement with the rules explained in Sec. 3.2. The weight of the edge is equal to the time necessary for the wave to pass between those points.

The graphs considered in the present paper will now be represented in the form of the corresponding points of the network, without considering the edges.

4. An algorithm for determining the length of the shortest path between any definite pair of vertices in a graph of a certain class of structure

Let us consider a graph, the set of vertices of which is

$$X = \{x_1, \ldots, x_n\}. \tag{4.1}$$

Let

$$X = X_1 \cup X_2, \tag{4.2}$$

$$X_3 = X_1 \cap X_2, \tag{4.3}$$

and let us assume that the structure of the edges satisfies the condition that there are no edges between vertices of the set $X_1 \backslash X_3$ and those of the set $X_2 \backslash X_3$. An example of such a graph is shown in Fig. 4. The algorithm to be

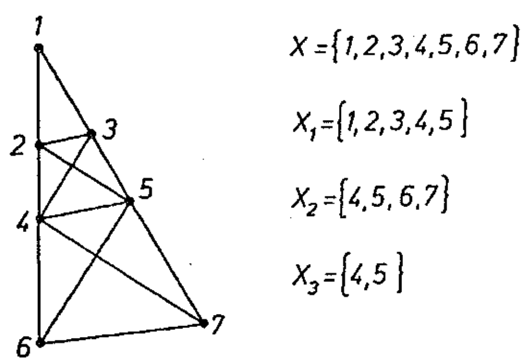

FIG. 4.

used for determining the lengths of the shortest paths between all the pairs of vertices of any subset of vertices $X' \subset X$ is as follows. Let

$$\begin{aligned} X_1' &= X' \cap X_1, \\ X_2' &= X' \cap X_2, \\ X'' &= X' \cup X_3, \\ X_1'' &= X_1' \cup X_3, \\ X_2'' &= X_2' \cup X_3. \end{aligned} \tag{4.4}$$

Let us now denote $d_X(x_i, x_j)$ – the length of the shortest path between the vertices x_i, x_j in the graph, the set of vertices of which is X; $s(x_i, x_j)$ – the weight of the edge between the vertices x_i, x_j.

The algorithm considered is as follows:

1. $s(x_i, x_j) := d_{X_1}(x_i, x_j)$ for all the pairs $x_i \in X_1''\backslash X_3,\ x_j \in X_1'',\ x_i \neq x_j$.
2. $s(x_i, x_j) := d_{X_2}(x_i, x_j)$ for all the pairs $x_i \in X_2''\backslash X_3,\ x_j \in X_2'',\ x_i \neq x_j$.
3. $s(x_i, x_j) := \min(d_{X_1}(x_i, x_j),\ d_{X_2}(x_i, x_j))$ for all the pairs $x_i, x_j \in X_3$.
4. $s(x_i, x_j) := \infty$ for $x_i \in X''\backslash X_3,\ x_j \in X_2''\backslash X_3$.
5. For every $x_i \in X'$ execute the steps 6 to 10.
6. $X_3' := X_3$, $c(x_i, x_j) = s(x_i, x_j)$ for $x_j \in X''$.
7. $x^* := \{x \in X_3' :\ c(x_i, x) = \min_{x_j \in X_3'}(c(x_i, x_j))\}$.
8. $c(x_i, x_j) := \min(c(x_i, x_j),\ c(x_i, x^*) + c(x^*, x_j))$ for $x_j \in X''$.
9. $X_3' := X_3' \backslash \{x^*\}$.
10. If $X_3' \neq \emptyset$, go to 6.
11. Stop.

As a result we find components of $c(x_i,\ x_j)$ being the lengths of the shortest paths between all the pairs of X' vertices. The algorithm can be speeded up if we take in p. 8 into account the fact that the graph is not oriented.

The correctness of the above algorithm can be verified as follows. The change of the weight of the edges, which was made in 1 to 4, does not change the lengths of the shortest paths between the vertices. This follows from the general properties of the shortest path. The correctness of the second part of the algorithm follows from the analysis of the Dijkstra algorithm [1, 6, 7, 8] for seeking for shortest paths and determining their lengths. The set of vertices considered has only been reeduced to X_3. If the remaining vertices of the graphs are considered, this fact does not change the labels in the Dijkstra algorithm, owing to the weights of the edges, having been introduced in particular fragments of the graph as the lengths of the shortest paths in sub-graphs – if, in addition, the following facts are taken into account. Each path between any vertex of the set $X_1'\backslash X_3$ and any vertex of the set $X_2'\backslash X_3$

must pass through a vertex of the set X_3. The shortest path, the near and far end of which belong to the same set X_1 or X_2, lies entirely in that set or contains sub-paths of the other set which must, in view of the structure of the connections in the original graph, begin and end at vertices of the set X_3.

5. Determination of shortest paths in the graphs assigned to the original problem

The estimated number of vertices in graph obtained as a result of discrete approximation to the original problem makes it impossible to use the algorithms available for determining the lengths of the shortest paths. As a consequence, it has been found to be necessary to work out an algorithm based on the properties of the graph obtained, in particular the fact of the connections between vertices being of a somewhat local character. This enables us to separate sub-graphs, the structure of which is discussed in Sec. 4. The algorithm described there will be used in an iterative manner. Vertices of a given set denoted, from now on, Y_w, between which the shortest distances are to be determined, and those which will enter the set X_3 at further iteration steps, are joined at each step. Iterations are performed by adding vertices of consecutive layers (Fig. 5). The double line represents, in

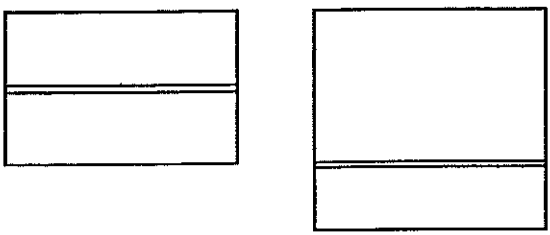

Fig. 5.

all the figures, the edges, the vertices of which constitute each time the set X_3. From the upper ($z = 0$), left-hand ($x = 0$) and right-hand boundary ($x = x_0$) we take the vertices belonging to the set Y_w. As regards the lower boundary they will all enter the set X_3. Once the last layer of the lower boundary has been included, we take only vertices from the set Y_w, if there are any. The reason for which the number of points of division admitted at the boundary between layers is different from that for the inside is now evident. The lengths of the shortest paths in a layer are determined by using, in an iterative manner, the algorithm described above according to the diagram represented in Fig. 6. The algorithm described in Sec. 4 is also used, in a similar manner, for particular subregions according to the diagrams shown in Figs. 7 to 10.

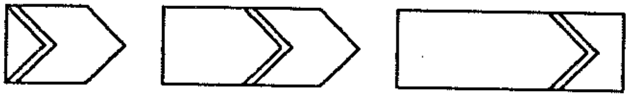

FIG. 6.

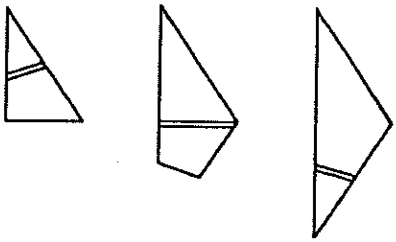

FIG. 7.

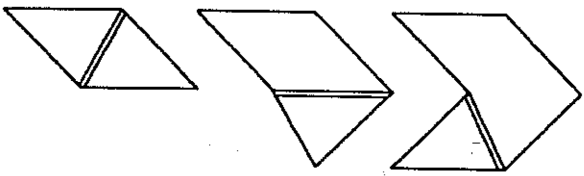

FIG. 8.

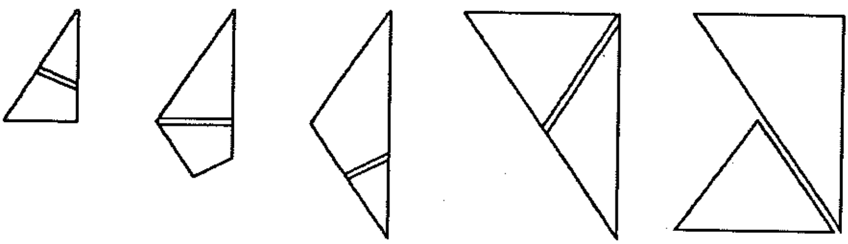

FIG. 9.

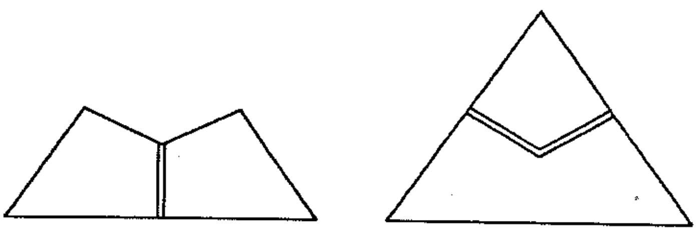

FIG. 10.

The algorithm used ensures economy of the computer memory and the computer time because (in the first case) we use only those vertices which lie at the boundaries if we pass to larger regions, and (in the second case) because the operations on the entire graph are represented by operations performed on subgraphs, the number of vertices of which is much smaller than the total number of vertices.

6. Example

A program for an IBM computer has been worked out in the Fortran language; the parameters of that program make possible its practical application. The parameters of the problem which will be used as an example are presented in Fig. 11.

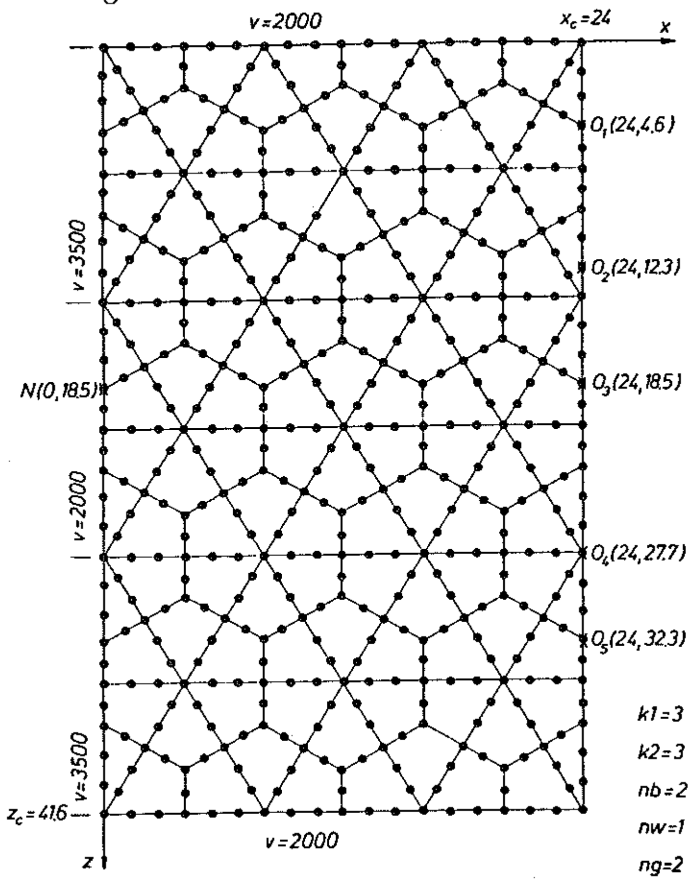

Fig. 11.

A three-layer body has been considered. It was assumed that the velocity of motion of a disturbance in the middle layer is much lower than those in the neighbouring layers. The time of travel of a disturbance from the point N of the left-hand boundary of the middle layer to points lying in the upper and middle layer, at the boundary between those layers and in the lower layer, was determined by analytical and numerical means. The analytical results were obtained directly from the relation (2.2). For the points O_1, O_2, O_4 and O_5 this was found to be reduced to the familiar laws of refraction. The case of O_4, in particular, was that of limit angle. In the case of O_3, the fact of its being symmetric to N about the vertical symmetry axis of the problem was made use of. As a result, the shortest trajectory in the sense of time between those points is as follows.

From N it runs to reach the boundary between the layers 1 and 2 at the limit angle, then, along the boundary at the velocity of the layer 1 and, at the limit angle, to the point O_3.

Table 1 contains distances between the point N (Fig. 11) and the points of reception.

Table 1.

Point of reception	Distance as obtained by numerical means	Distance as obtained by analytical means
O_1	932×10^{-5}	931×10^{-5}
O_2	886×10^{-5}	877×10^{-5}
O_3	1084×10^{-5}	1065×10^{-5}
O_4	1076×10^{-5}	1065×10^{-5}
O_5	1088×10^{-5}	1082×10^{-5}

This example may look very simple, though it illustrates the principal features of the propagation of disturbances, thus confirming the effectiveness of the algorithm used. Application for processing the results of practical measurements will be the subject of a separate paper.

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